



C^* -algebraic Smale Mean Value Conjecture and Dubinin-Sugawa Dual Mean Value Conjecture

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Abstract. We develop C^* -algebraic analogues of two central problems in complex analysis: the Smale Mean Value Conjecture and the Dubinin-Sugawa Dual Mean Value Conjecture. Building on Smale's classical mean value theorem and its dual formulation, we extend these conjectures to the setting of commutative C^* -algebras. The proposed framework defines algebraic differentiation for polynomials over C^* -algebras and formulates both standard and strong versions of the conjectures. We further introduce higher-order and dynamical extensions, including a C^* -algebraic analogue of the Miles-Leighton-Pilgrim dynamics conjecture. Rigorous verification is provided for degree two polynomials, establishing the validity of these conjectures in this foundational case. The results highlight the deep interplay between operator-algebraic structures and analytic inequalities, offering new perspectives on long-standing problems in polynomial theory. This work opens pathways for generalizations to Banach and von Neumann algebras, while emphasizing the challenges posed by noncommutative settings.

Key words: C^* -algebra, Smale mean value theorem, Smale mean value conjecture, Dubinin-Sugawa dual mean value theorem, Dubinin-Sugawa dual mean value conjecture.

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C^* -алгебраическая гипотеза Смейла о среднем значении и дуальная гипотеза о среднем значении Дубинина-Сугавы

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Аннотация. Мы разрабатываем C^* -алгебраические аналоги двух центральных проблем комплексного анализа: гипотезы Смейла о среднем значении и дуальной гипотезы о среднем значении Дубинина-Сугавы. Опираясь на классическую теорему о среднем значении Смейла и ее двойственную формулировку, мы распространяем эти гипотезы на случай коммутативных C^* -алгебр. Предлагаемая структура определяет алгебраическое дифференцирование для многочленов над C^* -алгебрами и формулирует как стандартные, так и усиленные версии гипотез. Мы также вводим обобщения высших порядков и динамические расширения, включая C^* -алгебраический аналог гипотезы о динамике Майлза-Лейтона-Пилигрима. Строгое обоснование проведено для многочленов второй степени, устанавливая справедливость гипотез в этом базовом случае. Результаты подчеркивают глубокую взаимосвязь между структурами операторных алгебр и аналитическими неравенствами, предлагая новые перспективы на давно стоящие проблемы теории многочленов. Работа открывает пути для обобщений на банаховы и фон Neumann алгебры, подчеркивая при этом трудности, связанные с некоммутативными постановками.

Ключевые слова: C^* -алгебра, теорема о среднем значении Смейла, гипотеза Смейла о среднем значении, дуальная теорема о среднем значении Дубинина-Сугавы, дуальная гипотеза о среднем значении Дубинина-Сугавы.

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1 Introduction

In 1981 Fields Medalist Prof. Steve Smale proved the following result using Koebe 1/4-theorem [1].

Theorem 1.1. (*Smale Mean Value Theorem*) Let $P(z) := a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P (i.e., $P'(z) \neq 0$), then

$$\min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right| \leq 4|P'(z)|. \quad (1)$$

Note that Theorem 1.1 can also be stated as follows.

Theorem 1.2. (*Smale Mean Value Theorem*) Let $P(z) := a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{C}$ of P such that

$$\left| \frac{P(z) - P(w)}{z - w} \right| \leq 4|P'(z)|. \quad (2)$$

Proof of Theorem 1.2 can also be found in book [2] and in the monograph [3]. Inequality (1) led Smale to ask the following problem: what is the best constant $4 > c > 0$ such that inequality (1) can be replaced by

$$\min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right| \leq c|P'(z)|? \quad (3)$$

Smale made the following conjecture with regard to inequality (3), which is open today.

Conjecture 1.1. (*Smale Mean Value Conjecture*) Let $P(z) := a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then

$$\min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right| \leq |P'(z)|$$

or

$$\min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right| \leq \frac{n-1}{n}|P'(z)| = \frac{\deg(P)-1}{\deg(P)}|P'(z)|.$$

Conjecture 1.1 also appears as Problem 1 in the Addenda after the list of 18 open problems (of which 3 are completely solved now, namely Problem 2: Poincaré Conjecture by Perelman in 2003 [4], Problem 14: Lorenz Attractor by Tucker in 2002 [5], Problem 17: Solving Polynomial Equations by Beltran and Pardo in 2009 [6, 7]) of Smale in *Mathematical Problems for the Next Century* [8]; see also [9–12]. Now for a polynomial P of degree $n \geq 2$, define

$$S(P) := \sup_{z \in \mathbb{C}, P'(z) \neq 0} \frac{1}{|P'(z)|} \min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right|$$

and

$$M := \sup_{P \text{ is a non linear polynomial}} S(P).$$

Smale conjecture then asks whether $M = 1$ or $S(P) = \frac{\deg(P)-1}{\deg(P)}$? It is known that Conjecture 1 holds for polynomials of degree 2, 3, 4 [2, 3, 13]. Through computer calculations, Smale conjecture is known to hold for polynomials of degree up to 10 [14].

