



Dedicated to the 85th anniversary of Professor P.P. Firstov

Mathematical Modeling of Radon Transport in Geological Media: Classical and Non-Classical Approaches

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Abstract. The article provides a review of mathematical models describing the diffusion-convective transport of radon (^{222}Rn) in porous geological media within the framework of the emanation method. The core of these mathematical models is the linear non-stationary diffusion-convection equation for radon, used to describe its transport in homogeneous porous soil or in a "soil-atmosphere" system under various initial and boundary conditions. Due to their linearity, the mathematical models are studied using the Laplace integral transform, which makes it possible to find an analytical form of the solution to the problem. The calculated curves obtained from the analytical solution allow one to establish the patterns of radon transport and to model it depending on various physical and geological parameters. The obtained solutions can be applied to practical problems. In particular, the article describes an algorithm for estimating the radon flux density from the Earth's surface, which has an increased sensitivity to deformation disturbances in the geological medium. This makes it possible to study, for example, the seismic activity of Southern Kamchatka. Further, generalizations of previously developed mathematical models are discussed; these are based on the concept of a fractal structure of the geomedium and the mathematical apparatus of fractional differentiation. In the stationary case, models of superdiffusion ($1 < \alpha < 2$) and subadvection ($0 < \alpha < 1$) are constructed, and their ability to describe anomalously fast radon migration towards the Earth's surface is demonstrated. A model with a variable order of the fractional derivative is considered, allowing the dynamics of radon volumetric activity to be correlated with seismic activity, as exemplified by Southern Kamchatka. Here, the key role is played by the order of the fractional derivative, which is related to the permeability (fractality) of the geomedium. The results obtained can be used in monitoring geodynamic processes, mineral exploration, or in solving environmental problems. It should be noted that the results described in the article were obtained jointly with Professor P.P. Firstov (06.04.1941–20.07.2021), a leading geophysicist who organized a network of observation sites for radon monitoring in Kamchatka. In 2026, P.P. Firstov would have turned 85. This article is dedicated to this significant date, and is also a tribute to the memory and respect for a colleague and an outstanding scientist whom P.P. Firstov was to the author.

Key words: radon, diffusion, convection, superdiffusion, seismic activity

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Математическое моделирование переноса радона в геологических средах: классические и неклассические подходы

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Аннотация. В статье проводится обзор математических моделей, которые описывают диффузионно-конвективный перенос радона в пористых геологических средах в рамках теории эманационного метода. Основу таких математических моделей составляет линейное нестационарное уравнение диффузии-конвекции радона для описания его переноса в однородном пористом грунте или системе «грунт-атмосфера» в зависимости от различных начальных и краевых условий. Основным инструментом исследования математических моделей является интегральное преобразование Лапласа, которое позволяет найти аналитическое решение поставленной задачи. Расчетные кривые, полученные согласно найденному аналитическому решению, позволяют установить закономерность процесса переноса радона и моделировать его в зависимости от различных физико-геологических параметров. Полученные решения можно использовать в прикладных задачах. В частности, в статье описывается алгоритм оценки плотности потока радона с поверхности Земли, обладающий повышенной чувствительностью к деформационным возмущениям в геологической среде. Далее обсуждаются обобщения ранее разработанных математических моделей, которые основаны на представлении о фрактальной структуре геосреды и математическом аппарате дробного дифференцирования. В стационарном случае построены модели супердиффузии ($1 < \alpha < 2$) и субадвекции ($0 < \alpha < 1$), показана их способность описывать аномально быструю миграцию радона к земной поверхности. Рассмотрена модель с переменным порядком дробной производной, позволяющая сопоставлять динамику объемной активности радона с сейсмической активностью на примере Южной Камчатки. Полученные результаты могут быть использованы при мониторинге геодинамических процессов, поиске полезных ископаемых или в решении задач экологии. Необходимо отметить, что представленные в статье результаты были получены совместно с д.ф.-м.н. Фирстовым П.П. (06.04.1941-20.07.2021), который организовал сеть пунктов наблюдений по мониторингу радона на Камчатке. В 2026 году Фирстову П.П. исполнилось бы 85 лет. Данная статья посвящена этой знаменательной дате, а также является данью памяти и уважения коллеге и выдающемуся ученому, которым для автора был Фирстов П.П.

Ключевые слова: радон, диффузия, конвекция, супердиффузия, сейсмическая активность

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1 Introduction

The study of mass transfer of emanations (radioactive gases) in geological media has been conducted for over 100 years. In Russia, pioneering work was carried out in 1922 under the leadership of A.P. Kirikov, who used emanation surveying to search for radium zones beneath Quaternary deposits within the ore field of the Tyuya-Muyun deposit in Central Asia [1]. In 1927, under the leadership of L.N. Bogoyavlensky, radiometric exploration of the Maikop oil field revealed an increase in radioactivity exceeding the margin of observational error [2]. A little later, in the 1930s, under the leadership of A.G. Grammakov, the emanation method was further developed in research on the prospecting of uranium ores [3, 4]. It should also be noted that the theory of the emanation method was being developed in other countries as well. For example, research was conducted by German scientists, with the works of O. Hahn, S. Flügge, and K. Siemens being well known [5, 6]. The emanation method gained widespread development after 1945 with the advent of nuclear power.

After the Tashkent earthquake of 1966, the emanation method received a new impetus for development. Staff members of the Institute of Seismology of the Academy of Sciences of the Uzbek SSR, under the leadership of G.A. Mavlyanov and V.I. Ulomov, recorded a sharp, anomalous increase in radon concentration in the water of deep artesian wells several months before the main shock [7, 8]. Just before the earthquake itself, the radon concentration dropped to background values, and after it, it rose again. For the first time in world practice, it was convincingly demonstrated that changes in radon content in subsurface fluids constitute a clear and detectable precursor to a seismic event. This discovery gave rise to an entire scientific field—earthquake monitoring using radon as an indicator of the stress-strain state of the geoenvironment [9–14, 14, 15].

In recent decades, radon monitoring for the purpose of studying seismic activity has been developed both in Russia [16, 17] and abroad [18–20]. In Russia, such monitoring was first organized in 1997 under the leadership of Dr. Sci. (Phys.-Math.) P.P. Firstov. A network of registration points was created for monitoring subsoil radon concentration at the Petropavlovsk-Kamchatsky Geodynamic Test Site, which is currently part of the structure of the Kamchatka Branch of the Federal Research Center "Unified Geophysical Service of the Russian Academy of Sciences." A registered short-term precursor before the Kronotsky earthquake of December 5, 1997, with a magnitude of 7.8, confirmed the potential of the emanation method [16]. The network has been fully operational since 2000. Over these years, a methodology for conducting observations of subsoil gases for earthquake prediction purposes was developed, data on precursors were obtained and generalized, and a methodology for earthquake forecasting, unparalleled in the world, was developed [21–23, 25–31].

It should be noted that the emanation method also has applications in other areas, for example, in environmental problems. In 1977, the United Nations Scientific Committee on the Effects of Atomic Radiation identified radon as the main source of danger to the population. The measurement of radon volumetric activity in soil air and buildings serves as the basis for classifying construction sites according to the degree of potential radon hazard [32].

Let us note the extensive bibliography (1,119 references) in Russian on the subject of radon, including applications in various applied problems, which was compiled by Russian radio engineer A.P. Zarubin for the period from 1990 to 2021 and is freely available on the website <https://www.prometeus.nsc.ru/partner/zarubin/radon.ssi>.

The mathematical apparatus of the emanation method theory is quite well developed and continues to evolve. In the 1950s–1960s, the experience of applying the emanation method theory for mineral prospecting in Russia and its mathematical basis were reflected in the fundamental books [33, 34]. These works presented the basic mathematical models of diffusion-convective transport of radon. The foundation of these mathematical models consisted of linear equations of classical mathematical physics (partial differential equations of parabolic type). The initial and boundary conditions for these equations were set based on the problem being solved, the structure of the geoenvironment, and the geological-physical parameters. The integral transforms of Laplace or Fourier typically served as the tools for investigating these models. Therefore, their solutions were expressed in analytical form. It is necessary to mention the monograph by authors V.V. Gudzenko and V.T. Dubinchuk [35], which presented mathematical models of radon transport in natural (ground) waters. Here, the mathematical models of radon transport consisted of ordinary differential equations describing the change in radon concentration over time upon its injection into groundwater.

Mathematical models of radon transport subsequently gained extensive development. In G.F. Novikov's 1989 textbook "Radiometric Prospecting," an entire chapter is devoted to models of radon transport in porous soils [36], which covers the mathematical formulations and solutions of various boundary value problems of radon transport in porous soils. One such problem was related to radon transport in layered porous soil. Based on this problem, the author of the article, jointly with P.P. Firstov and I.A. Ilyin, proposed mathematical models of diffusive and diffusive-convective radon transport in the soil-atmosphere system [37–39]. The importance of such a problem was conditioned by the fact that radon, being a natural ionizer of the atmospheric surface layer, influences its electric field strength [40]. Subsequently, the author proposed a more general model of radon transport in the soil-atmosphere system [41], for which an analytical solution was obtained. Another important task was the development of an algorithm for calculating the radon flux density from the earth's surface, based on a diffusive-convective model of radon transport [42, 43]. The algorithm represented the solution of the inverse problem of recovering the radon flux density taking into account experimental data on radon volumetric activity in porous soil. Calculations showed that radon flux density is a more sensitive characteristic than radon volumetric activity, which confirmed the results of article [44] and allowed proposing a methodology for a multifactorial experiment on radon transport processes in the lithosphere-atmosphere system [45].

However, these models had a certain limitation—they can only indicate the presence of bodies with an increased content of the parent element radium at depths of several tens of meters. Such a low migration capacity of radon is due to its relatively short half-

life (3.82 days) and high atomic weight (222). Nevertheless, in some cases, emanation surveying localizes radium-containing bodies located at depths of up to 500 m [36]. Therefore, a problem arose related to describing the mechanisms that lead to the high migration capacity of radon in the soil-atmosphere system. Various mechanisms exist that increase the migration capacity of radon, some of which are presented in a review article that also describes their limitations [46]. However, building on article [47], which was devoted to anomalous diffusion due to the fractal properties of the medium, the previously developed mathematical models of radon transport were generalized, laying the foundation for the non-classical theory of the emanation method [48]. The non-classical theory of the emanation method is a generalization of the classical one to the case of effects of temporal and spatial nonlocality (memory). Here, the mathematical apparatus used is fractional calculus [49, 50]. Such an approach leads us to integro-differential partial differential equations with power-law difference kernels, which can be written in terms of fractional derivatives. Therefore, mathematical models built on the basis of fractional derivatives will be called fractional models, and those using integer-order derivatives—classical models. The Laplace or Fourier integral transform can also serve as a tool for investigating linear fractional mathematical models.

In 2008, in the author's work [51] jointly with B.M. Shevtsov and P.P. Firstov, a new fractional stationary mathematical model of the emanation method was applied for the first time, describing the superdiffusive regime of radon transport in porous fractal soil. This model showed better results in approximating experimental data on radon concentration distribution with depth than the classical diffusion model. Subsequently, this mathematical model was further developed in the works of the author [52, 53]. Also, in 2008, the author proposed a fractional mathematical model of stationary diffusive radon transport in the soil-atmosphere system [54]. Later, this model was also developed in the author's articles [55–58]. Further development of fractional mathematical models of radon transport involved investigating the order of the fractional derivative and the seismic activity of Southern Kamchatka [59, 60]. It was shown that the order of the fractional derivative is related to the permeability of the geoenvironment; therefore, it can be used as an additional degree of freedom when studying the seismic activity of Southern Kamchatka. It should be noted that a more detailed description of the mathematical models of radon transport in the soil-atmosphere system, presented in this article, is given in the author's monographs [48, 61].

The purpose of this review article is to acquaint the reader with a systematized generalization of the results obtained in 2006–2012 by the author jointly with Dr. Sci. (Phys.-Math.) P.P. Firstov (04/06/1941–07/20/2021), whose 85th birthday is being commemorated this year. It should be noted that P.P. Firstov made a significant contribution to the development of the emanation method theory for the purpose of predicting earthquakes in Kamchatka, organized and maintained for three decades a network of radon monitoring observation points at the Petropavlovsk-Kamchatsky Geodynamic Test Site. Therefore, this review article also serves as a tribute to the memory and respect for the outstanding scientist whom Firstov P.P. was to the author.

To achieve the stated purpose, the review is structured as follows. The first section outlines the physical and geological principles underlying the emanation method, including the mechanisms of radon generation, its transport properties, and the main factors controlling its migration in porous and fractured media. The second section presents the classical mathematical models of radon transport based on diffusion and diffusion-convection equations, followed by a discussion of their applicability and limitations in describing field observations. The third section introduces the non-classical (fractional) approach to radon transport modeling, which accounts for the effects of temporal and spatial nonlocality associated with the fractal structure of geological media. Here, the mathematical formalization using fractional derivatives is described, and the corresponding boundary value problems are formulated. The fourth section is devoted to the application of the developed models to the analysis of radon monitoring data obtained at the Petropavlovsk-Kamchatsky Geodynamic Test Site. Particular attention is paid to the relationship between the order of the fractional derivative, reflecting the permeability of the geoenvironment, and the seismic activity of Southern Kamchatka. The final section summarizes the main findings, discusses the advantages and constraints of the proposed approach, and outlines possible directions for future research, including the integration of fractional transport models with geophysical and hydrogeological data for improving earthquake precursor detection.

