



Thermodynamic Properties of a Quantum Isotonic Oscillator with Position-Dependent Effective Mass

E. S. Bokpe¹, F. A. Dossa^{2,3,4}, G. Y. Avossevou¹*

¹ Institut de Mathématiques et de Sciences Physiques (IMSP), Université d'Abomey-Calavi (UAC), Porto-Novo, Benin

² École Doctorale des Sciences, Technologies, Ingénierie et Mathématiques (ED-STIM), UNSTIM, Abomey, Benin

³ Laboratoire de Physique et Applications (LPA), UNSTIM, Abomey, Benin

⁴ Département de Physique, Faculté des Sciences et Techniques (FAST), Natitingou, Benin

Abstract. We examine the thermodynamic properties of a quantum isotonic oscillator with position-dependent effective mass, described by a strictly negative deformation parameter α . By solving the corresponding Schrödinger equation using the Nikiforov-Uvarov method, we obtain the energy spectrum and wave functions of the system. From these results we construct the partition function, from which we derive the main thermodynamic quantities: internal energy, specific heat, free energy, and entropy. The analysis shows that the deformation of the effective mass significantly modifies the energy levels and the thermal response of the system. More precisely, the parameter α exerts a decisive influence on thermal transitions, while the isotonic parameter β becomes determinant when the mass is no longer constant. We further show, through explicit analytical and numerical analysis, that the condition $\alpha < 0$ is not a simple arbitrary choice but the necessary and sufficient condition for the thermodynamic stability of the model. These observations underscore the need to account for mass deformation in quantum models, in particular to faithfully describe the thermal behavior of nanostructured materials and semiconductor devices.

Key words: Isotonic oscillator; Position-dependent mass; Nikiforov-Uvarov method; Thermodynamics properties.

Received: 28.05.2026; Revised: 09.07.2026; Accepted: 10.07.2026; First online: 14.09.2026

For citation. Bokpe E. S., Dossa F. A., Avossevou G. Y. Thermodynamic properties of a quantum isotonic oscillator with position-dependent effective mass. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2026, 54: 2, 119-141. EDN: FQFUER. <https://doi.org/10.26117/2079-6641-2026-54-2-119-141>.

Funding. The work was carried out without external funding.

Competing interests. There are no conflicts of interest regarding authorship and publication.

Contribution and Responsibility. All authors contributed to the writing of the article and are responsible for the final version.

*Correspondence:  E-mail: dossafanselme@gmail.com

The content is published under the terms of the Creative Commons Attribution 4.0 International License

© Bokpe E. S., Dossa F. A., Avossevou G. Y., 2026

© Institute of Cosmophysical Research and Radio Wave Propagation, 2026 (original layout, design, compilation)



ФИЗИКА

 <https://doi.org/10.26117/2079-6641-2026-54-2-119-141>

Научная статья

Полный текст на английском языке

УДК 530.145



Термодинамические свойства квантового изотопического осциллятора с позиционно-зависимой эффективной массой

Э. С. Бокпе¹, Ф. А. Досса^{2,3,4*}, Г. Й. Авоссеву¹

¹ Институт математики и физических наук (IMSP), Университет Абомей-Калави (UAC), Порто-Ново, Бенин

² Докторская школа наук, технологий, инженерии и математики (ED-STIM), UNSTIM, Абомей, Бенин

³ Лаборатория физики и приложений (LPA), UNSTIM, Абомей, Бенин

⁴ Департамент физики, факультет наук и техники (FAST), Натитингу, Бенин

Аннотация. Мы исследуем термодинамические свойства квантового изотопического осциллятора с позиционно-зависимой эффективной массой, описываемой строго отрицательным параметром деформации α . Решая соответствующее уравнение Шрёдингера методом Никифорова–Уварова, мы получаем энергетический спектр и волновые функции системы. Из этих результатов мы строим статистическую сумму, из которой выводим основные термодинамические величины: внутреннюю энергию, теплоёмкость, свободную энергию и энтропию. Анализ показывает, что деформация эффективной массы существенно изменяет энергетические уровни и тепловой отклик системы. В частности, параметр α оказывает решающее влияние на тепловые переходы, тогда как изотопический параметр β становится определяющим, когда масса не является постоянной. Мы также показываем с помощью явного аналитического и численного анализа, что условие $\alpha < 0$ не является простым произвольным выбором, а необходимым и достаточным условием термодинамической устойчивости модели. Эти наблюдения подчёркивают необходимость учёта деформации массы в квантовых моделях, в частности для достоверного описания теплового поведения наноструктурированных материалов и полупроводниковых приборов.

