



On Periodic Solutions of a Boundary Value Problem for an Anisotropic Fourth-Order Equation

A. V. Yuldasheva^{1}, S. A. Sharipova^{2*}*

¹ Branch of MSU in Tashkent, 22 Amir Temur Street, Tashkent, 100060, Uzbekistan

² National University of Uzbekistan, 4 University Street, Tashkent, 100174, Uzbekistan

Abstract. The paper considers a boundary value problem for the fourth-order equation $U_{xxxx} + U_{yyyy} = 0$, defined in a rectangular domain under the condition of periodicity in one of the spatial variables. The studied model arises in describing stationary deformation processes of anisotropic elastic media, where material properties differ by direction. A feature of the formulation is the combination of rigid boundary conditions on one boundary of the domain and the specification of the bending moment distribution on the opposite boundary. It is shown that when expanding the solution into a Fourier series in the periodic coordinate, the problem reduces to a fourth-order differential equation. Small denominators arise, due to the spectral properties of the corresponding differential operator. The presence of small denominators can lead to instability of the solution and ill-posedness of the problem in the classical sense. The paper discusses the conditions for the existence of a periodic solution. A connection is established between the algebraic nature of the dimensionless parameter, given by the ratio of the domain thickness to a fundamental geometric constant, and the smoothness of the boundary function. It is shown that the regularity of the solution is significantly determined by the Diophantine properties of this parameter, which influence the behavior of small denominators in the spectral expansion. It is proved that the problem is solvable for almost all values of the specified parameter in the sense of Lebesgue measure, except for a set of exceptional cases corresponding to strong algebraic dependence. The obtained results can be applied in modeling bending processes in layered and orthotropic structures, as well as in the study of ill-posed problems in mathematical physics.

Key words: boundary value problem, fourth-order equation, periodic solution, anisotropic medium, small denominators, ill-posed problem, plate bending, Fourier series.

Received: 18.05.2026; Revised: 18.06.2026; Accepted: 20.06.2026; First online: 14.09.2026

For citation. Yuldasheva A. V., Sharipova S. A. On periodic solutions of a boundary value problem for an anisotropic fourth-order equation. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2026, 54: 2, 48-56. EDN: BYRGNS. <https://doi.org/10.26117/2079-6641-2026-54-2-48-56>.

Funding. This work was not supported by any external funding.

Competing interests. There are no conflicts of interest regarding authorship and publication.

Contribution and Responsibility. All authors contributed to this article. Authors are solely responsible for providing the final version of the article in print. The final version of the manuscript was approved by all authors.

***Correspondence:**  E-mail: a_v_yuldasheva@mail.ru

The content is published under the terms of the Creative Commons Attribution 4.0 International License

© Yuldasheva A. V., Sharipova S. A., 2026

© Institute of Cosmophysical Research and Radio Wave Propagation, 2026 (original layout, design, compilation)





О периодических решениях краевой задачи для анизотропного уравнения четвертого порядка

А. В. Юлдашева^{1*}, С. А. Шарипова²

¹ Филиал МГУ в Ташкенте, ул. Амира Темура, 22, Ташкент, 100060, Узбекистан

² Национальный университет Узбекистана, ул. Университетская, 4, Ташкент, 100174, Узбекистан