The review is based on previously published works by the author and co-authors, supplemented with the necessary theoretical background and references to the key developments in the field. The mathematical derivations are presented in a form that is intended to be accessible to both specialists in radiometrics and geophysicists interested in the application of fractional calculus to problems of mass transfer in complex media.

2 Radon and Mechanisms of Its Transport

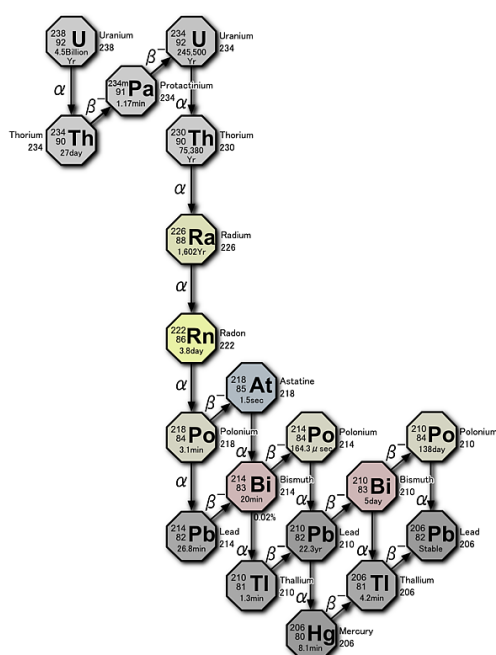


Fig. 1. Uranium series

Radon (^{222}Rn) is a chemical element of group zero of the periodic system of elements with atomic number 86, a daughter product of the decay of ^{226}Ra with a half-life of $T = 3.85$ days (Fig. ??). Under normal conditions, radon is a monatomic chemically inert gas, highly soluble in water and other liquids.

Due to its gaseous state, radon is capable of migrating to the day surface from deep layers of rocks [36].

Let us consider the main mechanisms of radon transport in the soil-atmosphere system [62]. The initial transport process is emanation, which refers to the release of radioactive gas (emanation) into the pore

space from a solid substance containing radium, which causes the appearance of radon in the geoenvironment and its subsequent mass transfer in groundwater and the atmosphere (Fig. 2).

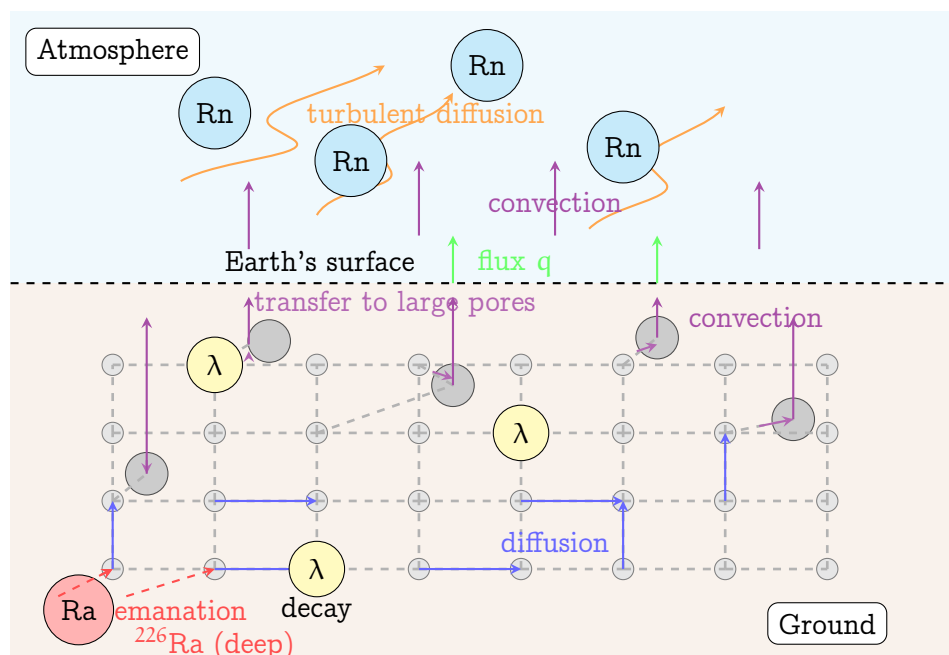


Fig. 2. Scheme of radon transport with an extended soil region: emanation from a deep ^{226}Ra source into small pores, diffusion with decay along connecting channels, transfer to large pores and convective rise to the surface, then turbulent diffusion and convection in the atmosphere.

At the second stage of the process, purely molecular diffusion propagation of radon occurs with consideration of decay in small soil pores until it enters larger pores, where convection (advection) can already take place [63]. Convection arises due to the temperature difference (and consequently air density) between the soil and the atmosphere. Warm air inside the soil, saturated with radon, rises upward, drawing gas from depth (chimney effect).

It should be noted here that filtration (advection) due to the atmospheric pressure gradient also affects radon transport to the earth's surface. Pressure drops characteristic of cyclones lead to an increase in pressure in soil pores. Excess soil air is squeezed to the surface, causing a sharp increase in radon flux (pump effect) [64]. Baric waves: pressure fluctuations cause the soil to "breathe acting like a piston, which increases radon's migration capacity [65].

At the third stage of the radon release process, a diffusion-convective transport process occurs through large soil pores, ending with its discharge into the atmosphere. In the atmosphere, radon is transported to the upper layers of the atmosphere during daytime due to turbulent diffusion and convection, and its concentration sharply drops to almost zero values. Radon in the surface atmosphere can also be transported horizontally by wind (horizontal advection). We will not consider the mechanism of

horizontal radon drift here, although it is important in problems of radon safety of territories [66].

It should also be mentioned here that the radon transport process in soil can be accelerated by deformation disturbances, which lead to the formation of microcracks, increasing the permeability of the medium and radon discharge into the atmosphere [67]. The presence of cracks can also facilitate the capture of radon by other tracer gases and its rapid transport to the earth's surface. Such radon transport is studied within the framework of geogas theory [46, 68]. We will not address this transport process in this review article.

Based on the above, molecular diffusion is responsible for the uniform distribution of radon in limited layers, but accelerated discharge to the day surface is almost always associated either with a pressure gradient (wind, temperature differences in buildings, weather, etc.) or with a sharp increase in the permeability of the geoenvironment (cracks, drainage, karst, etc.).

Therefore, we will focus on the mechanisms of diffusion and convection (advection) of radon, which will be taken into account in the mathematical models.

3 Mathematical Models of the Classical Theory of the Emanation Method

Mathematical models of the classical theory of the emanation method can be studied in fundamental books [33–36]. The basic equation of such models is the diffusion-convection equation of radon transport in a homogeneous porous medium. Let us consider it in more detail.

3.1 Mathematical Model of Diffusion-Convective Radon Transport in Homogeneous Porous Soil

Consider a semi-infinite homogeneous porous medium ($z > 0$) in which radon transport occurs. Let $A(z, t)$ be the volumetric activity of radon (Bq/m^3). The process is described by the diffusion-convection equation with a source (emanation) and sink due to radioactive decay [36]:

$$\frac{\partial A}{\partial t} = D_G \frac{\partial^2 A}{\partial z^2} - v_G \frac{\partial A}{\partial z} - \lambda(A - A_\infty), \quad (1)$$

where D_G is the true diffusion coefficient (m^2/s), v_G is the advection velocity (m/s), λ is the radon decay constant (s^{-1}), $A_\infty = \frac{K_{em} A_{Ra} \rho_d}{\eta}$ is the pore activity of radon in equilibrium with ^{226}Ra , K_{em} is the emanation coefficient, A_{Ra} is the specific activity of ^{226}Ra in the rock (Bq/g), ρ_d is the dry soil density (kg/m^3), η is the porosity. The relationship between diffusion coefficients: $D_G = \eta D_E = \eta^2 D_M$, where D_E is the effective diffusion coefficient, D_M is the molecular diffusion coefficient.

It should be noted that in the literature, it is often not specified which type of diffusion coefficient is used when modeling the mass transfer process, which can lead to confusion. It is necessary to distinguish three different diffusion coefficients: the

molecular diffusion coefficient (D_M), the effective diffusion coefficient (D_E), and the true diffusion coefficient (D_G). The molecular diffusion coefficient characterizes radon diffusion in the pore air of the soil. The effective diffusion coefficient or intermediate diffusion coefficient describes radon transport through the porous structure, i.e., the soil, and is related to molecular diffusion by the relation $D_E = D_M \eta$, where η is the soil porosity coefficient. The true diffusion coefficient includes the effects of porosity and tortuosity of the soil. In the literature, it is often defined as follows: $D_G = D_E \eta = D_M \eta^2 = D_M \eta / T$, where T is the soil tortuosity.

For equation (1), initial and boundary conditions are set:

$$A(z, 0) = A_\infty, \quad A(0, t) = 0, \quad \lim_{z \rightarrow -\infty} A(z, t) = A_\infty. \quad (2)$$

Here, the z axis is directed into the soil. The stationary solution ($\partial A / \partial t = 0$) of problem (1), (2) can be obtained, for example, using the Laplace integral transform [69], which has the form:

$$A(z) = A_\infty \left(1 - \exp \left[-z \left(-\frac{v_G}{2D_G} + \sqrt{\left(\frac{v_G}{2D_G} \right)^2 + \frac{\lambda}{D_G}} \right) \right] \right). \quad (3)$$

It can be seen that according to the stationary solution (3), the volumetric activity of radon decays exponentially.

The non-stationary solution of problem (1), (2) is expressed in terms of the error function (erfc) and can be written as:

$$A(z, t) = A_\infty \left[1 - \frac{1}{2} \left(e^{z \left(\frac{v_G}{2D_G} - \sqrt{\left(\frac{v_G}{2D_G} \right)^2 + \frac{\lambda}{D_G}} \right)} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_G t}} - \sqrt{\left(\frac{v_G^2}{4D_G} + \lambda \right) t} \right) + e^{z \left(\frac{v_G}{2D_G} + \sqrt{\left(\frac{v_G}{2D_G} \right)^2 + \frac{\lambda}{D_G}} \right)} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_G t}} + \sqrt{\left(\frac{v_G^2}{4D_G} + \lambda \right) t} \right) \right) \right]. \quad (4)$$

In the case when for equation (1) the impermeability condition (zero flux) is specified at the day surface ($z = 0$), due, for example, to its freezing [70]

$$\left. \frac{\partial A(z, t)}{\partial z} \right|_{z=0} = 0, \quad \lim_{z \rightarrow \infty} A(z, t) = A_\infty, \quad (5)$$

and the initial distribution corresponds to the stationary profile (3)

$$A(z, 0) = A_\infty (1 - \exp(-zm)),$$

where $m = \frac{v_G}{2D_G} + \sqrt{\left(\frac{v_G}{2D_G} \right)^2 + \frac{\lambda}{D_G}}$.

The analytical solution can be obtained using the integral Laplace transform in the form

$$A(z, t) = A_\infty \left(1 - \exp(-zm) + m \sqrt{D_G} e^{\frac{zv_G}{2D_G}} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_G t}} + \sqrt{\left(\frac{v_G^2}{4D_G} + \lambda \right) t} \right) \right), \quad (6)$$

$$\begin{aligned}
R = & \frac{e^{-z\sqrt{\frac{a}{D_G}}} \operatorname{erfc}\left(\frac{z}{2\sqrt{D_G t}} - \sqrt{a \cdot t}\right)}{2(b + \sqrt{a})} + \\
& + \frac{e^{z\sqrt{\frac{a}{D_G}}} \operatorname{erfc}\left(\frac{z}{2\sqrt{D_G t}} + \sqrt{a \cdot t}\right)}{2(b - \sqrt{a})} - \frac{be^{\frac{zb}{\sqrt{D_G}} + (b^2 - a)t} \operatorname{erfc}\left(\frac{z}{2\sqrt{D_G t}} + b\sqrt{t}\right)}{b^2 - a}, \\
& a = v_G^2/(4D_E) + \lambda, b = \frac{v_G}{2\sqrt{D_E}},
\end{aligned}$$

also containing exponentials and error integrals.

Note that in the special case when $v_G = 0$, i.e., when convective transport is absent and radon migrates solely due to molecular diffusion, solution (6) reduces to the well-known solution obtained in works [71, 72]. This limiting case is of particular methodological importance, since it allows one to isolate the contribution of the purely diffusion mechanism to the total radon flux and to verify the correctness of the more general diffusion–convection model. The solutions obtained in [71, 72] are widely used in the analysis of radon propagation from a plane source located at a given depth and in the estimation of radon flux density from the day surface under various boundary conditions. In particular, in work [72], a field experiment was conducted whose essence consisted in artificial shielding of the upper soil layer, which made it possible to exclude the influence of atmospheric factors and to determine the radon flux density from the earth's surface under conditions of a blocked free exhalation regime. The comparison of theoretical calculations with the experimental data obtained in that work showed good agreement and confirmed the adequacy of the diffusion model for describing radon transport in homogeneous porous media. It should be emphasised, however, that the purely diffusion approximation describes only the slow component of radon migration, whereas the full diffusion–convection model (1) makes it possible to take into account the additional advective transport mechanism, which, as will be shown below, plays a decisive role in the formation of anomalously high radon fluxes during periods of geodynamic activation.

According to solution (6), a family of calculated curves of radon concentration distribution in soil without advection was constructed (Fig. 3).

For numerical modeling, parameter values were selected for eluvial-deluvial loose sediments according to work [36].