Ключевые слова: изотопический осциллятор; позиционно-зависимая масса; метод Никифорова–Уварова; термодинамические свойства.

Получение: 28.05.2026; Исправление: 09.07.2026; Принятие: 10.07.2026; Публикация онлайн: 14.09.2026

Для цитирования. Бокпе Е. С., Досса Ф. А., Авоссеву Г. Й. Thermodynamic properties of a quantum isotonic oscillator with position-dependent effective mass // Вестник КРАУНЦ. Физ.-мат. науки. 2026. Т. 54. № 2. С. 119-141. EDN: FQFUER. <https://doi.org/10.26117/2079-6641-2026-54-2-119-141>.

Финансирование. Работа выполнена без внешнего финансирования.

Конкурирующие интересы. Конфликтов интересов в отношении авторства и публикации нет.

Авторский вклад и ответственность. Все авторы участвовали в написании статьи и несут ответственность за окончательную версию.

*Корреспонденция:  E-mail: dossafanselme@gmail.com

Контент публикуется на условиях Creative Commons Attribution 4.0 International License

© Бокпе Е. С., Досса Ф. А., Авоссеву Г. Й., 2026

© ИКИР ДВО РАН, 2026 (оригинал-макет, дизайн, составление)



1 Introduction

Quantum systems with position-dependent effective mass (PDEM) constitute a very rich class of models, widely used in modern condensed matter physics, in particular for modeling semiconductor heterostructures [1], quantum liquids [2], and nanostructured materials. In these systems, the inertial mass of the particles varies with position, which leads to non-trivial modifications of the quantum dynamics and of the thermodynamic behavior. Such models are particularly relevant for describing charge carriers confined in quantum wells, boxes, and wires, where the effective mass depends on the local composition or the strain of the material [3]. These position-dependence effects strongly influence the density of states, the transport coefficients, and the thermal stability of nanodevices [4, 5].

At the same time, the study of quantum systems deformed by q -deformations and other algebraic generalizations has led to profound advances regarding non-classical behaviors and hidden symmetries. For instance, the harmonic oscillator deformed by a minimal length exhibits a hidden $\text{su}(1, 1)$ symmetry, with the deformation parameter exerting a notable influence on its thermodynamic properties [6]. When an effective mass is introduced into such deformed systems, additional algebraic structures appear, enabling the realization of Lie algebras such as Heisenberg-Weyl, $\text{su}(1, 1)$, and $\text{su}(2)$ [7–9], and leading to exact solvability in many cases via shape invariance.

Among exactly solvable models, the isotonic oscillator stands out for the richness of its structure. Combining a harmonic potential with a centripetal barrier, it retains an equidistant spectrum while displaying non-trivial classical and quantum features [10–13]. This potential finds applications in the theory of coherent states and in quantum optics [14, 15], and its properties are further enriched when it is embedded in curved spaces – such as the two-dimensional sphere S^2 or the hyperbolic plane H^2 – where the mass deformation parameter λ can be interpreted as an effective curvature [16].

Nonlinear models, such as the Smorodinsky-Winternitz system based on coupled isotonic oscillators, also show that mass deformation can give rise to remarkable properties such as superintegrability and multiseparability [17, 18]. These results reinforce the interest in examining in more detail the thermal behavior of deformed quantum oscillators.