Usually the study of Smale mean value conjecture goes with the following conjecture, which is known as normalization equivalence to Conjecture 1.

Conjecture 1.2. (Normalized Smale Mean Value Conjecture) *Let P be a degree $n \geq 2$ polynomial over $\mathbb{C}$ which is normalized, i.e., $P(0) = 0$ and $P'(0) = 1$. Then*

$$S_0(P) := \min_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(w)}{w} \right| = \frac{\deg(P) - 1}{\deg(P)}$$

or

$$M_0 := \sup_{P \text{ is a nonlinear polynomial}, P(0)=0, P'(0)=1} S_0(P) = 1.$$

In 1996 Andrievskii and Ruscheweyh showed that if critical points of a normalized polynomial P lie on the unit circle, then $S_0(P) \leq 1$ [15]. In 2009 Hinkkanen and Kayumov showed that if all critical points of a normalized polynomial P are real, then $S_0(P) \leq \frac{2}{3}$ [16]. In 2010 Hinkkanen and Kayumov proved that if the critical points of normalized polynomial P lie on the sector $\{re^{i\theta} : r > 0, |\theta| \leq \pi/6\}$, then $S_0(P) \leq \frac{1}{2}$ and if the critical points of normalized polynomial P lie in the union of two rays $\{1 + re^{\pm i\theta} : r \geq 0\}$ for each $0 < \theta \leq \pi/2$, then $S_0(P) \leq \frac{2}{3}$ [17]. In [2] it is shown that if the critical points of a normalized polynomial are all real, then $S_0(P) \leq \frac{\deg(P)-2}{\deg(P)} \left[\left(\frac{\deg(P)-1}{\deg(P)-2} \right)^{\deg(P)-1} - 2 \right]$ and in [3] it is shown that $S_0(P) \leq e - 2$.

In 2002 Beardon, Minda and Ng showed that $S(P) \leq 4^{\frac{\deg(P)-2}{\deg(P)-1}}$ [18]. In 2006 Fujikawa and Sugawa proved that $S(P) \leq 4^{\frac{1+(\deg(P)-2)4^{\frac{-1}{\deg(P)-1}}}{\deg(P)+1}}$ [19]. In 2007 Conte, Fujikawa and Lakic showed that $S(P) \leq 4^{\frac{\deg(P)-1}{\deg(P)+1}}$ [20]. In 2007 Crane showed that there exists $n_0 \geq 2$ such that for all $n \geq n_0$, $S(P) \leq 4 - \frac{2}{\sqrt{\deg(P)}}$ [21] (also see [22]). In 2003 Ng proved that Conjecture 1 holds with $M = 2$ for all odd polynomials with nonzero linear terms [23]. In 2006 Dubinin proved that if the critical points of a normalized polynomial P have equal modulus, then $S_0(P) \leq \frac{\deg(P)-1}{\deg(P)}$ [24].

At the other side, in 2002 Sendov formulated a generalization of Conjecture 1.2 [25]. In 2003 Sendov and Nikolov gave some variations of Conjecture 1.2 [26]. In 2005 Tyson showed that Tischler's strong form of Smale mean value conjecture [13, 27] fails [28]. Further, it is important to record that one can interpret Conjecture 1.2 in the form of finite Blaschke products [3, 29, 30]. It is also possible to connect Conjecture 1.2 with electrostatics [31].

In [1] Smale also derived the following higher order mean value theorem.

Theorem 1.3. (Higher Order Smale Mean Value Theorem) *Let $P(z) := a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{C}$ of P such that*

$$\frac{|P^{(k)}(z)|}{k!} \frac{|P(z) - P(w)|^{k-1}}{|P'(z)|^k} \leq 4^{k-1}, \quad \forall 2 \leq k \leq n. \quad (4)$$

A conjecture regarding the convergence of iterates of critical point satisfying inequality (2) has been made by Miles–Leighton and Pilgrim in 2012 [32].