The values of pore radon activity $A_\infty = 10 \text{ kBq/m}^3$ were chosen according to observations at the radon monitoring station network located in the area of the Petropavlovsk-Kamchatsky geodynamic test site.

At a radon diffusion coefficient value in porous soil of $D_G = 5 \cdot 10^{-7} \text{ m}^2/\text{s}$, radioactive equilibrium of radon with parent radium is established within 4–5 days, which is in good agreement with the results of work [36]. Curves of radon concentration distribution in porous soil with advection velocity $v_G = 10^{-6} \text{ m/s}$ with the same diffusion coefficient were also constructed (Fig. 3b). In this case, the radon advection velocity significantly increases the contribution to the intensity of the transport process, and the time for establishing radioactive equilibrium decreases by a factor of 2 and amounts to 2 days.

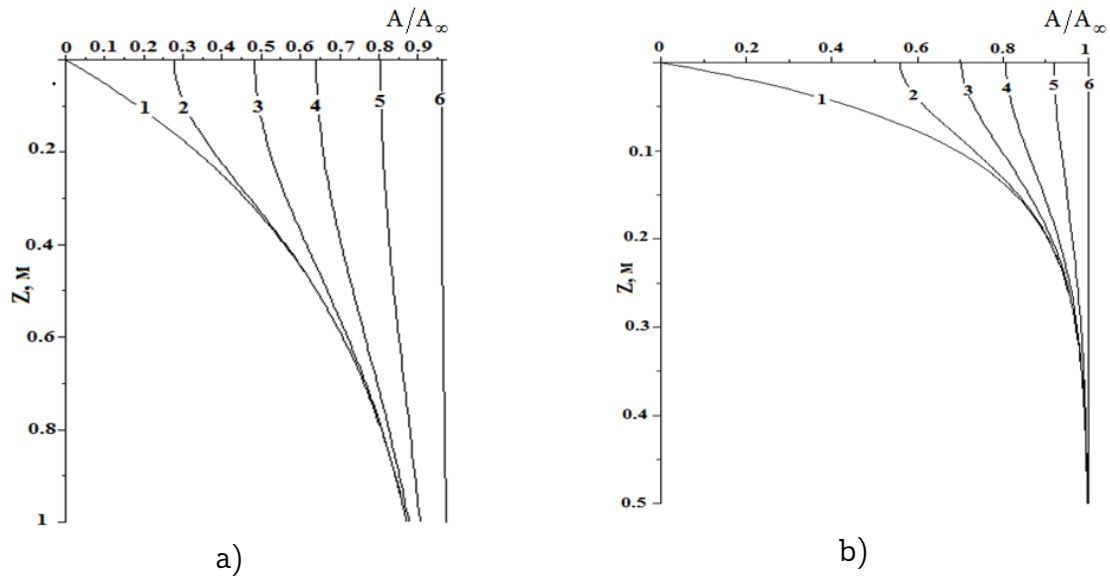


Fig. 3. Family of calculated curves of radon concentration distribution in soil, constructed at a diffusion coefficient value of $D_G = 5 \cdot 10^{-7} \text{ m}^2/\text{s}$ at various times: 1- $t=0$; 2- $t=8\text{h}$; 3- $t=1\text{day}$; 4- $t=2\text{days}$; 5- $t=4\text{days}$; 6- $t=11\text{days}$: a) $v_G = 0 \text{ m/s}$; b) $v_G = 10^{-6} \text{ m/s}$.

Therefore, when modeling radon transport in specific geological media and systems, it is necessary to consider both transport mechanisms: diffusion and advection.

3.2 Algorithm for Estimating Radon Flux Density from the Earth's Surface

The model equation (1) can be applied to solve the problem of estimating radon flux density from the earth's surface [71, 72]. It is known that radon flux density is a more sensitive characteristic of the stress-strain state of the geological environment than its pore activity [44]. Therefore, an algorithm for calculating radon flux density from the day surface based on the diffusion-convection equation (1) was proposed [42, 43]. Let us consider it in more detail.

The problem is formulated as follows: for equation (1) and conditions

$$\begin{aligned} A(z, 0) = A_\infty, q(t) = -D_G \eta \left. \frac{\partial A(z, t)}{\partial z} \right|_{z=0} + v_G \eta A(0, t), \\ A(z, t)|_{z=l_j} = A_i^{\text{obs}}, i = 1, \dots, M, J = 1, \dots, N. \end{aligned} \quad (7)$$

estimate the flux density $q(t)$ from available data on radon volumetric activity A^{obs} at some point in homogeneous porous soil.

Let $q(t) = q_c$, then problem (1), (7) has the solution

$$\begin{aligned} A(z, t) = A_\infty + \frac{(v_G \eta A_\infty - q_c) m}{2\lambda \eta} I(m, n, D_G, \lambda, z, t), \\ I(m, n, D_G, \lambda, z, t) = I_1 + I_2 - I_3, I_1 = \frac{m-n}{n} e^{(m-n)z} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_G t}} - n\sqrt{D_G t} \right), \end{aligned} \quad (8)$$

$$I_2 = \frac{m+n}{n} e^{(m+n)z} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_G t}} + n\sqrt{D_G t} \right), I_3 = 2e^{-zm-\lambda t} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_G t}} + m\sqrt{D_G t} \right),$$

$$m = \frac{v_G}{2D_G}, n = \frac{\sqrt{v_G^2 + 4D_G \lambda}}{2D_G}.$$

Solution (8) describes the process of change in radon volumetric activity values in soil under the influence of constant flux density q_c at times t . If this solution is differentiated with respect to q_c , the expression for the sensitivity coefficient $\phi(z, t)$ is obtained [73]

$$\phi(z, t) = -\frac{m}{2\lambda\eta} I(m, n, D_G, \lambda, z, t). \quad (9)$$

Since the sensitivity coefficient $\phi(z, t)$ does not depend on $q(t)$, the problem of estimating radon flux density is linear and, consequently, the superposition principle can be applied to solution (8).

A consequence of the superposition principle for the solution is Duhamel's theorem; therefore, it is valid only for linear problems. Duhamel's theorem can be obtained using the Laplace transform or the superposition principle [73]. We will rely on the superposition principle, which allows obtaining the most convenient numerical approximation. According to Duhamel's theorem, the values of radon volumetric activity at some point at some time are determined from individual solution blocks according to the superposition principle. One such solution, for example $\phi(z, t)$, corresponds to the change in radon volumetric activity values at point z , which is subjected to constant radon flux density q_c .

We will approximate the radon flux density on the earth's surface as follows. On time intervals $\tau_{1/2}, \tau_{3/2}, \dots, \tau_{M-1/2}$, flux density values are used at times: from 0 to τ_1 , from τ_1 to τ_2 , from τ_{M-1} to τ_M . The radon volumetric activity at the initial time is taken equal to the volumetric activity at which radon is in radioactive equilibrium with radium, i.e., A_∞ .

Using the superposition principle, according to which the radon volumetric activity values at point z at times t_M are determined by the contribution of individual components, each corresponding to one or another component of radon flux density from q_1 to q_M taking into account (9):

$$A_M = A_\infty + \sum_{n=1}^M q_n \Delta\varphi_{M-n}, \Delta\varphi_{M-n} = \varphi_{M-n+1} - \varphi_{M-n}.$$

If we have J sensors for measuring radon volumetric activity located at different depths. Then for each observation point we obtain:

$$\begin{aligned} A_{1M} &= \bar{A}_{1M}|_{q_M=0} + \varphi_{11} q_M, \\ A_{2M} &= \bar{A}_{2M}|_{q_M=0} + \varphi_{21} q_M, \\ &\dots \\ A_{JM} &= \bar{A}_{JM}|_{q_M=0} + \varphi_{J1} q_M. \end{aligned} \quad (10)$$

It is difficult to establish exact agreement between measured and calculated radon volumetric activity values because the single component of radon flux density is difficult

to choose so that the equality $A_{jM} = A_{jM}^{\text{obs}}$ holds; therefore, agreement must be achieved in some averaged approximation. Estimate q_M taking into account (10) can be done using the least squares method:

$$S = \sum_{j=1}^J \left(A_{jM}^{\text{obs}} - \bar{A}_{jM}|_{q_M=0} - \varphi_{j1} q_M \right)^2. \quad (11)$$

We minimize the residual functional (11) with respect to the flux density component q_M . This problem reduces to finding the zeros of the derivatives with respect to the parameter q_M . We replace q_M with the estimate:

$$\bar{q}_M = \frac{\sum_{j=1}^J \left(A_{jM}^{\text{obs}} - \sum_{i=1}^{M-1} q_i \Delta \varphi_{jM-i} - A_{\infty} \right) \varphi_{j1}}{\sum_{k=1}^J \varphi_{k1}^2}, \quad (12)$$

where $\Delta \varphi_{M-n} = \varphi_{M-n+1} - \varphi_{M-n}$.

The algorithm for estimating radon flux density from the earth's surface was implemented in the program "REKSEM" developed by the author [74], which was subsequently registered in the Industry Fund of Algorithms and Programs (OFAP). The program was tested at the network of stations for monitoring subsoil radon at the Petropavlovsk-Kamchatsky geodynamic test site in Kamchatka.

For example, the most interesting results were obtained at the radon registration point "Institute," which is located near the building of the Institute of Volcanology and Seismology of the Far Eastern Branch of the Russian Academy of Sciences (Petropavlovsk-Kamchatsky). Radon volumetric activity registration at this point is carried out in an underground bunker (8 m³) at two points: 1) in subsoil air at a depth of one meter from the bunker floor (aeration zone); 2) from the bunker floor surface. Above the bunker on the earth's surface, a metal container is installed, and the underground bunker is ventilated through the container and pipe system due to natural air convection. Temperature and atmospheric pressure sensors are installed in the above-ground container and underground bunker. As comparison of meteorological quantities in the underground bunker and above-ground container showed, in the underground bunker during summer, due to large temperature variations in the above-ground container, daily atmospheric pressure fluctuations are clearly traced. Registration of all parameters is carried out using two ALMEMO 2590-9 measuring instruments with a sampling frequency of 6.0 hour⁻¹ [17].

Fig. 4a shows the dynamics of radon volumetric activity for the period July 20 - August 29, 2006 at a depth of one meter and from the earth's surface. On all curves, daily atmospheric pressure variation is traced, and from August 13 to 22, an increase in amplitude is observed on all channels. In the aeration zone, the excess is 15%, and on the surface 30% of the background.

Fig. 4b shows the calculated radon flux density curve according to the above-proposed methodology. In the calculated radon flux density curves, the amplitude excess over background for the anomalous period is more than 60%, which confirms the higher sensitivity of flux density compared to the dynamic parameter—radon volumetric activity.

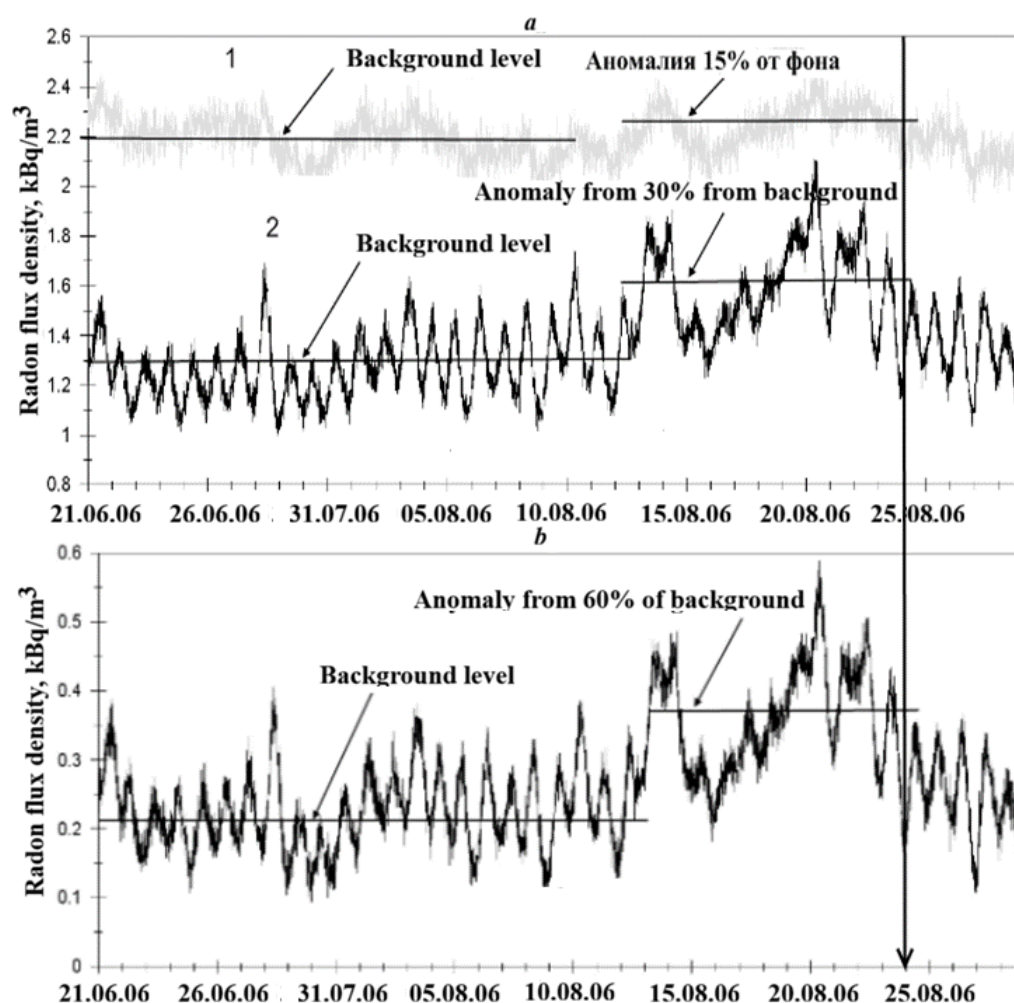


Fig. 4. a - dynamics of radon volumetric activity at sensor depth of 0.1 m (1) and 1.0 m (2); b – calculated radon flux density. The arrow indicates the earthquake with $M = 6.2$, $R = 190$ km.