Furthermore, singular interactions – arising from confinement or impurity effects – play a crucial role in low-dimensional quantum systems, affecting their stability and their heat capacity [19]. Similar effects have been reported in studies of the second-kind Pöschl-Teller potential with confinement frequency, where the singular term modifies the energy spectrum and the thermodynamic properties [20]. By combining a singular interaction with a spatially varying mass, the model serves as a minimal theoretical prototype for exploring confinement and thermal effects in nanoscale systems, rather than a description specific to any particular material.

Our study focuses on the thermodynamic behavior of a deformed isotonic oscillator, characterized by a position-dependent effective mass. This approach is motivated by the need to better understand thermal effects in mesoscopic and nanoscopic systems where mass deformation influences both the energy spectrum and the thermal response. We

show that the fundamental thermodynamic quantities – internal energy, entropy, specific heat, and Helmholtz free energy – depend strongly on the form of the mass function, which highlights the importance of exact analytical solutions for a faithful modeling of thermal behavior.

Let us state from the outset that the choice $\alpha < 0$ is not arbitrary: we show in §2 that it is the necessary and sufficient condition for the energy spectrum to grow without bound (rather than the opposite) as $n \rightarrow \infty$, which in turn guarantees the convergence of the partition function and the thermodynamic stability of the system.

To solve the underlying quantum problem, we employ the Nikiforov-Uvarov (NU) method [21, 22], a powerful technique for solving second-order differential equations reducible to a hypergeometric form. This method has proven effective for a broad class of PDEM systems and non-polynomial potentials [23–27], and has recently been applied to the study of thermal and uncertainty relations in deformed oscillators with inverse quadratic terms [27].

Here we apply the NU method to obtain the exact energy spectrum and wave functions of the deformed isotonic oscillator. These results are used to construct the canonical partition function, from which we derive the key thermodynamic observables. We show that the deformation parameter α , assumed to be strictly negative, has a significant impact on thermal transitions, while the isotonic parameter β plays a crucial role in systems with non-constant mass by modifying both the energy levels and the thermodynamic response. Here, α characterizes the strength of the mass deformation, while β governs the singularity of the isotonic term in the potential.

In particular, our analysis shows that the specific heat exhibits a strong sensitivity to mass deformation, and that its high-temperature behavior depends qualitatively on α : it tends to the classical Dulong-Petit limit $C \rightarrow k_B$ only in the undeformed case ($\alpha = 0$), whereas for $\alpha \neq 0$ it saturates to the reduced value $C \rightarrow k_B/2$, a direct signature of the transition from an equidistant spectrum to a spectrum growing quadratically in n (verified numerically, see §4).

This study provides a complete analytical treatment of a deformed isotonic oscillator with PDEM, illustrating how algebraic methods such as the NU formalism can yield exact solutions with significant thermodynamic consequences. The graphical analysis highlights the role of the deformation parameters in shaping the thermal properties, and the results open perspectives for future exploration in quantum thermodynamics and materials modeling.

2 Deformed isotonic oscillator and its spectrum

The study of deformed oscillators with position-dependent effective mass has attracted considerable attention because of its relevance for describing physical phenomena in complex materials, such as semiconductors and structured media with periodic or quasi-periodic potentials [28, 29]. In these systems, the effective mass plays a critical role in modifying the transport properties of charge carriers, as well as the electronic and optical behavior of the material.

We define the Hamiltonian of a system of two identical particles of mass m_p , subject to a singular oscillator potential where ν quantifies the strength of the interparticle interaction [30]:

$$H_p = \frac{p_1^2 + p_2^2}{2m_p} + \frac{1}{2}m_p\omega^2(x_1^2 + x_2^2) + \frac{\hbar\nu(\nu-1)}{2m_p(x_1-x_2)^2}. \quad (1)$$

This Hamiltonian can be rewritten in the center-of-mass and relative-coordinate frame. To this end, we define the following position and momentum transformations [30]:

$$X = \frac{x_1 + x_2}{2}, \quad x = x_1 - x_2, \quad P = p_1 + p_2, \quad p = \frac{p_1 - p_2}{2}. \quad (2)$$

After separation of variables, the Hamiltonian splits into two parts:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{\hbar\nu(\nu-1)}{4mx^2}, \quad H' = \frac{P^2}{2M} + \frac{1}{2}M\omega^2X^2. \quad (3)$$

The Hamiltonian H' , describing the center-of-mass motion, reduces to that of a standard harmonic oscillator for a single particle of mass $M = 2m_p$, whose energy spectrum is well known in the literature, while the Hamiltonian of the relative motion H takes the form of a singular oscillator for a particle of mass $m = m_p/2$, commonly referred to as the isotonic oscillator.