Conjecture 1.3. (*Miles–Leighton–Pilgrim Dynamics Conjecture*) Let $P(z) := z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. Then there exists a critical point $w \in \mathbb{C}$ of P such that

$$\left| \frac{P(w)}{w} \right| \leq 1$$

and

$$P^m(w) \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.$$

Miles–Leighton and Pilgrim succeeded in showing that Conjecture 1.3 holds for polynomials of degree 2 and 3.

On the other side, in 2009 Dubinin and Sugawa derived the following striking result which is a dual result for Smale mean value theorem [33] (see also the survey [34]).

Theorem 1.4. (*Dubinin–Sugawa Dual Mean Value Theorem*) Let $P(z) := a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then there exists a critical point $w \in \mathbb{C}$ of P such that

$$\frac{|P'(z)|}{n4^n} \leq \left| \frac{P(z) - P(w)}{z - w} \right|.$$

In other words,

$$\frac{|P'(z)|}{n4^n} \leq \max_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right|.$$

Similar to Smale conjecture, Dubinin and Sugawa made the following conjecture in 2009.

Conjecture 1.4. (*Dubinin–Sugawa Dual Mean Value Conjecture*) Let $P(z) := a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 2$ over $\mathbb{C}$. If $z \in \mathbb{C}$ is not a critical point of P , then

$$\frac{|P'(z)|}{n} \leq \max_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right|.$$

This conjecture is discussed in detail in [34]. Similar to earlier quantities, for a polynomial P of degree $n \geq 2$, define

$$DS(P) := \inf_{z \in \mathbb{C}, P'(z) \neq 0} \frac{1}{|P'(z)|} \max_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(z) - P(w)}{z - w} \right|.$$

Similar to Conjecture 1.1, we have the following normalized version of Conjecture 1.4.

Conjecture 1.5. (*Normalized Dubinin–Sugawa Dual Mean Value Conjecture*) Let P be a degree $n \geq 2$ polynomial over $\mathbb{C}$ which is normalized, i.e., $P(0) = 0$ and $P'(0) = 1$. Then

$$DS_0(P) := \max_{w \in \mathbb{C}, P'(w)=0} \left| \frac{P(w)}{w} \right| = \frac{1}{n} = \frac{1}{\deg(P)}.$$

In 2016 Ng and Zhang showed that if P is a normalized polynomial, then $DS_0(P) > \frac{1}{4^{\deg(P)}} [30]$. In 2019 Dubinin showed that $DS(P) \geq \frac{1}{n} \tan\left(\frac{\pi}{4^{\deg(P)}}\right) [35]$. In 2019 Hinkkanen, Kayumov and Khammatova showed that Conjecture 1.5 holds for polynomials of degree six [36].

In this paper we formulate C*-algebraic versions of Conjectures 1.1 and 1.4. We show that they are valid for degree 2 C*-algebraic polynomials. We also formulate C*-algebraic version of Conjecture 1.3 and Conjecture 1.4.

2 C*-algebraic Smale Mean Value Conjecture

Let $\mathcal{A}$ be a C*-algebra. For $P(z) := (z - a_1)(z - a_2) \cdots (z - a_n)$ for all $z \in \mathcal{A}$ with $a_1, a_2, \dots, a_n \in \mathcal{A}$, we define

$$P'(z) := \sum_{j=1}^n (z - a_1) \cdots \widehat{(z - a_j)} \cdots (z - a_n), \quad \forall z \in \mathcal{A}$$

where the term with cap is missing. We are not considering Fréchet or Gâteaux differentiability. Instead, it is a derivative strictly in the algebraic sense—that is, a derivative considered for polynomials over any ring.

We state the C*-algebraic version of Conjecture 1.1 as follows.

Conjecture 2.1. (*C*-algebraic Smale Mean Value Conjecture*) *Let $\mathcal{A}$ be a commutative C*-algebra. Let $P(z) := (z - a_1) \cdots (z - a_n)$ be a polynomial of degree $n \geq 2$ over $\mathcal{A}$, $a_1, \dots, a_n \in \mathcal{A}$. If $z \in \mathcal{A}$ is not a critical point of P (i.e., $P'(z) \neq 0$), then there exists a critical point $w \in \mathcal{A}$ of P such that*

$$\frac{\|P(z) - P(w)\|}{\|z - w\|} \leq 1 \|P'(z)\|$$

or

$$\frac{\|P(z) - P(w)\|}{\|z - w\|} \leq \frac{n-1}{n} \|P'(z)\| = \frac{\deg(P)-1}{\deg(P)} \|P'(z)\|.$$

We can formulate a stronger conjecture than Conjecture 2.1 as follows.