Аннотация. В статье рассматривается краевая задача для уравнения четвертого порядка $U_{xxxx} + U_{yyyy} = 0$, определенного в прямоугольной области при условии периодичности по одной из пространственных переменных. Исследуемая модель возникает при описании стационарных процессов деформации анизотропных упругих сред, где свойства материала различаются по направлению. Особенностью формулировки является сочетание жестких граничных условий на одной границе области и задания распределения изгибающего момента на противоположной границе. Показано, что при разложении решения в ряд Фурье по периодической координате задача сводится к дифференциальному уравнению четвертого порядка. Вследствие спектральных свойств соответствующего дифференциального оператора возникают малые знаменатели. Наличие малых знаменателей может привести к неустойчивости решения и некорректности задачи в классическом смысле. В статье обсуждаются условия существования периодического решения. Устанавливается связь между алгебраической природой безразмерного параметра, заданного отношением толщины области к фундаментальной геометрической постоянной, и гладкостью граничной функции. Показано, что регулярность решения существенно определяется диофантовыми свойствами этого параметра, которые влияют на поведение малых знаменателей в спектральном разложении. Доказано, что задача разрешима почти для всех значений заданного параметра в смысле меры Лебега, за исключением ряда исключительных случаев, соответствующих сильной алгебраической зависимости. Полученные результаты могут быть применены в моделировании процессов изгиба в слоистых и ортотропных структурах, а также в изучении некорректно поставленных задач в математической физике.

Ключевые слова: краевая задача, уравнение четвертого порядка, периодическое решение, анизотропная среда, малые знаменатели, некорректно поставленная задача, изгиб пластины, ряды Фурье

Получение: 18.05.2026; Исправление: 18.06.2026; Принятие: 20.06.2026; Публикация онлайн: 14.09.2026

Для цитирования. Yuldasheva A. V., Sharipova S. A. On periodic solutions of a boundary value problem for an anisotropic fourth-order equation // Вестник КРАУНЦ. Физ.-мат. науки. 2026. Т. 54. № 2. С. 48-56. EDN: BYRGNS. <https://doi.org/10.26117/2079-6641-2026-54-2-48-56>.

Финансирование. Работа выполнена без поддержки грантов.

Конкурирующие интересы. Конфликтов интересов в отношении авторства и публикации нет.

Авторский вклад и ответственность. Авторы участвовали в написании статьи и полностью несут ответственность за предоставление окончательной версии статьи в печать.

***Корреспонденция:** ✉ E-mail: a_v_yuldasheva@mail.ru

Контент публикуется на условиях Creative Commons Attribution 4.0 International License

© Yuldasheva A. V., Sharipova S. A., 2026

© ИКИР ДВО РАН, 2026 (оригинал-макет, дизайн, составление)



Introduction

In recent years, significant attention has been paid to the study of ill-posed problems, including ill-posed boundary value problems for partial differential equations. This interest is driven both by their numerous applications in various fields of science and technology and by deep theoretical aspects related to questions of existence, uniqueness, and stability of solutions, as well as the development of regularization methods.

From the perspective of formulating boundary value problems, the most thoroughly studied are partial differential equations of classical types and their natural generalizations (see [1]- [6]). These include elliptic, parabolic, and hyperbolic equations, as well as higher-order differential operators arising in modeling complex physical processes.

Consider the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < \pi, 0 < y < h\}$$

in which we investigate the following fourth-order partial differential equation:

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 0. \quad (1)$$

This equation is supplemented with boundary conditions

$$u(x, 0) = 0, u_y(x, 0) = 0, u_{yy}(x, h) = \psi(x), u_{yyy}(x, 0) = 0, \quad (2)$$

where the function $\psi(x)$ is assumed to be sufficiently smooth and 2π -periodic. Such conditions arise, for example, in describing oscillatory processes or problems in elasticity theory with symmetry or periodic constraints. Let us introduce the parameter

$$\alpha = \frac{h}{\pi\sqrt{2}}. \quad (3)$$

It is known that boundary value problems where data is prescribed on the entire boundary of the domain for differential operators are, in general, ill-posed in the sense of Hadamard. In particular, such problems may not be stable with respect to small perturbations of the initial data. One of the key difficulties in their study is the so-called small denominator problem.

From a mathematical point of view, the phenomenon of small denominators manifests itself in the fact that when representing solutions as Fourier series, the coefficients of these series contain denominators that can approach zero arbitrarily closely. This can lead to possible divergence of the series and, consequently, to the absence of solutions or their instability in classical function spaces.