According to operational data from the Kamchatka Branch of the Geophysical Service of the Russian Academy of Sciences, after a two-month seismic lull, on July 17 and 20, earthquakes with local magnitudes of 5.9 and 5.8 occurred off the coast of Kamchatka, respectively. The foci of these earthquakes were located in the north (55.54° N) and south (49.6° N) of the Kamchatka Peninsula at a distance of more than 350 km from the registration point. In the southern Kamchatka region (51.19° N), on August 24 at a distance of 190 km from the PRT point, the strongest earthquake with $ML = 6.2$ occurred, and on September 1-2, a series of earthquakes occurred in the south of Kronotsky and Avacha Bays. The epicenter of the strongest shock with magnitude $M = 5.6$ was located 110 km from the registration point. The activation of seismicity along the entire length of the northern flank of the Kuril-Kamchatka island arc gives reason to assume an increase in the movement speed of the Pacific Plate, which could lead to a restructuring of the regional stress field. Therefore, the anomaly well expressed in radon flux density can apparently be explained by an increase in advection in radon transport in the soil.

It should be noted that the proposed algorithm for solving the inverse problem of finding radon flux density on the earth's surface has one drawback—sensitivity to measurement errors at small time steps (seconds) [73]. However, the registration of experimental data from the Kamchatka radon monitoring network is carried out with a sampling interval of 6.0 hour⁻¹, so when implementing this algorithm, sensitivity to measurement errors is not high.

3.3 Mathematical Model of Non-Stationary Diffusion-Convective Radon Transport in the Soil-Atmosphere System

Let us now consider the mathematical model of radon transport in the soil-atmosphere system. The model will consist of two diffusion-convection equations (1) with matching conditions at the medium boundary—equality of fluxes and radon concentrations [41]:

$$\begin{aligned}\frac{\partial A(z, t)}{\partial t} &= D_g \frac{\partial^2 A(z, t)}{\partial z^2} - v_g \frac{\partial A(z, t)}{\partial z} - \lambda (A(z, t) - A_\infty), z > 0, \\ \frac{\partial A(z, t)}{\partial t} &= D_a \frac{\partial^2 A(z, t)}{\partial z^2} - v_a \frac{\partial A(z, t)}{\partial z} - \lambda A(z, t), z < 0, \\ D_g \frac{\partial A(z, t)}{\partial z} \Big|_{z=0+0} - v_g A(z, t) \Big|_{z=0+0} &= D_a \frac{\partial A(z, t)}{\partial z} \Big|_{z=0-0} - v_a A(z, t) \Big|_{z=0-0}, \\ A(z, t) \Big|_{z=0+0} &= A(z, t) \Big|_{z=0-0},\end{aligned}\quad (13)$$

$$\lim_{z \rightarrow \infty} A(z, t) = A_\infty, \lim_{z \rightarrow -\infty} A(z, t) = 0, A(z, 0) = A_\infty \text{ for } z > 0.$$

Here D_g, D_a are radon diffusion coefficients in soil and atmosphere, m²/s; v_g, v_a are radon advection velocities in soil and atmosphere, m/s.

The solution of problem (13) has the form:

$$A(z, t) = A_\infty \left(1 - \left(\frac{\left(\gamma + \xi \sqrt{\lambda} \right) e^{-z \sqrt{\frac{\lambda}{D_g}}} \operatorname{erfc} \left(\frac{z}{2 \sqrt{D_g t}} - \sqrt{\lambda t} \right)}{2 \left(\gamma + \sqrt{\lambda} \right)} + \frac{\left(\gamma - \xi \sqrt{\lambda} \right) e^{z \sqrt{\frac{\lambda}{D_g}}} \operatorname{erfc} \left(\frac{z}{2 \sqrt{D_g t}} + \sqrt{\lambda t} \right)}{2 \left(\gamma - \sqrt{\lambda} \right)} - \frac{(1 - \xi) \gamma^2 e^{\frac{\gamma z}{\sqrt{D_g}} + \gamma^2 t - \lambda t} \operatorname{erfc} \left(\frac{z}{2 \sqrt{D_g t}} + \gamma \sqrt{t} \right)}{\gamma^2 - \lambda} \right) \right), z > 0,$$

$$A(z, t) = A_{\infty} \left(\frac{\left((1 - \xi)\sqrt{\lambda} \right) e^{-z\sqrt{\frac{\lambda}{D_a}}} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_a t}} - \sqrt{\lambda t} \right)}{2(\gamma + \sqrt{\lambda})} + \frac{\left((1 - \xi)\sqrt{\lambda} \right) e^{z\sqrt{\frac{\lambda}{D_a}}} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_a t}} + \sqrt{\lambda t} \right)}{2(\gamma - \sqrt{\lambda})} + \frac{(1 - \xi) \gamma^2 e^{\frac{\gamma z}{\sqrt{D_a}} + \gamma^2 t - \lambda t} \operatorname{erfc} \left(\frac{z}{2\sqrt{D_a t}} + \gamma \sqrt{t} \right)}{\gamma^2 - \lambda} \right), z < 0.$$

where $a = v_g - v_a$, $b = -\sqrt{D_a}$, $c = -\sqrt{D_a} - \sqrt{D_g}$. $\gamma = a/c$, $\xi = b/c$.

Note that in the author's works [37–39], some special cases of model (13) were studied.

Model (13) represents a two-layer radon transport model, where the first layer is porous soil and the second is the surface layer of the atmosphere. It should be noted that the case of soil with a multilayer structure was considered in [75, 76]. These papers used numerical analysis methods. Also worth noting are papers devoted to radon transport in cellular anisotropic media [77].

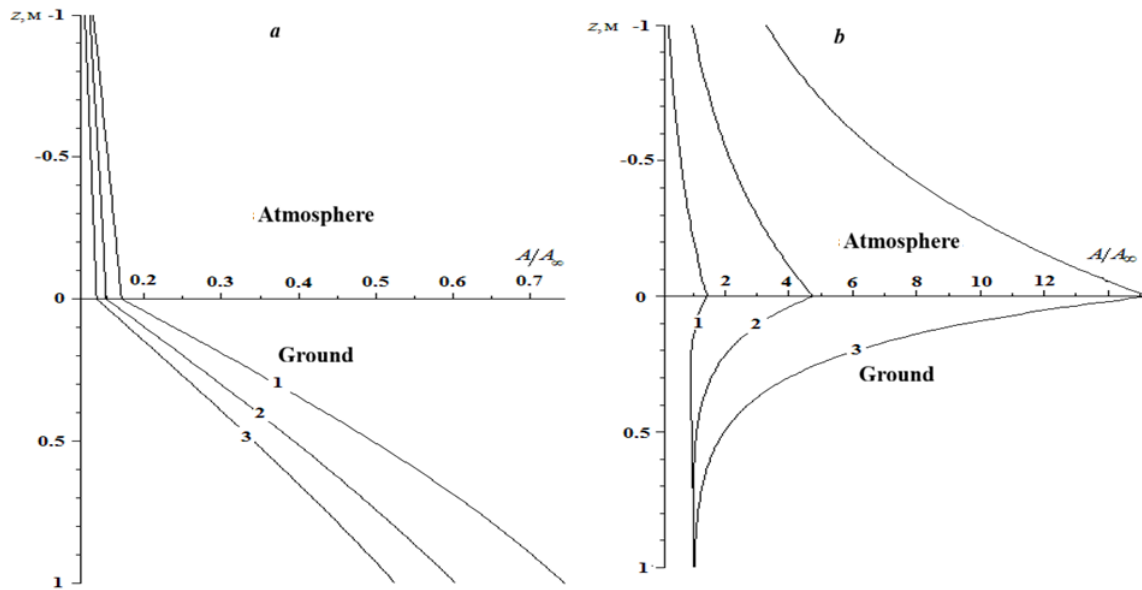


Fig. 5. Curves of radon concentration distribution in soil and atmosphere, constructed at various times: 1-1000 s; 2-2000 s; 3-3000 s: a) $D_a = 5 \cdot 10^{-3} \text{ m}^2/\text{s}$, $D_g = 5 \cdot 10^{-4} \text{ m}^2/\text{s}$, $v_a = 10^{-3} \text{ m/s}$, $v_g = 5 \cdot 10^{-4} \text{ m/s}$; b) $D_a = 5 \cdot 10^{-3} \text{ m}^2/\text{s}$, $D_g = 5 \cdot 10^{-4} \text{ m}^2/\text{s}$, $v_a = 0 \text{ m/s}$, $v_g = 5 \cdot 10^{-2} \text{ m/s}$

Fig. 5 shows calculated curves of radon concentration distribution in soil and atmosphere. Fig. 5a shows that radon concentration decreases with time toward the earth's surface. Relative volumetric activity values at the interface range from 0.05 to 0.15, which is about 10% of the background radon volumetric activity value in soil. In the atmosphere, due to diffusion and advection, radon concentration tends to zero.

Fig. 5b shows calculated radon concentration distribution curves constructed under the assumption that convection in the atmosphere is absent, i.e., $v_a = 0$ m/s, and the radon convection velocity in soil takes the value $v_g = 5 \cdot 10^{-2}$ m/s. Here we see that radon accumulates near the day surface due to high convection velocity, and turbulent diffusion does not have time to transport radon to the upper layers.

4 Mathematical Models of the Non-Classical Theory of the Emanation Method

According to the mathematical model of diffusion-convective transport on the surface of loose sediments, radon volumetric activity can indicate the presence of bodies with increased content of the parent element radium at depths of several tens of meters. Such low migration capacity of radon is due to its relatively short half-life (3.82 days) and high atomic weight (222), which limits the diffusion length. However, using emanation surveying, radium-containing bodies located at depths up to 500 m are localized in some cases [36].

One approach to explaining the increase in radon migration is provided by the "geogas" theory [46], the essence of which is that in water-saturated fractured rocks, microbubbles of deep gas (geogas) rise through fluid-filled fractures, significantly increasing the speed of its migration to the day surface. However, both within the framework of the classical diffusion-convective model and using the "geogas" theory, it is difficult to explain the high migration capacity of radon in loose sediments.

In a number of works, it has been shown that the geological environment has fractal properties. In this case, radon migration due to superdiffusion should proceed much more intensively. The transport properties of randomly inhomogeneous media within the framework of fractal representation are studied by percolation theory [78]. It is known that short-term anomalies in hydrogeochemical fields, which are considered as precursors of strong earthquakes, can be caused by percolation effects. The essence of these effects is that the medium where transport occurs is considered as a percolation cluster, i.e., a system of nodes (pores) and connections between them. The properties of nodes and connections are closely related to the fractal dimension, which characterizes the "complexity" of the medium under consideration, and the distribution of pores and connections obeys a power law with a fractional exponent and, as a rule, has a self-similar structure. In the classical theory of the emanation method, radon concentration during migration to the surface decreases exponentially. If the medium has fractal properties that characterize the process of anomalous diffusion, then radon concentration during migration to the earth's surface should decrease according to a power law. Such processes are studied within the framework of the non-classical theory of the emanation method [48].

4.1 Mathematical Model of Stationary Radon Superdiffusion in Porous Soil with Fractal Properties

In a number of works, it has been shown that the geological environment has fractal properties, leading to anomalous diffusion (superdiffusion), when the mean square displacement grows faster than a linear law. To describe such processes, the apparatus of fractional calculus is used. Consider the stationary radon transport equation with the Riemann-Liouville fractional derivative of order $1 < \alpha < 2$ [51]:

$$\begin{aligned} D_G D_{0z}^\alpha A(z) - \lambda A(z) + \lambda A_\infty &= 0, \\ \lim_{z \rightarrow \infty} D_{0z}^{\alpha-1} A(z) &= A_\infty (\lambda/D_G)^{1-1/\alpha}, \quad \lim_{z \rightarrow \infty} D_{0z}^{\alpha-2} A(z) = 0. \end{aligned} \quad (14)$$

The Riemann-Liouville fractional derivative of order $1 < \alpha < 2$ [49, 50]:

$$D_{0z}^\alpha A(z) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dz^2} \int_0^z \frac{A(\xi) d\xi}{(z-\xi)^{\alpha-1}}.$$

The solution of problem (14) is given in terms of the Mittag-Leffler special function:

$$A(z) = A_\infty \left(1 + (\lambda/D_G)^{1-1/\alpha} z^{\alpha-1} E_{\alpha,\alpha}(z^\alpha \lambda/D_G) - E_{\alpha,1}(z^\alpha \lambda/D_G) \right), \quad (15)$$

where $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + \beta)$, $\Gamma(\cdot)$ is the gamma function.

When $\alpha = 2$, solution (15) transforms into the classical exponential solution (3) at $v_G = 0$,

$$A(z) = A_\infty \left(1 - e^{-\sqrt{\lambda/D_G} z} \right),$$

and when $\alpha \rightarrow 1$ — into the advective regime.