Conjecture 2.2. (*C*-algebraic Smale Mean Value Conjecture — strong form*) *Let $\mathcal{A}$ be a commutative C*-algebra. Let $P(z) := (z - a_1) \cdots (z - a_n)$ be a polynomial of degree $n \geq 2$ over $\mathcal{A}$, $a_1, \dots, a_n \in \mathcal{A}$. If $z \in \mathcal{A}$ is not a critical point of P , then there exists a critical point $w \in \mathcal{A}$ of P such that*

$$(P(z) - P(w))(P(z) - P(w))^* \leq (z - w)P'(z)P'(z)^*(z - w)^* \quad (5)$$

or

$$(P(z) - P(w))(P(z) - P(w))^* \leq \left(\frac{n-1}{n}\right)^2 (z - w)P'(z)P'(z)^*(z - w)^*. \quad (6)$$

Proposition 2.1. Conjecture 2.2 implies Conjecture 2.1.

Proof. Taking norm in inequality (5) we get

$$\|P(z) - P(w)\|^2 = \|(P(z) - P(w))(P(z) - P(w))^*\| \leq \|(z - w)P'(z)P'(z)^*(z - w)^*\| =$$

$$= \|(z - w)P'(z)\|^2 \leq \|z - w\|^2 \|P'(z)\|^2.$$

Similarly, by taking norm in inequality (6) we get the second inequality in Conjecture 2.1. $\square$

Theorem 2.1. *Conjecture 2.1 holds for degree 2 C^* -algebraic polynomials over commutative C^* -algebras.*

Proof. We prove that Conjecture 2.2 holds and hence Conjecture 2.1 holds. Let $\mathcal{A}$ be a commutative C^* -algebra and let $P(z) := (z - a)(z - b)$, $a, b \in \mathcal{A}$. Then $P'(z) = 2z - (a + b)$ for all $z \in \mathcal{A}$. Let $z \in \mathcal{A}$ be such that $P'(z) \neq 0$. Note that P' has only one zero, namely $c := \frac{a+b}{2}$. To show inequality (6), we need to show that

$$(P(z) - P(c))(P(z) - P(c))^* \leq \frac{1}{4}(z - c)P'(z)P'(z)^*(z - c)^*.$$

Define $x := z - a$ and $y := z - b$. Then

$$z - c = z - \frac{a+b}{2} = \frac{z - a + z - b}{2} = \frac{x + y}{2}, \quad P'(z) = z - a + z - b = x + y,$$

$$P(c) = \left(\frac{a+b}{2} - a\right) \left(\frac{a+b}{2} - b\right) = \frac{-(b-a)^2}{4} = \frac{-(x-y)^2}{4},$$

and

$$(P(z) - P(c))(P(z) - P(c))^* = \left(xy + \frac{(x-y)^2}{4}\right) \left(xy + \frac{(x-y)^2}{4}\right)^* = \frac{(x+y)^2(x^* + y^*)^2}{16}.$$

Therefore

$$\begin{aligned} & \frac{1}{4}(z - c)P'(z)P'(z)^*(z - c)^* - (P(z) - P(c))(P(z) - P(c))^* = \\ & = \frac{1}{16}((x+y)^2(x^* + y^*)^2 - (x+y)^2(x^* + y^*)^2) = 0. \end{aligned}$$

$\square$

Based on Theorem 1.3 we formulate the following higher order conjecture.

Conjecture 2.3. *(Higher Order C^* -algebraic Smale Mean Value Conjecture) Let $\mathcal{A}$ be a commutative C^* -algebra. Let $P(z) := (z - a_1) \cdots (z - a_n)$ be a polynomial of degree $n \geq 2$ over $\mathcal{A}$, $a_1, \dots, a_n \in \mathcal{A}$. If $z \in \mathcal{A}$ is not a critical point of P , then there exists a critical point $w \in \mathcal{A}$ of P such that*

$$\frac{\|P^{(k)}(z)\|}{k!} \frac{\|P(z) - P(w)\|^{k-1}}{\|P'(z)\|^k} \leq 4^{k-1}, \quad \forall 2 \leq k \leq n.$$