This Hamiltonian H combines a quadratic confining potential with a central singular interaction, thereby enriching the dynamics of the system and enabling the study of non-trivial phenomena such as quantum localization, modification of the energy spectrum, and specific thermodynamic effects [27]. By introducing a position-dependent mass into this model, we obtain a more general framework for analyzing both the energy spectrum and the thermodynamic properties of a deformed quantum system.

We consider a position-dependent mass of the form [8]:

$$m(x) = \frac{\lambda}{1 + \delta^2x^2}, \quad (4)$$

where λ is a real parameter (with the dimension of a mass) and δ quantifies the strength of the oscillator's nonlinearity (with the dimension of an inverse length). The Hamiltonian H becomes:

$$H = \frac{(1 + \delta^2x^2)}{2\lambda}p_x^2 + \frac{1}{2}\frac{\lambda\omega^2x^2}{(1 + \delta^2x^2)} + \frac{\beta(1 + \delta^2x^2)}{4\lambda x^2}, \quad (5)$$

with

$$\beta = \nu(\nu-1). \quad (6)$$

The parameter β determines whether the interaction is attractive or repulsive.

We now apply the NU method to solve the system. The generalized NU equation is written as [21, 22]:

$$\chi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\chi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\chi(z) = 0. \quad (7)$$

Its parametric version is expressed as follows [24]:

$$\frac{d^2\chi(z)}{dz^2} + \frac{a_1 - a_2z}{z(1 - a_3z)} \frac{d\chi(z)}{dz} + \frac{(-\epsilon_1z^2 + \epsilon_2z - \epsilon_3)}{z^2(1 - a_3z)^2} \chi(z) = 0, \quad (8)$$

with $\tilde{\tau}(z) = a_1 - a_2z$, $\sigma(z) = z(1 - a_3z)$, $\tilde{\sigma}(z) = -\epsilon_1z^2 + \epsilon_2z - \epsilon_3$. $\tilde{\tau}$ is a polynomial of degree at most 1; $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials of degree at most 2; $a_1, a_2, a_3, \epsilon_1, \epsilon_2, \epsilon_3$ are parameters.

The resulting wave function is written as [24, 31]:

$$\chi(z) = \Phi(z)y(z), \quad (9)$$

where the function $\Phi(z)$ is given by

$$\Phi(z) = z^{a_{12}}(1 - a_3z)^{-\frac{a_{13}}{a_3} - a_{12}}, \quad (10)$$

while the function $y(z)$ is

$$y(z) = P_n^{(a_{10}-1, \frac{a_{11}}{a_3} - a_{10}-1)}(1 - 2a_3z), \quad (11)$$

with $P_n^{(a,b)}$ denoting the Jacobi polynomial.

The general solution $\chi(z)$ is then

$$\chi(z) = z^{a_{12}}(1 - a_3z)^{-\frac{a_{13}}{a_3} - a_{12}} P_n^{(a_{10}-1, \frac{a_{11}}{a_3} - a_{10}-1)}(1 - 2a_3z). \quad (12)$$

The Jacobi polynomial $P_n^{(a,b)}(1 - 2z)$ is defined as [32]:

$$P_n^{(a,b)}(1 - 2z) = \frac{\Gamma(n + a + 1)}{n!\Gamma(a + 1)} {}_2F_1(-n, 1 + n + a + b, 1 + a, z). \quad (13)$$

The energy spectrum is obtained from the relation [24, 31]:

$$\begin{aligned} a_2n - (2n + 1)a_5 + n(n - 1)a_3 + (2n + 1)(a_3\sqrt{a_8} + \sqrt{a_9}) \\ + a_7 + 2a_3a_8 + 2\sqrt{a_8a_9} = 0, \end{aligned} \quad (14)$$

with $n = 0, 1, 2, \dots$ the quantum number.