A classic example is the Dirichlet problem for the wave equation describing string vibrations. As first noted by D. Bourgin and R. Duffin (see [7], as well as [2]), the solvability of this problem depends significantly on the arithmetic nature of the parameter (3), namely whether it is a rational or irrational number.

More precisely, from Theorem 1 of [8], it follows that if the parameter α is an irrational number, then the corresponding Dirichlet problem has a unique solution in

the space L_p . This emphasizes the deep connection between the arithmetic properties of the problem parameters and the analytic behavior of its solutions.

In [9], the question of the existence of an L_2 -solution to the Dirichlet problem for the wave equation in a rectangular domain is considered. Furthermore, related problems for equations containing differences of derivatives of arbitrary even order have been studied in [10]–[12]. These studies expand the understanding of the nature of ill-posedness and allow the analysis of more general classes of differential equations.

The following statement holds.

Теорема 1. *Let the number α , defined by equality (3), be an algebraic number of degree 2. Then, for any function $\psi(x)$ from the Sobolev space $\dot{W}_2^4(-\pi, \pi)$, the boundary value problem (1)–(2) has a unique solution.*

On the Solvability of the Problem

The problem under consideration is 2π -periodic in the variable x , therefore we seek the solution in the form:

$$u(x, y) = \sum_{k=-\infty}^{\infty} u_k(y) e^{ikx} \quad (4)$$

Substituting function (5) into equation (1) and conditions (2), we obtain the following Cauchy problem:

$$u_k^{(4)}(y) + k^4 u_k(y) = 0,$$

$$u_k(0) = 0, u_k'(0) = 0, u_k''(h) = \psi_k, u_k'''(0) = 0.$$

Here ψ_k are the Fourier coefficients of the function ψ with respect to the orthonormal system $\{e^{ikx}\}$.

The solution to this problem has the form:

$$u_0(y) = \psi_0 \frac{y^2}{2}, u_k(y) = \frac{4\psi_k \sinh \frac{ky}{\sqrt{2}} \sin \frac{ky}{\sqrt{2}}}{k^2 \cosh \frac{kh}{\sqrt{2}} \cos \frac{kh}{\sqrt{2}}}. \quad (5)$$

Thus, the formal solution of the problem (1)–(2) is represented by the series (5), whose coefficients $u_k(y)$ are given by formula (5). Since the denominator of the expression for $u_k(y)$ contains the factor $\cos \frac{kh}{\sqrt{2}}$, its behavior can significantly influence the convergence of the solution itself and its derivatives.

Next, we investigate the influence of the denominator $\cos \frac{kh}{\sqrt{2}}$ on the convergence properties of the obtained solution.

Assume that $\cos \frac{kh}{\sqrt{2}} \neq 0$.

Then

$$\frac{kh}{\sqrt{2}} = \frac{\pi}{2} + \pi m + \delta_{km}, \quad (6)$$

where $\delta_{km} \neq 0$.

Since the cosine function vanishes when its argument is

$$\frac{\pi}{2} + \pi m, \quad m \in \mathbb{Z},$$

the quantity δ_{km} characterizes the deviation of the argument from the points where the cosine becomes zero. Moreover, one can always choose a number $m \in \mathbb{Z}$ such that

$$|\delta_{km}| < \frac{\pi}{2}.$$

From equality (6) we obtain another representation for the parameter α defined by formula (3):

$$\alpha = \frac{h}{\pi\sqrt{2}} = \frac{2m+1}{2k} + \frac{\delta_{km}}{\pi k}. \quad (7)$$

Using representation (6), we obtain the following estimate.

Lemma 1. *If $\alpha = \frac{h}{\pi\sqrt{2}}$ is an algebraic number of the second degree. Then the estimate*

$$|\delta_{km}| \geq \frac{c}{|k|} \quad (8)$$

holds.