Fig. 6 shows a comparison with experimental data from article [79] for different values of α .

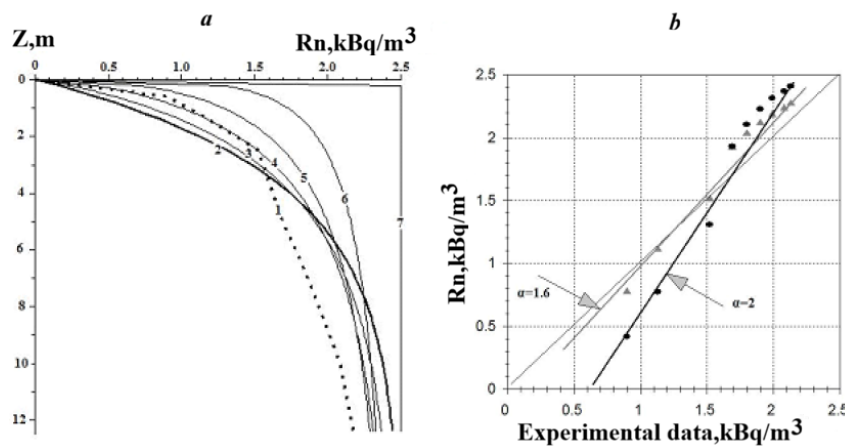


Fig. 6. Experimental data (curve 1) and calculated curves for different α (2 — $\alpha = 2$, 3 — 1.8, 4 — 1.6, 5 — 1.4, 6 — 1.2, 7 — 1).

In work [79], experimental data on radon registration at depths of 0.8, 5.5, 9, and 11.5 m are presented, which are shown in Fig. 6 (curve 1). The shape of the curve is

explained by the fact that there may be a distributed source (presence of radon sources in the medium along the migration path), or there is nonlinear diffusion when D_G is a function of concentration, with a convective component—filtration flow. The work gives preference to the second explanation, but with a certain degree of convention, one can assume the presence of a distributed source and consider the curve of radon volumetric activity variation with depth according to the considered model (14).

4.2 Mathematical Model of Stationary Radon Superdiffusion and Advection in Fractal Porous Soil

Model (14) received further development in article [53]. Consider the stationary model equation of superdiffusion and anomalous advection, which can be written in terms of the Riemann-Liouville fractional derivative as follows:

$$D_{0x}^\alpha A(z) + aD_{0x}^\beta A(z) - b(A(z) - A_\infty) = 0, \quad (16)$$

$$\lim_{z \rightarrow \infty} D_{0z}^{\alpha-1} A(z) = C, \quad \lim_{z \rightarrow \infty} D_{0z}^{\alpha-2} A(z) = 0,$$

where $a = -\frac{v_G}{D_G}$, $b = \frac{\lambda}{D_G}$, $0 < \beta < \alpha$, $C > 0$ is a constant.

Note that the special case $\beta = 1$ was considered in the author's article [52].

$$\begin{aligned} A(z) = & \frac{A_\infty b}{K} \sum_{n=0}^{\infty} \frac{b^n z^{\alpha(n+1)-1}}{n!} {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ (\alpha(n+1), \alpha-\beta) \end{matrix} \right] az^{\alpha-\beta} - \\ & - A_\infty \sum_{n=0}^{\infty} \frac{b^{n+1} z^{\alpha(n+1)}}{n!} {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ (\alpha(n+1)+1, \alpha-\beta) \end{matrix} \right] az^{\alpha-\beta}. \end{aligned} \quad (17)$$

where ${}_1\Psi_1 \left[\begin{matrix} (a_1, \alpha_1) \\ (b_1, \beta_1) \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + \alpha_1 k)}{\Gamma(b_1 + \beta_1 k)} \frac{z^k}{k!}$ is the generalized Wright function [50],

K is the solution of the equation $p^\alpha + ap^\beta - b = 0$.

Solution (17) at parameter value $\alpha = 2$ transforms into the classical solution (3) up to a factor. Let us show this. Let $\alpha = 2$, then the solution of the quadratic equation $p^2 - ap - b = 0$ will be the positive root $K = k_1 = \frac{a + \sqrt{a^2 + 4b}}{2}$. Taking into account that $-b = k_1 k_2$, where $k_2 = \frac{a - \sqrt{a^2 + 4b}}{2}$, we obtain:

$$C = \frac{A_\infty b}{k_1} = -\frac{A_\infty k_1 k_2}{k_1} = -A_\infty k_2.$$

Solution (17) is written as:

$$\begin{aligned} A(z) = & -A_\infty k_2 \sum_{n=0}^{\infty} \frac{b^n z^{2(n+1)-1}}{n!} {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ (2(n+1), 1) \end{matrix} \right] az - \\ & - A_0 \sum_{n=0}^{\infty} \frac{b^{n+1} z^{2(n+1)}}{n!} {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ (2(n+1)+1, 1) \end{matrix} \right] az. \end{aligned} \quad (18)$$

Note that the generalized Wright function can be expressed through the confluent hypergeometric Kummer function [80]:

$${}_1\Psi_1 \left[\begin{matrix} (\sigma, 1) \\ (\mu, 1) \end{matrix} \right] z = \frac{\Gamma(\sigma)}{\Gamma(\mu)} {}_1F_1(\sigma, \mu, z). \quad (19)$$

Therefore, solution (18) using expression (19) transforms to:

$$\begin{aligned} A(z) = & -A_\infty k_2 \sum_{n=0}^{\infty} \frac{b^n z^{2(n+1)-1} {}_1F_1(n+1, 2(n+1), az)}{\Gamma(2(n+1))} - \\ & - A_\infty \sum_{n=0}^{\infty} \frac{b^{n+1} z^{2(n+1)} {}_1F_1(n+1, 2(n+1)+1, az)}{\Gamma(2(n+1)+1)}. \end{aligned} \quad (20)$$

Applying successively the representation of the Kummer function through the Bessel function of the first kind of imaginary argument first for the function:

$${}_1F_1(n+1, 2(n+1), az) = \Gamma(n+3/2) 2^{2n+1} (az)^{-n-1/2} e^{\frac{az}{2}} I_{n+1/2}\left(\frac{az}{2}\right),$$

and then for the function:

$${}_1F_1(n+1, 2(n+1)+1, az) = \Gamma(n+3/2) 2^{2n+1} (az)^{-n-1/2} e^{\frac{az}{2}} \left(I_{n+1/2}\left(\frac{az}{2}\right) - I_{n+3/2}\left(\frac{az}{2}\right) \right),$$

from solution (20) we proceed to the solution:

$$\begin{aligned} A(z) = & -A_\infty e^{\frac{az}{2}} k_2 \sum_{n=0}^{\infty} \frac{b^n z^{2(n+1)-1} \Gamma(n+3/2) 2^{2n+1} (az)^{-n-1/2} I_{n+1/2}\left(\frac{az}{2}\right)}{\Gamma(2(n+1))} - \\ & - A_\infty e^{\frac{az}{2}} \sum_{n=0}^{\infty} \frac{b^{n+1} z^{2(n+1)} \Gamma(n+3/2) 2^{2n+1} (az)^{-n-1/2} \left(I_{n+1/2}\left(\frac{az}{2}\right) - I_{n+3/2}\left(\frac{az}{2}\right) \right)}{\Gamma(2(n+1))} \end{aligned}$$

taking into account that $\frac{\Gamma(n+3/2) 2^{2n+1}}{\Gamma(2(n+1))} = \frac{\sqrt{\pi}}{n!}$, $\frac{\Gamma(n+3/2) 2^{2n+1}}{\Gamma(2(n+1))} = \frac{\sqrt{\pi}}{2(n+1)!}$ and representations [80]:

$$\sum_{n=0}^{\infty} \frac{t^n I_{n+\nu}(\xi)}{n!} = \left(\frac{2t}{\xi} + 1 \right)^{\frac{\nu}{2}} I_\nu \left(\sqrt{\xi^2 + 2t\xi} \right)$$

and

$$\sum_{n=0}^{\infty} \frac{t^n I_{n-1/2}(\xi)}{n!} = \sqrt{\frac{2}{\pi\xi}} \operatorname{ch} \left(\sqrt{\xi^2 + 2t\xi} \right)$$

we obtain:

$$A(z) = A_\infty \left(1 - e^{-\sqrt{a^2+4b}z/2} \right) = A_\infty \left(1 - e^{\left(-\frac{v_G}{2D_G} + \sqrt{\frac{v_G^2}{4D_G^2} + \frac{\lambda}{D_G}} \right) z} \right).$$

This solution coincides up to a factor with the classical solution (3).

4.3 Mathematical Model of Radon Superdiffusion and Superadvection in Fractal Porous Soil with Sink into the Surface Atmosphere Layer

Continuing the development of model (14), consider stationary radon transport in the superdiffusion regime with a sink into the surface atmosphere layer without a convective component [54]:

$$\begin{aligned}
 D_g D_{0z}^\alpha A(z) - \lambda A(z) + \lambda A_\infty &= 0, z > 0, \\
 D_a \frac{d^2 A(z)}{dz^2} - \lambda A(z) &= 0, z < 0, \\
 \text{For the case } 1 < \alpha < 2 \\
 \lim_{z \rightarrow 0+0} D_{0z}^{\alpha-2} A(z) &= A(z)|_{z=0-0}, \\
 D_g \lim_{z \rightarrow 0+0} D_{0z}^{\alpha-1} A(z) &= D_a \frac{dA(z)}{dz} \Big|_{z=0-0}, \\
 \text{For the case } 0 < \alpha < 1 \\
 \lim_{z \rightarrow 0+0} D_{0z}^{\alpha-1} A(z) &= A(z)|_{z=0-0}, \\
 \lim_{z \rightarrow \infty} A(z) &= A_\infty, \lim_{z \rightarrow -\infty} A(z) = 0.
 \end{aligned} \tag{21}$$

For the case $1 < \alpha < 2$ (superdiffusion), the solution has the form:

$$\begin{aligned}
 A(z) &= A_\infty \left(1 + \frac{\mu^{1-1/\alpha} (\sigma \chi^{\alpha-1} E_{\alpha,\alpha}(\mu z^\alpha) + z^{\alpha-2} E_{\alpha,\alpha-1}(\mu z^\alpha))}{\mu^{1/\alpha} + \sigma} - E_{\alpha,1}(\mu z^\alpha) \right), z > 0, \\
 A(z) &= \frac{A_\infty \mu^{1-1/\alpha} \exp(z/l_a)}{\mu^{1/\alpha} + \sigma}, z < 0.
 \end{aligned} \tag{22}$$

For the case $0 < \alpha < 1$ (superadvection), the solution is written as:

$$\begin{aligned}
 A(z) &= A_\infty (1 + \mu^{1-1/\alpha} z^{\alpha-1} E_{\alpha,\alpha}(\mu z^\alpha) - E_{\alpha,1}(\mu z^\alpha)), z > 0 \\
 A(z) &= A_\infty \mu^{1-1/\alpha} \exp(z/l_a), z < 0,
 \end{aligned} \tag{23}$$

$$l_a = \sqrt{D_a/\lambda}, \mu = \lambda/D_g, \sigma = \lambda l_a/D_g.$$

In the case of superdiffusion ($1 < \alpha < 2$), near the interface, the radon distributions are located between the curves for $\alpha = 2$ and 1 (Fig. 7).

The first of them corresponds to ordinary diffusion ($\alpha = 2$), and the second to the limiting advective regime ($\alpha = 1$). However, deeper into the medium, the lines are rearranged in reverse order, and with decreasing α , the calculated curves stretch, which means that the effective diffusion length increases and radon penetrates more easily through porous soil. This non-monotonic behavior of the concentration profiles is a characteristic feature of anomalous diffusion in media with fractal properties. Physically, it reflects the fact that in such media the transport process is no longer governed by a single characteristic scale: near the surface, the concentration gradient is determined mainly by the boundary conditions and the sink into the atmosphere, while at greater depths the long-range correlations inherent in the fractal structure begin to dominate, leading to a slower decay of the concentration with depth compared with the exponential

law characteristic of classical diffusion. As a result, even at significant distances from the surface, the volumetric activity of radon remains relatively high, which increases the probability of detecting deep radium sources and extends the effective depth of the emanation method.