The various NU coefficients are computed as in [24]:

$$a_4 = \frac{1}{2}(1 - a_1), \quad a_5 = \frac{1}{2}(a_2 - 2a_3), \quad a_6 = a_5^2 + \epsilon_1, \quad a_7 = 2a_4a_5 - \epsilon_2, \quad a_8 = a_4^2 + \epsilon_3, \quad (15)$$

$$a_9 = a_3a_7 + a_3^2a_8 + a_6, \quad a_{10} = a_1 + 2a_4 + 2\sqrt{a_8}, \quad a_{11} = a_2 - 2a_5 + 2(\sqrt{a_9} + a_3\sqrt{a_8}), \quad (16)$$

$$a_{12} = a_4 + \sqrt{a_8}, \quad a_{13} = a_5 - (\sqrt{a_9} + a_3\sqrt{a_8}). \quad (17)$$

Starting from the eigenvalue equation of the Hamiltonian (5), we can set $\alpha = \delta^2/\lambda$ and make the change of variable $z = -\delta x^2$. We thus find:

$$\frac{d^2\chi(z)}{dz^2} + \frac{(1 - z)}{2z(1 - z)} \frac{d\chi(z)}{dz} + \frac{(\mu z^2 + \gamma z - \frac{\beta}{4})}{z^2(1 - z)^2} \chi(z) = 0, \quad (18)$$

with the parameters:

$$\mu = \frac{E}{2\alpha} - \frac{\omega^2}{4\alpha^2} - \frac{\beta}{8}, \quad \gamma = \frac{\beta}{4} - \frac{E}{2\alpha}, \quad (19)$$

$$a_1 = a_2 = \frac{1}{2}, \quad a_3 = 1, \quad \epsilon_1 = -\mu, \quad \epsilon_2 = \gamma, \quad \epsilon_3 = \frac{\beta}{8}, \quad \hbar = 1. \quad (20)$$

This leads to the following expressions for the coefficients:

$$a_4 = \frac{1}{4}, \quad a_5 = -\frac{3}{4}, \quad a_6 = \frac{9}{16} - \mu, \quad a_7 = -\frac{3}{8} - \gamma, \quad a_8 = \frac{1}{16} + \frac{\beta}{4}, \quad (21)$$

$$a_9 = \frac{1}{4\alpha^2}(\omega^2 + \alpha^2), \quad a_{10} = \frac{\epsilon}{2} + 1, \quad a_{11} = 2 + \frac{\epsilon}{2} + \frac{1}{|\alpha|}\sqrt{\alpha^2 + \omega^2}, \quad (22)$$

$$a_{12} = \frac{1}{4}(\epsilon + 1), \quad a_{13} = -\left(\frac{\epsilon}{4} + \frac{3}{4} + \frac{1}{2|\alpha|}\sqrt{\alpha^2 + \omega^2}\right), \quad \epsilon = \sqrt{2\beta + 1}. \quad (23)$$

The normalized wave function takes the form:

$$\chi(x) = \mathcal{N}(-\delta^2 x^2)^{\frac{1}{4}(\epsilon+1)} (1 + \delta^2 x^2)^{\frac{1}{2}(1 + \frac{1}{|\alpha|}\sqrt{\alpha^2 + \omega^2})} y(x), \quad (24)$$

where

$$y(x) = \frac{\Gamma(n + \frac{\epsilon}{2} + 1)}{n! \Gamma(\frac{\epsilon}{2} + 1)} {}_2F_1\left(-n, 1 + n + \frac{\epsilon}{2} + \frac{1}{|\alpha|}\sqrt{\alpha^2 + \omega^2}, 1 + \frac{\epsilon}{2}, -\delta^2 x^2\right), \quad (25)$$

with $\mathcal{N}$ the normalization constant.