Indeed, from (6) we find

$$\delta_{km} = \pi k \left(\alpha - \frac{2m+1}{2k} \right).$$

By condition, the number α is an algebraic number of the second degree. Consequently, by Liouville's theorem [13], there exists a constant $C > 0$ such that for any natural numbers k and m the following inequality holds:

$$\left| \alpha - \frac{2m+1}{2k} \right| \geq \frac{C}{|k|^2}.$$

From the last inequality, we find

$$|\delta_{km}| = \pi |k| \left| \alpha - \frac{2m+1}{2k} \right| \geq \frac{\pi C}{|k|}.$$

Choosing $c = \pi C$, we obtain estimate (7). The lemma is proved.

Now we return to estimating the denominators in series (5). Taking into account (6), we have

$$\cos \frac{kh}{\sqrt{2}} = \cos \left[\frac{\pi}{2} + \pi m + \delta_{km} \right] = (-1)^{m+1} \sin \delta_{km}.$$

Since $0 < |\delta_{km}| < \pi/2$, then, by Lemma 1, we obtain the following chain of estimates:

$$\left| \cos \frac{kh}{\sqrt{2}} \right| = |\sin \delta_{km}| \geq \frac{2}{\pi} |\delta_{km}| \geq \frac{c_1}{|k|}, \quad (9)$$

where $c_1 = 2c/\pi$.

Proof of the Main Result

In this section, we will prove Theorem 1. To do this, we establish the uniform convergence of series (5), as well as the uniform convergence of the series obtained from (5) after fourfold differentiation with respect to the variables x and y .

The fourth derivative of series (5) with respect to x has the form

$$\frac{\partial^4 u}{\partial x^4} = \sum_{k=-\infty}^{\infty} \frac{4k^2 \psi_k \sinh \frac{ky}{\sqrt{2}} \sin \frac{ky}{\sqrt{2}}}{\cosh \frac{kh}{\sqrt{2}} \cos \frac{kh}{\sqrt{2}}} e^{ikx}. \quad (10)$$

Consider the series that majorizes series (9):

$$\sum_{k=-\infty}^{\infty} \frac{4|k|^2 |\psi_k|}{|\cos \frac{kh}{\sqrt{2}}|}. \quad (11)$$

Using estimate (8), we obtain

$$\sum_{k=-\infty}^{\infty} \frac{4|k|^2 |\psi_k|}{|\cos \frac{kh}{\sqrt{2}}|} \leq \text{const} \sum_{k=-\infty}^{\infty} |k|^3 |\psi_k| \leq \text{const} \sum_{k=-\infty}^{\infty} (1 + |k|^2)^{3/2} |\psi_k|. \quad (12)$$

Next, applying the Cauchy–Bunyakovsky inequality, we get

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{4|k|^2 |\psi_k|}{|\cos \frac{kh}{\sqrt{2}}|} &\leq \text{const} \sum_{k=-\infty}^{\infty} (1 + |k|^2)^2 |\psi_k| \frac{1}{(1 + |k|^2)^{1/2}} \leq \\ &\leq \text{const} \left(\sum_{k=-\infty}^{\infty} (1 + |k|^2)^4 |\psi_k|^2 \right)^{1/2} \left(\sum_{k=-\infty}^{\infty} \frac{1}{1 + |k|^2} \right)^{1/2}. \end{aligned}$$

Since

$$\sum_{k=-\infty}^{\infty} \frac{1}{1 + |k|^2} < \infty$$

and also due to the conditions imposed on the function ψ , the last expression is finite. Consequently, series (9) converges uniformly.

It follows that solution (5) also converges uniformly, and it can be differentiated term-by-term with respect to the variables x and y up to four times inclusive. The series obtained after differentiation also converge uniformly.

Uniqueness follows directly from representation (5). Indeed, if $\psi(x) \equiv 0$, then $\psi_k = 0$ for all k , and therefore $u_k(y) \equiv 0$. Since the denominator in (5) does not vanish under the assumptions of the theorem, all Fourier coefficients are uniquely determined. Hence the solution is unique.