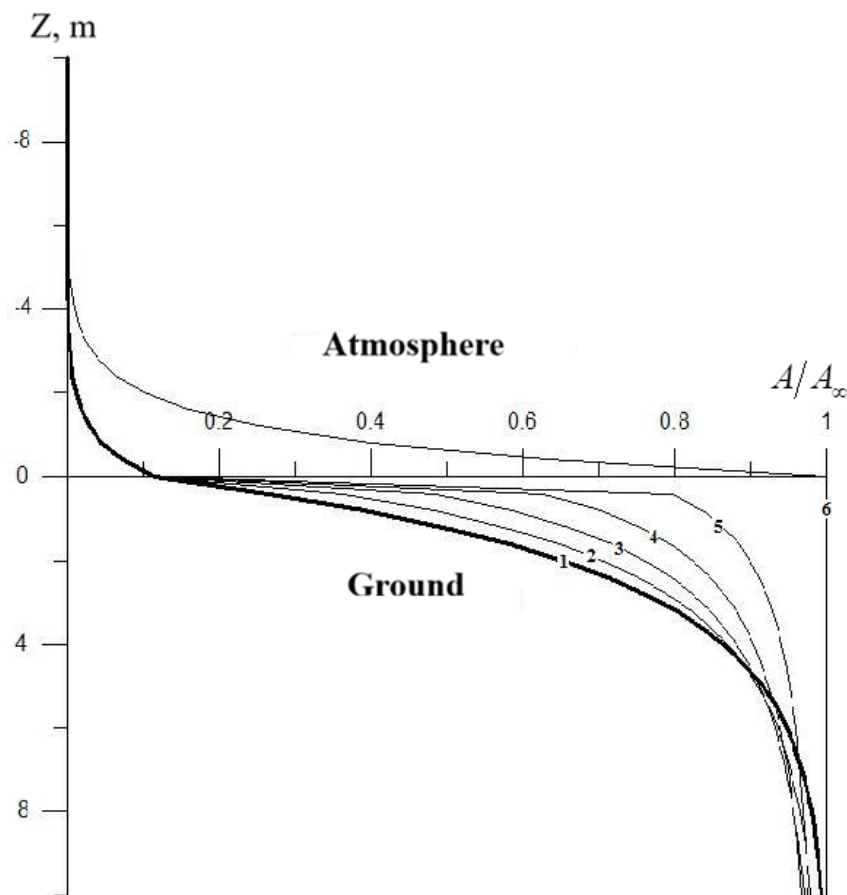


Fig. 7. Profiles of radon volumetric activity during superdiffusion ($1 < \alpha < 2$): 1 – $\alpha = 2$, 2 – 1.8, 3 – 1.6, 4 – 1.4, 5 – 1.2, 6 – 1.

Calculations show that in the case of superadvection ($0 < \alpha < 1$), radon is transported out of the fractal medium even more intensively than during superdiffusion and accumulates near the boundary with the atmosphere (Fig. 8).

This accumulation of radon concentration is a consequence of the anisotropy of the geoenvironment. The anisotropy of the geoenvironment is associated with decompaction (dilatancy) of rocks, which occurs as a result of their destruction. This means that superadvection can explain radon emissions during periods of strong deformation disturbances.

Superadvection, as noted above, can be explained by transport accompanied by the release of radon from pore fractal structures into the pore space. However, what are the properties of these structures, how do they depend on the state of the medium, and how is the fractal dimension related to the fractional exponent of the derivative in the transport equation—these questions require separate studies.

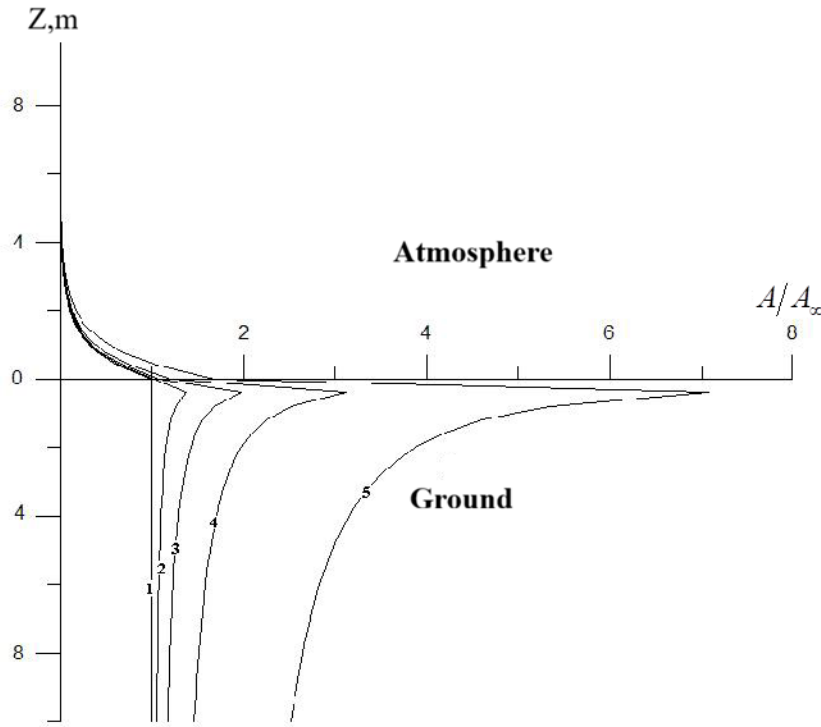


Fig. 8. Profiles of radon volumetric activity during superadvection ($0 < \alpha < 1$): 1 – $\alpha = 1$, 2 – 0.8, 3 – 0.6, 4 – 0.4, 5 – 0.2.

Note that Fig. 8 is similar to Fig. 5b, which describes classical non-stationary diffusion-convective radon transport with a high convection velocity value. Therefore, one can speak here in some sense about the equivalence of models (20) for $0 < \alpha < 1$ and (13).

Model (21) received further development in works [55–58]. Here we will focus on a more general non-stationary fractional model of diffusion-convective radon transport in the soil-atmosphere system proposed in article [58].

4.4 Fractional Mathematical Model of Non-Stationary Radon Diffusion-Convection in the Soil-Atmosphere System

It is necessary to find the solution $A(z, t)$ in the domain ($t > 0, -\infty < z < \infty$):

$$\lim_{z \rightarrow \infty} A(z, t) = 0,$$

$$\frac{\partial A(z, t)}{\partial t} = D_a \frac{\partial^2 A(z, t)}{\partial z^2} + v_a \frac{\partial A(z, t)}{\partial z} - \lambda A(z, t), \quad z > 0 \quad (24)$$

$$\lim_{z \rightarrow 0-0} D_{z0}^{\alpha-2} A(z, t) = A(z, t)|_{z=0+0},$$

$$\lim_{z \rightarrow 0-0} \left[D_g D_{z0}^{\alpha-1} A(z, t) + v_g D_{z0}^{\beta-1} A(z, t) \right] = D_a \frac{\partial A(z, t)}{\partial z} \Big|_{z=0+0} + v_a A(z, t)|_{z=0+0},$$

$$\frac{\partial^\gamma A(z, t)}{\partial t^\gamma} = D_g D_{z0}^\alpha A(z, t) + v_g D_{z0}^\beta A(z, t) - (\lambda A(z, t) - A_\infty), \quad z < 0,$$

$$\lim_{z \rightarrow -\infty} A(z, t) = A_\infty.$$

where $D_{z0}^\alpha, D_{z0}^\beta$ are Riemann-Liouville fractional differentiation operators ($1 < \alpha < 2, 0 < \beta < 1$):

$$D_{z0}^\alpha A(z, t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dz^2} \int_z^0 \frac{A(y, t) dy}{(z-y)^{\alpha-1}}, D_{z0}^\beta A(z, t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dz} \int_z^0 \frac{A(y, t) dy}{(z-y)^\beta},$$

where $\Gamma(\cdot)$ is the gamma function and ∂_{0t}^γ is the operator in the sense of Gerasimov-Caputo [81, 82]:

$$\frac{\partial^\gamma A(z, t)}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)} \int_z^0 \frac{A'_\theta(z, \theta) d\theta}{(t-\theta)^\gamma}, 0 < \gamma < 1,$$

D_a, D_g are the diffusion coefficients of matter in the atmosphere and loose sediments, $m^2/s, m^2/s$; λ is the decay constant of matter, $1/s$; A_∞ is the volumetric activity of matter in radioactive equilibrium with the emanation source at a given depth, Bq/m^3 ; $A(z, t)$ – volumetric activity of the substance in loose sediments, Bq/m^3 , $A_\infty = KA_M \rho(1-\eta)/\eta$, where K – substance emanation coefficient, rel. units; A_M – specific activity of the emanation source, Bq/kg ; ρ – density of solid particles of loose sediments, kg/m^3 ; η – porosity of loose sediments.

The boundary value problem (24) describes non-local and non-stationary diffusion-convective transfer of radioactive substance in the lithosphere-atmosphere system. We will not discuss the derivation of model equations (24) here. We will say that the introduction of fractional derivatives with respect to time and space to describe the diffusion-convective transfer in the soil indicates the non-locality of this medium. The non-locality of the medium in our case is associated with its permeability, which is responsible for the intensity of the transfer process.

External boundary conditions indicate that at some depth in the soil there is a source of radioactive substance, and in the atmosphere with height due to turbulent diffusion there is a sharp decrease in its concentration. Internal boundary conditions – equality of concentrations and flows at the boundary of the media provide the necessary condition for gluing solutions.

It should be noted that the paper [41] considered the boundary value problem (24) for radioactive radon gas under the condition $\alpha = 2, \beta = \gamma = 1$. The analytical solution to the problem was obtained using the Laplace integral equation in terms of integral error functions. The paper [55] considered the stationary case of the problem (24), when the partial derivatives with respect to time on the right-hand side are equal to zero.

Let us introduce at the characteristic time t_0 both the scale z_0 and the corresponding dimensionless coordinates $\tau = t/t_0, \xi = z/z_0$. Then the equation (24) will be written in dimensionless coordinates as:

$$\frac{\partial A(\xi, \tau)}{\partial \tau} = \bar{D}_a \frac{\partial^2 A(\xi, \tau)}{\partial \xi^2} + \bar{v}_a \frac{\partial A(\xi, \tau)}{\partial \xi} - \bar{\lambda} A(\xi, \tau), \quad \xi > 0 \quad (25)$$

$$\frac{\partial^\gamma A(\xi, \tau)}{\partial \tau^\gamma} = \bar{D}_g D_{\xi 0}^\alpha A(\xi, \tau) + \bar{v}_g D_{\xi 0}^\beta A(\xi, \tau) - (\bar{\lambda} A(\xi, \tau) - A_\infty), \quad \xi < 0$$

$$\lim_{\xi \rightarrow 0-0} D_{0\xi}^{\alpha-2} A(\xi, \tau) = A(\xi, \tau)|_{\xi=0+0},$$

$$\lim_{\xi \rightarrow 0-0} [\bar{D}_g D_{\xi 0}^{\alpha-1} A(\xi, \tau) + \bar{v}_g D_{\xi 0}^{\beta-1} A(\xi, \tau)] = \bar{D}_a \frac{\partial A(\xi, \tau)}{\partial \xi} \Big|_{\xi=0+0} + \bar{v}_a A(\xi, \tau)|_{\xi=0+0},$$

Transfer parameters in the soil and in the atmosphere will also be dimensionless quantities.

We will seek a solution to equation (25) in the form of a traveling wave with a velocity of V . We will make a replacement in (25): $A(\xi, \tau) = f(x)$, $x = \xi - V\tau$, $W_a = V - \bar{v}_a$, $s = -V\tau$. Then equation (25) will become the equation:

$$\bar{D}_a \frac{d^2 f(x)}{dx^2} + W_a \frac{df(x)}{dx} - \bar{\lambda} f(x) = 0, \quad x > s \quad (26)$$

$$\bar{D}_g D_{xs}^\alpha f(x) + \bar{v}_g D_{xs}^\beta f(x) + V^\gamma \frac{d^\gamma f(x)}{dx^\gamma} - (\bar{\lambda} f(x) - A_\infty) = 0, \quad x < s$$

$$\lim_{x \rightarrow s-0} D_{xs}^{\alpha-2} f(x) = f(x)|_{x=s+0},$$

$$\lim_{x \rightarrow s-0} [\bar{D}_g D_{xs}^{\alpha-1} f(x) + \bar{v}_g D_{xs}^{\beta-1} f(x)] = \bar{D}_a \frac{df(x)}{dx} \Big|_{x=s+0} + \bar{v}_a f(x)|_{x=s+0}.$$

The solution of the first equation (26) for the atmosphere is known, it can be written as follows:

$$f(x) = C_1 e^{x \left(\frac{-r_1 + \sqrt{r_1^2 + 4r_2}}{2} \right)} + C_2 e^{-x \left(\frac{r_1 + \sqrt{r_1^2 + 4r_2}}{2} \right)}, \quad (27)$$

where $r_1 = W_a/\bar{D}_a$, $r_2 = \bar{\lambda}/\bar{D}_a$.