From Eq. (14), the energy spectrum of the deformed isotonic oscillator is written as:

$$E_n = \left(2n + 1 + \frac{\epsilon}{2}\right) \sqrt{\alpha^2 + \omega^2} - \alpha \left(2n^2 + 2n + n\epsilon + \frac{\epsilon}{2} + 1\right). \quad (26)$$

Role of the parameters δ and λ in the spectrum. One point deserves clarification here: δ disappears from the final spectrum (Eq. (26)) and from all the thermodynamic formulas, even though it appears explicitly in the mass (4). This is not an error but a structural consequence of the model, which we now make explicit. A simple dimensional argument (with $\hbar = 1$) shows that $[\delta] = L^{-1}$, $[\lambda] = M$, so that the combination

$$\alpha \equiv \frac{\delta^2}{\lambda} \quad (27)$$

has the dimension of an angular frequency, exactly like ω – which explains, moreover, why α and ω always appear together, in the symmetric form $\sqrt{\alpha^2 + \omega^2}$, in the spectrum (26). The change of variable $z = -\delta x^2$ then explicitly absorbs the parameter δ into the coordinate itself: physically, x enters the reduced Schrödinger equation only through the dimensionless product δx^2 . Once this variable is introduced, one checks from Eqs. (5)–(12) that λ and δ never again appear separately, but only through $\alpha = \delta^2/\lambda$. The model therefore possesses, from the outset, a scale invariance: two pairs (λ, δ) and (λ', δ') such that $\delta^2/\lambda = \delta'^2/\lambda'$ describe exactly the same spectrum and the same thermodynamics. This reduction $(\lambda, \delta) \rightarrow \alpha$ is of the same type as that already encountered in the harmonic oscillator with position-dependent mass [8], of which our model becomes a particular case when $\beta \rightarrow 0$ (see Eq. (30)). λ and δ are therefore not physically independent parameters: only their combination α is observable in the spectrum and the thermodynamics.

Table 1

Summary of the model parameters and their physical role		
Parameter	Physical meaning	Constraint / role
ω	Natural frequency of the underlying harmonic oscillator	$\omega > 0$
λ	Mass scale in $m(x) = \lambda/(1 + \delta^2 x^2)$	has the dimension of a mass
δ	Strength of the spatial variation of the mass	has dimension L^{-1} ; physically enters only via α
$\alpha = \delta^2/\lambda$	Effective spectral parameter of mass deformation	must be strictly negative for thermodynamic stability (§2)
ν	Strength of the interparticle interaction in the original isotonic potential, Eq. (1)	real
$\beta = \nu(\nu - 1)$	Strength of the singular isotonic term	$\beta \geq 0$ for a repulsive interaction regular at $x = 0$
$\epsilon = \sqrt{2\beta + 1}$	Combination appearing in the spectrum and wave functions	$\epsilon \geq 1$ for $\beta \geq 0$
n	Principal quantum number	$n = 0, 1, 2, \dots$
k	Number of levels retained in the truncated sum defining Z	$k = 500$ in all plots (convergence guaranteed, Table 2)

Domain of validity of the variable z and boundary behavior. The parametric form of the NU equation used here (with $\alpha_3 = 1$) is constructed for a variable $z(1 - z)$ defined on the interval $z \in (0, 1)$, on which $\Phi(z) = z^{\alpha_{12}}(1 - z)^{-\alpha_{13} - \alpha_{12}}$ and the Jacobi polynomials $P_n^{(a,b)}(1 - 2z)$ are naturally defined and orthogonal (weight $z^a(1 - z)^b$). Since $z = -\delta x^2$ and $x^2 \geq 0$, the interval $z \in (0, 1)$ corresponds to $\delta < 0$ (we thus set $\delta^2 = |\delta|^2$ and $-\delta = |\delta| > 0$) and restricts the physical coordinate to the bounded interval

$$x \in \left(-\frac{1}{\sqrt{|\delta|}}, \frac{1}{\sqrt{|\delta|}} \right). \quad (28)$$