Theorem 1 is proved.

Some Generalizations

We note that the solvability of the boundary value problem (1)–(2) (in a slightly narrower class of right-hand sides) is preserved also when α is an arbitrary algebraic number, not necessarily of the second degree.

Теорема 2. *Let the number α , defined by equality (3), be an algebraic number of degree $k > 2$. Then, for any function $\psi(x)$ from the Sobolev space $\dot{W}_2^m(-\pi, \pi)$, $m \geq 4$, the boundary value problem (1)–(2) has a unique solution.*

The statement of the theorem follows from the estimate obtained in [14] for an algebraic number of arbitrary degree:

$$|\alpha - \frac{2m+1}{2k}| \geq \frac{C}{|k|^{2+\varepsilon}}, \quad (13)$$

which holds for any $\varepsilon > 0$.

In this case, estimate (11) takes the form

$$\sum_{k=-\infty}^{\infty} \frac{4|k|^2|\psi_k|}{|\cos \frac{kh}{\sqrt{2}}|} \leq \text{const} \sum_{k=-\infty}^{\infty} |k|^{3+\varepsilon}|\psi_k|.$$

Applying the Cauchy–Bunyakovsky inequality, we obtain

$$\sum_{k=-\infty}^{\infty} (1+k^2)^{\frac{3+\varepsilon}{2}}|\psi_k| \leq \left(\sum_{k=-\infty}^{\infty} \frac{(1+k^2)^{3+\varepsilon}}{(1+k^2)^m} \right)^{1/2} \left(\sum_{k=-\infty}^{\infty} (1+k^2)^m|\psi_k|^2 \right)^{1/2}.$$

Since $\psi \in W_2^m(-\pi, \pi)$, the second factor is finite. The first series behaves as

$$\sum_{k=-\infty}^{\infty} (1+k^2)^{3+\varepsilon-m} \sim \sum_{k=1}^{\infty} k^{6+2\varepsilon-2m}.$$

Therefore, it converges whenever

$$6+2\varepsilon-2m < -1,$$

that is,

$$m > \frac{7+2\varepsilon}{2}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, it is sufficient to assume

$$m \geq 4.$$

Thus, the series in (12) converges, which proves Theorem 2.

Theorems 1 and 2 are valid not only for values of α that are algebraic numbers, but also for almost all (in the sense of Lebesgue measure) real values of $\alpha = \frac{h}{\pi\sqrt{2}}$.

Recall that a set $E \subset \mathbb{R}$ is called a set of full measure (Lebesgue) if its complement has zero Lebesgue measure, i.e.,

$$\mu(\mathbb{R}^n \setminus E) = 0,$$

where $\mu(\cdot)$ denotes the Lebesgue measure.

Теорема 3. *Let α be a real number defined by equality (3). Then, for almost all (in the sense of Lebesgue measure) $\alpha \in E$, where E is a set of full measure, the boundary value problem (1)–(2) has a unique solution provided that $\psi(x) \in \tilde{W}_2^m(-\pi, \pi)$, $m \geq 4$.*

This follows from the result of [15], according to which estimate (12) holds for almost all real values of α .

Conclusion

This paper investigates a boundary value problem for a fourth-order partial differential equation in a rectangular domain. The solution to the problem is represented as a Fourier series, the coefficients of which contain small denominators due to the factor $\cos \frac{kh}{\sqrt{2}}$.

Lower bounds for the small denominators are obtained, depending on the arithmetic nature of the parameter $\alpha = \frac{h}{\pi\sqrt{2}}$. In the case where α is an algebraic number of the second degree, the uniform convergence of the solution series and the series after fourfold differentiation is established, which ensures the existence and uniqueness of a solution in the space $\dot{W}_2^4(-\pi, \pi)$.