From physical considerations, at $x \rightarrow \infty$ the solution of equation (27) tends to zero. Indeed, the radon concentration decreases toward the earth's surface, and in the atmosphere it is close to zero. Therefore, the constant $C_1=0$ and then the boundary conditions at the soil-atmosphere boundary are rewritten as follows:

$$\lim_{x \rightarrow s-0} D_{xs}^{\alpha-2} f(x) = R_1 C_2,$$

$$\lim_{x \rightarrow s-0} [\bar{D}_g D_{xs}^{\alpha-1} f(x) + \bar{v}_g D_{xs}^{\beta-1} f(x)] = R_2 C_2,$$

$$R_1 = \exp \left(-s \left(r_1 + \sqrt{r_1^2 + 4r_2} \right) / 2 \right), R_2 = -R_1 (\bar{D}_a (r_1 + \sqrt{r_1^2 + 4r_2}) / 2 + \bar{v}_a)$$

Let us consider the second equation (26) in more detail. Applying the integral Laplace transform to it, we obtain the following equation:

$$f(p) = \frac{C_2 R_2 p + p^2 C_2 R_1 - \bar{\lambda} A_\infty}{p (\bar{D}_g p^\alpha + V^\gamma p^\gamma - \bar{v}_g p^\beta - \bar{\lambda})}. \quad (28)$$

Let us separately consider the expression: $1/(\bar{D}_g p^\alpha + V^\gamma p^\gamma - \bar{v}_g p^\beta - \bar{\lambda})$. Here the following cases are possible: $\gamma < \beta$, $\gamma = \beta$, $\gamma > \beta$. The case when $\gamma = \beta$ leads us to the

solution obtained in [83], but with slightly different coefficients. Let $\gamma > \beta$, then we can write

$$1/(p^\alpha - \sigma p^\gamma - \mu p^\beta - b) = p^{-\gamma} / [(p^{\alpha-\gamma} - \sigma)(1 - (\mu p^{\beta-\gamma} + b p^{-\gamma})/(p^{\alpha-\gamma} - \sigma))],$$

$$\sigma = -V^\gamma / \bar{D}_g, \quad \mu = \bar{v}_g / \bar{D}_g, \quad b = \bar{\lambda} / \bar{D}_g.$$

Taking into account the condition $|(\mu p^{\beta-\gamma} + b p^{-\gamma})/(p^{\alpha-\gamma} - \sigma)| < 1$ it can be written as follows [50]:

$$\sum_{n=0}^{\infty} p^{-\gamma} (\mu p^{\beta-\gamma} + b p^{-\gamma})^n / (p^{\alpha-\gamma} - \sigma)^{n+1} = \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{n! \mu^m b^l}{m! l!} p^{-1-l-(\gamma-\beta)m} / (p^{\alpha-\gamma} - \sigma)^{n+1}.$$

According to the work [50], the inverse Laplace transform of this expression for the first term of the equation (28) gives:

$$f_1(x) = \frac{C_1 R_2}{\bar{D}_g} \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{n! \mu^m b^l x^{(\alpha-\gamma)n + \alpha - 2 + l + (\gamma-\beta)m}}{m! l!} F_1,$$

$$F_1 = {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ ((\alpha-\gamma)n + \alpha - 1 + l + (\gamma-\beta)m, \alpha-\gamma) \end{matrix} \middle| (\sigma x^{\alpha-\gamma}) \right],$$

where ${}_1\Psi_1(x)$ is the generalized Wright function [50]. Similarly for the second and third terms of the equation (28):

$$f_2(x) = \frac{C_1 R_1}{\bar{D}_g} \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{n! \mu^m b^l x^{(\alpha-\gamma)n + \alpha - 1 + l + (\gamma-\beta)m}}{m! l!} F_2,$$

$$F_2 = {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ ((\alpha-\gamma)n + \alpha + l + (\gamma-\beta)m, \alpha-\gamma) \end{matrix} \middle| (\sigma x^{\alpha-\gamma}) \right]$$

$$f_3(x) = -A_\infty b \sum_{n=0}^{\infty} \sum_{l+m=n} \frac{n! \mu^m b^l x^{(\alpha-\gamma)n + \alpha + l + (\gamma-\beta)m}}{m! l!} F_3,$$

$$F_3 = {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ ((\alpha-\gamma)n + \alpha + 1 + l + (\gamma-\beta)m, \alpha-\gamma) \end{matrix} \middle| (\sigma x^{\alpha-\gamma}) \right]$$

The solution in this case is a superposition:

$$f(x) = f_1(x) + f_2(x) + f_3(x). \quad (29)$$

Let's find the constant C_1 . To do this, we use the method proposed in [55]. We write the equation (28) as follows:

$$f(p) = \frac{C_2 R_1 (p^2 + R_2/R_1 p - A_\infty b/(R_1 C_2))}{\bar{D}_g p (p^\alpha - \mu p^\beta - \sigma p^\gamma - b)} = \quad (30)$$

$$= \frac{C_2 R_1}{\bar{D}_g p} \frac{\left[p - \left(\frac{-R + \sqrt{R^2 + 4A_\infty b / (R_1 C_2)}}{2} \right) \right]}{p^\alpha - \mu p^\beta - \sigma p^\gamma - b} \left[p - \left(\frac{-R - \sqrt{R^2 + 4A_\infty b / (R_1 C_2)}}{2} \right) \right],$$

$$R = R_2 / R_1.$$

Since the denominator of the equation (30) $p^\alpha - \mu p^\beta - \sigma p^\gamma - b$ is an infinitely increasing function, and its values are in the right half-plane, then the equation has a positive root. Let us denote this root as K , then the denominator of the equation (29) can be expanded in powers of $p - K$:

$$p^\alpha - \mu p^\beta - \sigma p^\gamma - b = \sum_{i=0}^{\infty} c_i (p - K)^i \quad (31)$$

Substituting (30) into (29) we obtain:

$$f(p) = \frac{C_2 R_1}{\bar{D}_g p} \frac{\left[p + \left(\frac{R - \sqrt{R^2 + 4A_\infty b / (R_1 C_1)}}{2} \right) \right]}{(p - K) \sum_{i=0}^{\infty} c_i (p - K)^{i-1}} \left[p + \left(\frac{R + \sqrt{R^2 + 4A_\infty b / (R_1 C_1)}}{2} \right) \right] \quad (32)$$

In order for $f(p)$ to be analytic in the right half-plane, it is necessary to choose the constant C_1 in such a way that the first bracket in the numerator is canceled with the bracket in the denominator. This is achieved when

$$C_2 = \frac{A_\infty b}{(K^2 + RK)R_1} \quad (33)$$

Then the solution (32) according to (33) will be written as follows:

$$A(\xi, \tau) = \frac{A_\infty b e^{-\left(\xi - V\tau\right) \left(\frac{r_1 + \sqrt{r_1^2 + 4r_2}}{2} \right)}}{(K^2 + RK)R_1}, \quad \xi > 0$$

$$A(\xi, \tau) = A_\infty b \left[\frac{R_2 \theta_2 + R_1 \theta_1}{\bar{D}_g (K^2 R_1 + K R_2)} - \theta_0 \right], \quad \xi < 0$$

$$\theta_i = \sum_{n=0}^{\infty} \sum_{m+l=n} \frac{\mu^m b^l (\xi - V\tau)^{(\alpha-\gamma)n + \alpha - 2 + i + l + (\gamma-\beta)m}}{m!l!} F_i, \quad (34)$$

$$F_i = {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ ((\alpha-\gamma)n + \alpha - 2 + i + l + (\gamma-\beta)m, \alpha-\gamma) \end{matrix} \middle| \sigma (\xi - V\tau)^{\alpha-\gamma} \right],$$

$$R_1 = \exp \left(V\tau \left(r_1 + \sqrt{r_1^2 + 4r_2} \right) / 2 \right), R_2 = -R_1 (\bar{D}_a (r_1 + \sqrt{r_1^2 + 4r_2}) / 2 + \bar{v}_a),$$

K is the solution of the equation $y^\alpha - \sigma y^\gamma - \mu y^\beta - b = 0$, $i = 1, 2, 3$.

Similar considerations can be used for the case $\gamma < \beta$. The parameters in (34) will be redefined.

It can be shown using the methodology of the works [55], that in the case when $\alpha = 2, \beta = \gamma = 1$ solution (34) goes over to the classical solution obtained in the article [41].

From solution (34), it can be seen that its structure is complex, and constructing its graph is a separate independent task that has important significance.

Next, we will consider a single-point mathematical model of radon transfer at a certain point in a fractal geoenvironment.

5 Mathematical Model of Radon Transport in Inhomogeneous Geoenvironment with Variable Memory and Its Application to Seismic Activity Monitoring in Kamchatka

In work [84], a generalized equation of radioactive decay of matter in a fractal medium was presented

$$\partial_{0t}^{\alpha} N(t) = -\lambda_1 N(t), \quad (35)$$

$N(t)$ is the number of radon particles that have not decayed by time t , recorded by the detector, λ_1 is the decay constant, the differentiation operator of order $0 < \alpha < 1$ on the right-hand side is $\partial_{0t}^{\alpha} N(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{N'(\eta) d\eta}{(t-\eta)^{\alpha}}$ is the Gerasimov-Caputo fractional derivative, α is a parameter related to the fractal properties (permeability: measure of tortuosity and dead-end pores) of the geoenvironment.

Law (35) can be interpreted as follows: the ensemble-average rate of radioactive decay of matter in a medium with memory, i.e., the number of particles decaying in a fractal geological medium per unit time, is proportional to the number of undecayed particles in the medium. The proportionality coefficient λ_1 has the meaning of the decay constant. Note that the authors of article [84] associate the order of the fractional derivative with the nuclear structure, while in our model, it serves as an effective parameter characterizing the degree of heterogeneity and memory effects in the geological medium, specifically linked to the heavy-tailed distribution of radon residence times in a hierarchically organized pore network.

Let the parameter α vary with time t . This physically means that the fractal properties of the medium change with time. In relation to geophysics, the stress-strain state of the earth's crust changes. It is important to emphasize the physical mechanism linking the micro-geometry of the medium and the order of the fractional derivative α in equation (35). The complex, fractal structure of pores (the presence of dead-end branches, caverns) gives rise to a broad, often power-law, spectrum of delay (trapping) times for radon atoms in the medium on their way to the detector. From the perspective of continuous-time random walk (CTRW) theory, such a distribution of waiting times leads to non-Markovian kinetics and effective macroscopic equations with fractional time derivatives. The parameter α in such an equation is the exponent in the asymptotic behavior of this distribution. When the pore structure is simple and well-connected ($\alpha \rightarrow 1$), delays are minimal, and the kinetics of the registration process follows a classical exponential pattern. Conversely, a highly fractalized medium ($\alpha < 1$)

is characterized by an anomalously slow, non-exponential "release" of radon from traps and its subsequent decay, which is precisely reflected in the equation with a fractional-order derivative.

Let there be a radon source with parameter λ_2 at some point in the fractal geological medium, then the generalized law of radioactive decay is written as follows:

$$\partial_{0t}^{\alpha(t)} N(t) = -\lambda_1 N(t) + \lambda_2 N(0). \quad (36)$$

The value of parameter λ_2 can be determined using experimental data. In the work, we assume it is known.

The radon registration sensor makes measurements in units of volumetric activity (Bq/m^3), so multiplying the equation by λ_1 and assuming $A(t) = \lambda_1 N(t)$, we write equation (36) in the form

$$\partial_{0t}^{\alpha(t)} A(t) = -\lambda_1 A(t) + \lambda_2 A(0), \quad (37)$$

$A(0)$ is the initial radon distribution.

In equation (37), the unknown functions are $A(t)$ and $\alpha(t)$. It is known that at some point in the fractal geological medium, radon volumetric activity values are recorded. We arrive at the following problem: find functions $A(t)$ and $\alpha(t)$ satisfying the conditions:

$$A(t_k) = \varphi(t_k), \quad k = 0, 1, 2, \dots, n, \quad (38)$$

where $\varphi(t_k) = \varphi_k$ are experimental data.

Let us approximate the fractional differentiation operator on the right-hand side of equation (37).

To do this, we represent the operator as:

$$\begin{aligned} \partial_{0t}^{\alpha(t)} A(t) &= \partial_{0t}^{\alpha(t)-1} \frac{d}{dt} [A(t) - A(0)] = D_{0t}^{\alpha(t)} [A(t) - A(0)] = \\ &= D_{0t}^{\alpha(t)} A(t) - \frac{A(0)t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}. \end{aligned} \quad (39)$$

$D_{0t}^{\alpha(t)} A(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_0^t \frac{A(\eta)d\eta}{(t-\eta)^{\alpha(t)}}$ is the Riemann-Liouville operator.

Since at each time step, due to the "frozen" order approximation, α is considered constant, the relationship between the Gerasimov-Caputo and Riemann-Liouville derivatives retains its classical form.

On the other hand, introduce a uniform time grid with step $\tau = t/n$, $t_k = k\tau$, $A(t_k) = A_k$, $k = 1, 2, \dots, n$.

Due to the low rate of change of the stress-strain state of the medium compared to the characteristic decay time of radon emanation memory, we use the adiabatic approximation of "frozen" fractional order. At each time step k , the integral operator is approximated assuming $\alpha(t) \approx \alpha(t_k) = \alpha_k$ for all $t \in [0, t_k]$. This approach, also known as the step-by-step constant order technique, allows reducing the nonlinear inverse problem to a sequence of one-dimensional optimizations and is standard in applied geophysical calculations.

We obtain

$$\begin{aligned}
 D_{0t}^{\alpha(t)} A(t) &= \frac{A(0)t^{-\alpha(t)}}{\Gamma(1-\alpha(t))} + \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{A'(\eta)d\eta}{(t-\eta)^{\alpha(t)}} \approx \\
 &\approx \frac{A(0)(k\tau)^{-\alpha_k}}{\Gamma(1-\alpha_k)} + \frac{1}{\Gamma(1-\alpha_k)} \int_0^t \frac{A'(t-y)}{y^{\alpha_k}} dy = \\
 &= \frac{A(0)(k\tau)^{-\alpha_k}}{\Gamma(1-\alpha_k)} + \frac{1}{\Gamma(1-\alpha_k)} \sum_{j=0}^{k-1} \frac{A_{k-j} - A_{k-j-1}}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{dy}{y^{\alpha_k}} = \\
 &= \frac{A(0)(k\tau)^{-\alpha_k}}{\Gamma(1-\alpha_k)} + \frac{\tau^{-\alpha_k}}{\Gamma(2-\alpha_k)} \sum_{j=0}^{k-1} (A_{k-j} - A_{k-j-1}) [(j+1)^{1-\alpha_k} - j^{1-\alpha_k}].
 \end{aligned} \tag{40}$$

Substituting (40) into expression (39), we obtain the following result:

$$\partial_{0t}^{\alpha_k} A(t) \approx \frac{\tau^{-\alpha_k}}{\Gamma(2-\alpha_k)} \sum_{j=0}^{k-1} [(j+1)^{1-\alpha_k} - j^{1-\alpha_k}] (A_{k-j} - A_{k-j-1}). \tag{41}$$

Substituting expression (41) into equation (37), we obtain the following expression

$$\frac{\tau^{-\alpha_k}}{\Gamma(2-\alpha_k)} \sum_{j=0}^{k-1} [(j+1)^{1-\alpha_k} - j^{1-\alpha_k}] (A_{k-j} - A_{k-j-1}) = -\lambda_1 A_k + \lambda_2 A_0. \tag{42}$$

Let us find the solution A_k of equation (42) as a function of the unknown values α_k .

$$A_k(\alpha_k) = \frac{A_{k-1} - \sum_{j=1}^{k-1} [(j+1)^{1-\alpha_k} - j^{1-\alpha_k}] (A_{k-j} - A_{k-j-1}) + \lambda_2 \Gamma(2-\alpha_k) \tau^{\alpha_k} A_0}{1 + \Gamma(2-\alpha_k) \lambda_1 \tau^{\alpha_k}}. \tag{43}$$

Condition (39) for solution (43) is written as:

$$A_k(\alpha_k) = \varphi_k. \tag{44}$$

To identify the parameter α_k , we form the residual according to (44) $\eta_k(\alpha_k) = \varphi_k - A_k(\alpha_k)$, which we minimize using the one-dimensional optimization method. As a result, we obtain a series of values α_k , which we will subsequently investigate for the purpose of studying seismic activity in Southern Kamchatka [59, 60].