Theorem 2.2. *Conjecture 2.3 holds for degree 2 C^* -algebraic polynomials.*

Proof. We continue from the proof of Theorem 2.1. Let $z \in \mathcal{A}$ be not a zero of P' . We have $P''(z) = 2$. Hence

$$\|P''(z)\| = 2, \quad \|P'(z)\| = \|x + y\|,$$

$$\|P(z) - P(c)\|^2 = \|(P(z) - P(c))(P(z) - P(c))^*\| = \frac{1}{16}\|(x+y)^2(x^* + y^*)^2\| =$$

$$= \frac{1}{16} \| (x+y)^2 \|^2 \leq \frac{1}{16} \| x+y \|^4.$$

Therefore

$$\frac{\|P^{(2)}(z)\|}{2!} \frac{\|P(z) - P(c)\|^{2-1}}{\|P'(z)\|^2} = \frac{2}{2} \frac{\left(\frac{\|(x+y)^2\|}{4}\right)}{\|x+y\|^2} \leq \frac{1}{4} \frac{\|x+y\|^2}{\|x+y\|^2} = \frac{1}{4} = 4^{2-1}.$$

□

We state Conjecture 1.3 for C*-algebraic polynomials as follows.

Conjecture 2.4. (*C*-algebraic Miles–Leighton–Pilgrim Dynamics Conjecture*) Let $\mathcal{A}$ be a commutative C*-algebra. Let $P(z) := (z - a_1) \cdots (z - a_n)$ be a polynomial of degree $n \geq 2$ over $\mathcal{A}$, $a_1, \dots, a_n \in \mathcal{A}$. If $P(0) = 0$ and $P'(0) = 1$, then there exists a critical point $w \in \mathcal{A}$ of P such that

$$\frac{\|P(w)\|}{\|w\|} \leq 1$$

and

$$P^m(w) \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.$$

3 C*-algebraic Dubinin–Sugawa Dual Mean Value Conjecture

We state the C*-algebraic version of Conjecture 1.4 as follows.

Conjecture 3.1. (*C*-algebraic Dubinin–Sugawa Dual Mean Value Conjecture*) Let $\mathcal{A}$ be a commutative C*-algebra. Let $P(z) := (z - a_1) \cdots (z - a_n)$ be a polynomial of degree $n \geq 2$ over $\mathcal{A}$, $a_1, \dots, a_n \in \mathcal{A}$. If $z \in \mathcal{A}$ is not a critical point of P , then there exists a critical point $w \in \mathcal{A}$ of P such that

$$\frac{\|P'(z)\|}{\deg(P)} = \frac{\|P'(z)\|}{n} \leq \frac{\|P(z) - P(w)\|}{\|z - w\|}.$$

Similar to the strong form of Conjecture 2.1, we have the following strong form of Conjecture 3.1.

Conjecture 3.2. (*C*-algebraic Dubinin–Sugawa Dual Mean Value Conjecture — strong form*) Let $\mathcal{A}$ be a commutative C*-algebra. Let $P(z) := (z - a_1) \cdots (z - a_n)$ be a polynomial of degree $n \geq 2$ over $\mathcal{A}$, $a_1, \dots, a_n \in \mathcal{A}$. If $z \in \mathcal{A}$ is not a critical point of P , then there exists a critical point $w \in \mathcal{A}$ of P such that

$$\frac{(z - w)P'(z)P'(z)^*(z - w)^*}{(\deg(P))^2} = \frac{(z - w)P'(z)P'(z)^*(z - w)^*}{n^2} \leq (P(z) - P(w))(P(z) - P(w))^*.$$

Theorem 3.1. Conjecture 3.1 holds for degree 2 C*-algebraic polynomials.

Proof. We prove Conjecture 3.2 holds. We continue from the proof of Theorem 2.1. Consider

$$(P(z) - P(c))(P(z) - P(c))^* - \frac{(z - c)P'(z)P'(z)^*(z - c)^*}{2^2}$$

$$= \frac{(x+y)^2(x^*+y^*)^2}{16} - \frac{1}{4} \frac{(x+y)^2(x^*+y^*)^2}{4} = 0.$$

□

Remark. Conjectures 2.1, 2.3, 2.4 and 3.1 can be stated for Banach algebras. The author believes that Conjectures 2.2 and 3.2 hold at least for von Neumann algebras.

Even though we formulated conjectures for arbitrary C^* -algebras, we used commutativity in all the proofs of the theorems in this paper. Thus, the techniques will not carry over to noncommutative C^* -algebras.

Conclusion

For 45 years, Smale's mean value conjecture has remained an important problem in the theory of polynomials. The Dubinin–Sugawa dual mean value conjecture has been an important problem for 17 years. In this article, we formulated a far-reaching general formulation of both conjectures and verified them for degree 2 polynomials over commutative C^* -algebras.

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