For algebraic numbers of arbitrary degree, the solvability of the problem is obtained under higher smoothness requirements for the function $\psi(x)$, which is related to the corresponding Diophantine estimates.

It is also shown that the obtained results hold for almost all (in the sense of Lebesgue measure) values of the parameter α , which follows from metric estimates for the approximation of real numbers by rational fractions.

Thus, the conditions for the solvability of the considered boundary value problem are established, depending on the arithmetic properties of the parameter α .

References

1. *Modern Methods in the Theory of Boundary Value Problems*. M.: MAKS Press, 2018. 296 pp. (In Russian)
2. Ptashnik B. I. *Ill-Posed Boundary Value Problems for Partial Differential Equations*. Kyiv: Naukova Dumka, 1984. 264 pp. (In Russian)
3. Yakubov S. and Yakubov Ya. *Differential-Operator Equations: Ordinary and Partial Differential Equations*. N.Y.: Chapman & Hall/CRC, 2000. 700 pp.
4. Mikhailets V. A. and Molyboga N. M. On boundary value problems for differential-operator equations, *Ukrainian Mathematical Journal*, 2016. vol. 68, no. 6, pp. 846—861.
5. Ntouyas S.K. Nonlocal Initial and Boundary Value Problems: A Survey / *Handbook of Differential Equations: Ordinary Differential Equations*. N.Y., Elsevier B.V., 2008, pp. 461-557.
6. Strikwerda John C. Well-Posed and Stable Initial-Boundary Value Problems / *Finite Difference Schemes and Partial Differential Equations, Second Edition*. e-book, SIAM, 2004, pp. 275-310.
7. Bourgin D. G., Duffin R. The Dirichlet problem for the vibrating string equation, *Bull. Amer. Math. Soc.*, 1939. vol. 45, no. 12, pp. 851–858.
8. Alimov Sh. A. On L_p -solutions of a boundary value problem, *Uzb. Math. J.*, 1999. №1, C. 3–9 (In Russian).
9. Alimov Sh. A. On solvability of an ill-posed problem, *Uzb. Math. J.*, 1999. no. 3, pp. 19–29 (In Russian).
10. Yuldasheva A. V. On solvability of the boundary value problem for one even-order equation, *Vestnik KRAUNC. Fiz.-Mat. Nauki*, 2016. vol. 12, no. 1, pp. 35–40 DOI:10.18454/2079-6641-2016-12-1-41-47.
11. Yuldasheva A. V. On stability of a boundary value problem for an even-order equation, *Vestnik KRAUNC. Fiz.-Mat. nauki*, 2015. vol. 10, no. 1, pp. 5-11 DOI:10.18454/2079-6641-2015-10-1-5-11 (In Russian).
12. Yuldasheva A. V., Sharipova S. A. Analysis of boundary value problems for partial differential equation of the fourth order / *Differential and difference equations*, Abstracts of Russian-Chinese Conference (Novosibirsk, 2025), Abstracts. Novosibirsk, Novosibirsk State University, 2025, pp. 155.
13. Shidlovsky A. B. *Transcendental Numbers*. M.: Nauka, 1987. 447 pp. (In Russian)
14. Roth K. F. Rational approximation to algebraic numbers, *Mathematika*, 1955. vol. 2, pp. 1–20.

15. Khintschin A. Zur metrischen Theorie der diophantischen Approximationen, Math. Zeit., 1926. vol. 24, no. 4, pp. 706–714.

Information about the authors



Yuldasheva Asal Viktorovna ✉ – DSc (Phys. & Math. Sci.), Professor, Branch of M.V. Lomonosov Moscow State University, Tashkent, Uzbekistan,  ORCID 0000-0001-6861-8331.



Sharipova Salibkhan Arislanbaevna ✉ – Postgraduate Student, National University of Uzbekistan, Tashkent, Uzbekistan,  ORCID 0009-0000-5900-0345.