In the radon volumetric activity values before the event with $M = 5.6$ that occurred on August 4, 2004, at points LVCh and IKIR, bay-shaped anomalies with a lead time of about one day stand out (Fig. 9).

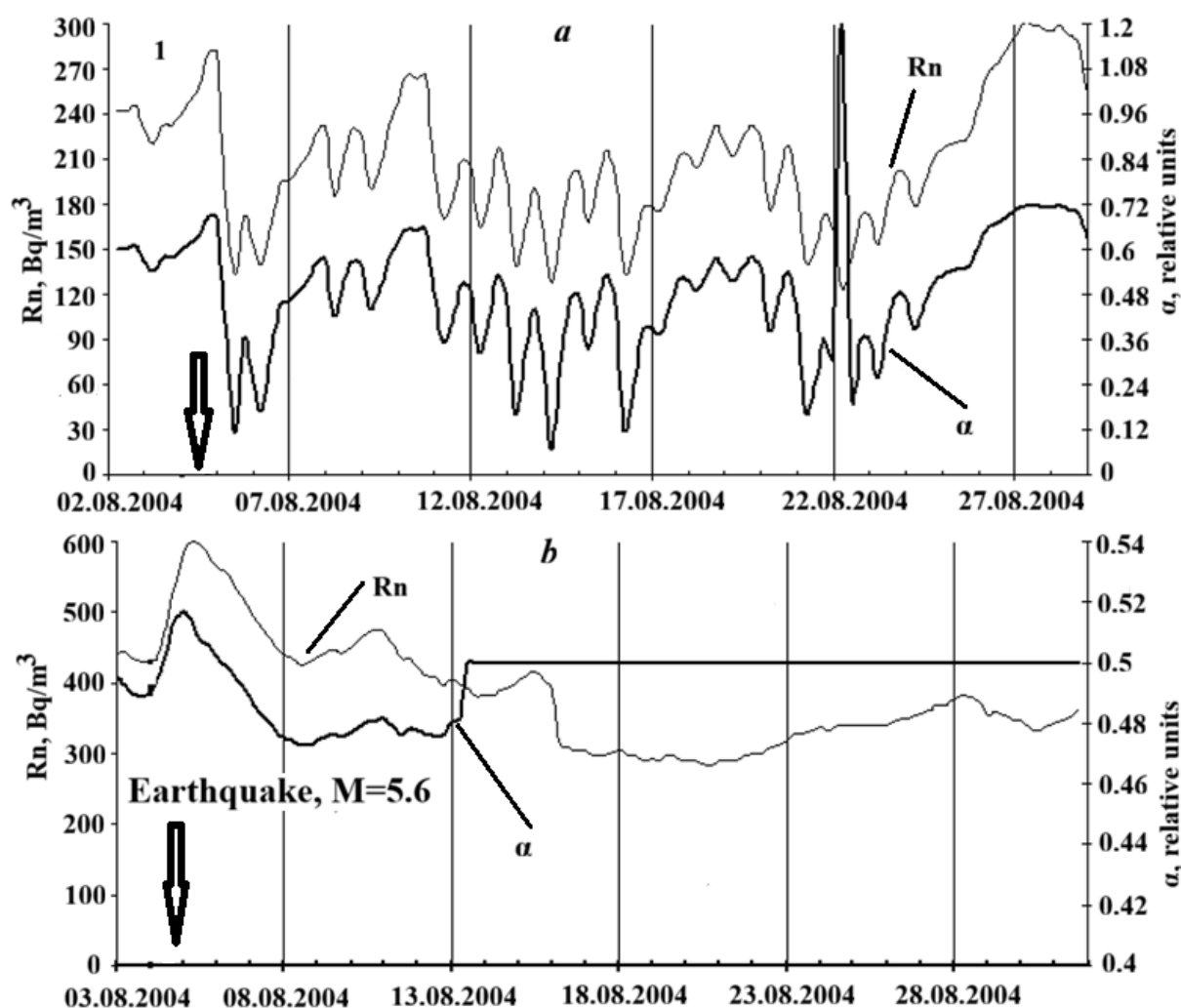


Fig. 9. Curves of radon volumetric activity and reconstructed α values: a) Levaia Avacha station; b) IKIR station. The arrow indicates the earthquake of August 4, 2004, $M=5.6$, $R=115$ km.

These anomalies, based on their in-phase appearance at three points—IKIR, LVCh, PRT—are associated with a geodeformation wave that arose at the final stage of earthquake preparation as a result of quasi-plastic flow of geomaterial. At a geodeformation wave propagation speed of 30 km/day, the wave emerged four days before the event [14].

At the LVCh point, synchronous behavior of curves for both parameters is observed (Fig. 9a), and for the INS point, the precursor anomaly in the exponent values looks quite convincing (Fig. 9b), reaching a stationary level on August 13. This indicates that the porous properties of loose sediments change when a geodeformation wave passes, and the fractional exponent can serve as another parameter for identifying and formalizing precursor anomalies.

Fig. 10 shows radon volumetric activity values and fractional exponent α values for the period 2005–2010 according to PRT point data.

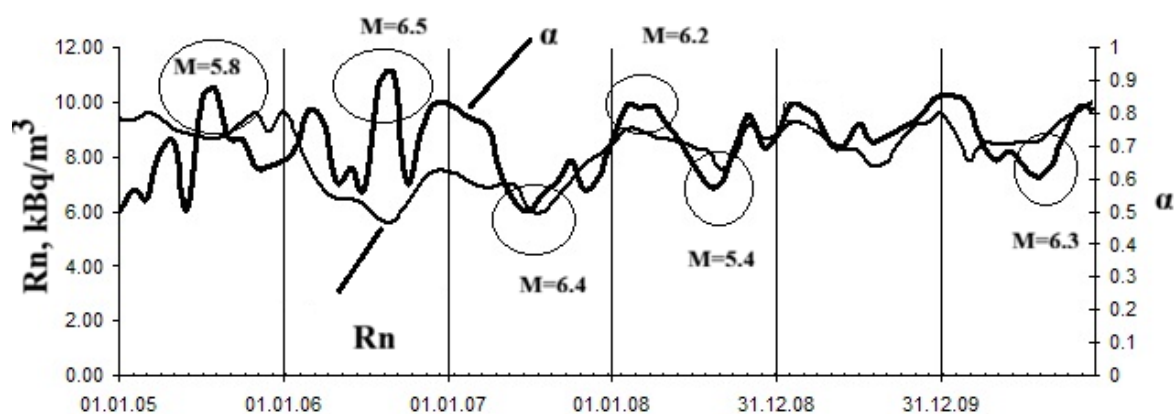


Fig. 10. Seismic activity of Southern Kamchatka in the period 2005–2010 according to PRT point data

Circles mark earthquakes in Southern Kamchatka with $M > 5.5$. In 2005, on July 7, an earthquake with $M = 5.8$ occurred in Avacha Bay (52.87 N, 160.10 E). In 2006, seismic activity mainly manifested in July–August. A strong earthquake during this period occurred on August 24 with $M = 6.5$ off the southern coast of Kamchatka (51.15 N, 157.52 E). In 2007, a strong earthquake occurred on May 30 on the territory of the Kamchatka Peninsula (52.14 N, 157.29 E) with $M = 6.4$. In 2008, seismic activity increased: three events with $M \geq 5.5$ occurred. On March 6 (51.52 N, 156.55 E) $M = 5.5$, July 24 (50.97 N, 157.58 E) $M = 6.2$, September 18 (51.94 N, 158.36 E) $M = 5.9$. In 2009, there were no earthquakes with $M > 5.5$, but on January 14, an earthquake with $M = 5.4$ occurred (52.27 N, 159.29 E). In 2010, seismic activity intensified in July; on July 30, an earthquake with $M = 6.3$ occurred in the Avacha Bay area (52.50 N, 159.84 E).

In Fig. 10, it can be seen that the times of the above-mentioned earthquakes coincide with either an increase or a decrease in the values of the fractional exponent α . Given that this exponent is related to the fractal dimension of the medium, it reflects the structural change in the geoenvironment under the influence of deformations. The opposite polarity of anomalies indicates that the deformation structure of the medium is mosaic and fractal.

As shown in work [85], sharp changes, including short-term ones, in the stress-strain state of the contact between rock blocks lead to interblock displacements. These, in turn, manifest as "quasi-viscous" flow of the earth's crust material. Apparently, the creep process due to quasi-viscous flow of rock masses generates "geodeformation waves" caused by the interaction of individual blocks of the geological environment. Geodeformation waves change the structure of loose sediments, which is reflected in changes in the exponent α and radon volumetric activity. Depending on the location of registration points relative to block boundaries, anomalies have different shapes (bay-shaped, stepwise) and different polarity depending on the geological-tectonic conditions of a given registration point relative to the earthquake focus.

6 Conclusion

In this review article, dedicated to the systematization of the results of many years of research by the author on mathematical modeling of radon transport in geological media, the evolution of models from classical ones based on the diffusion-convection equation with constant coefficients to non-classical ones taking into account fractal properties of the geoenvironment and memory effects using the apparatus of fractional differentiation has been considered.

Let us briefly formulate the main results presented in the work.

1. Within the framework of the classical theory of the emanation method, the basic model of non-stationary diffusion-convective radon transport in homogeneous porous soil has been analyzed, for which an exact analytical solution in terms of error integrals has been obtained. It has been shown that taking advection into account significantly reduces the time for establishing radioactive equilibrium and increases the contribution to the intensity of the transport process compared to the purely diffusion mechanism.

2. Based on the classical model, an algorithm for estimating radon flux density from the earth's surface has been developed and tested on real data. The application of the algorithm to data from the radon monitoring network at the Petropavlovsk-Kamchatsky geodynamic test site confirmed that flux density is a more sensitive indicator of deformation disturbances in the geoenvironment than radon volumetric activity. This result is important for improving the efficiency of monitoring geodynamic processes and searching for earthquake precursors.

3. A non-stationary two-layer model of diffusion-convective radon transport in the "soil-atmosphere" system has been proposed and investigated, for which an analytical solution has also been found. This model allows correctly describing radon discharge into the surface atmosphere layer and taking into account the difference in transport parameters in two media, which is necessary for a comprehensive study of geophysical and ecological processes.

4. To explain the anomalously high migration capacity of radon observed in some cases, models of stationary superdiffusion (for $1 < \alpha < 2$) and superadvection (for $0 < \alpha < 1$) in a porous medium with a fractal structure have been constructed and studied. The obtained analytical solutions in terms of special functions (Mittag-Leffler, Wright) demonstrate the power-law nature of radon concentration profiles, which agrees better with some experimental data than the classical exponential model. It has been established that in the superadvection regime, intensive radon accumulation occurs near the interface, which can serve as an explanation for sharp preseismic emissions.

5. The non-classical theory of the emanation method has been generalized to a model with a time-varying order of the fractional derivative. Based on this approach, a methodology for identifying the dynamics of the fractal parameter $\alpha(t)$ from time series of radon volumetric activity has been developed. The application of the methodology to monitoring data in Southern Kamchatka for the period 2005–2010 showed the presence of anomalies in the behavior of the parameter α correlating with strong seismic events. This confirms the hypothesis that the order of the fractional derivative can serve as a quantitative measure of changes in the permeability (fractal structure) of the

geoenvironment under the influence of deformation disturbances and can be useful in developing earthquake prediction systems.

In general, the results obtained contribute to the development of the theory of the emanation method and demonstrate the effectiveness of applying classical and fractional mathematical models for solving fundamental and applied problems of geophysics. The developed models and algorithms can be used not only in the field of seismic activity monitoring but also in solving a wide range of problems, such as mineral exploration, assessment of radon hazard in territories, and study of the ecological state of the environment.

Further development of the topic is seen in the transition to more complex models taking into account the multiscale heterogeneity of the geoenvironment, the mutual influence of hydrogeological and geothermal fields, as well as in the development and verification of models based on fractional derivatives of variable order in space for describing nonlocal transport processes in the "lithosphere-atmosphere" system.

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