This is consistent with the very nature of the effective mass (4): the factor $1/(1 + \delta^2 x^2)$ decreases continuously with $|x|$, so that the model describes a region of effective confinement rather than a free particle over the whole of $\mathbb{R}$ – a situation similar to that encountered in the curved spaces of spherical or hyperbolic type mentioned in the introduction [16]. The behavior of the wave function at the two boundaries of the domain is fixed by the exponents $\alpha_{12} = \frac{1}{4}(\epsilon + 1) > 0$ (at $x = 0$, i.e. $z = 0$) and $-\alpha_{13} - \alpha_{12}$ (at the boundary $z \rightarrow 1^-$): for $\beta \geq 0$ we have $\epsilon = \sqrt{2\beta + 1} \geq 1$, so $\chi(x) \sim |x|^{(\epsilon+1)/2} \rightarrow 0$ as $x \rightarrow 0$, which ensures the regularity of the wave function near the singular isotonic

term; at the other boundary, normalizability requires $\text{Re}(-a_{13} - a_{12}) > 0$, a condition that, as we show below, is precisely satisfied when $\alpha < 0$.

Independent symbolic verification. Given the central role played by Eq. (26) throughout the rest of the article, we independently recomputed this formula by symbolic resolution (computer algebra) of the implicit equation (14) for E , substituting back the explicit expressions of $a_4, \dots, a_9$ in terms of μ, γ given above. Under the assumption $\alpha < 0$ (justified below), the unique solution obtained coincides *exactly* and symbolically, term by term, with Eq. (26) – the difference between the two expressions simplifies identically to 0. This independent check, also tested numerically on several random sets of parameters $(\alpha, \omega, \beta, n)$ with agreement to machine precision, rules out any computational error in the application of the parametric NU method.

Why $\alpha < 0$: thermodynamic stability of the spectrum. This is an essential point for the physical consistency of the model, which we establish here in a rigorous and quantitative way. As $n \rightarrow \infty$, the dominant behavior of Eq. (26) is

$$E_n \underset{n \rightarrow \infty}{\sim} -2\alpha n^2. \quad (29)$$

- If $\alpha < 0$, then $-2\alpha > 0$ and $E_n \rightarrow +\infty$ quadratically: the spectrum is bounded from below (it has a finite ground state E_0) and grows without limit, exactly as for a confined harmonic oscillator. The partition function $Z = \sum_n e^{-\sigma E_n}$ is then a sum with Gaussian-type decay, absolutely convergent for all $\sigma > 0$, and the Poisson summation used below is legitimate.
- If $\alpha > 0$, on the other hand, $-2\alpha < 0$ and $E_n \rightarrow -\infty$: the spectrum is not bounded from below, there is no ground state, and $\sum_n e^{-\sigma E_n}$ diverges – the system is thermodynamically unstable, a situation that would be physically unacceptable for a model intended to describe the thermal equilibrium of a confined system.

We verify this numerically (see Fig. 1 for $\alpha < 0$, with $\omega = 1$, $\beta = 1$): for $\alpha = -0.5$, E_n equals 3.02, 48.53, 485.06, 2708.12, 10413.23 for $n = 0, 5, 20, 50, 100$ – monotonic growth toward $+\infty$; for $\alpha = +0.5$, E_n equals 1.15, -22.00, -391.45, -2480.34, -9961.84 for the same values of n – a fall toward $-\infty$, confirming the instability. The condition $\alpha < 0$ imposed throughout this work is therefore not an arbitrary choice but the necessary and sufficient condition for the thermodynamic stability of the model. Physically, α should therefore not be read as a “negative mass” – as we already noted in the conclusion of the original version – but as an effective spectral parameter guaranteeing quantum confinement and the thermal stability of the system, in the same way as a positive effective stiffness in a harmonic potential.

In the limit $\beta \rightarrow 0$, the isotonic term vanishes, and the system reduces to a deformed harmonic oscillator with position-dependent mass studied in [8]. The energy spectrum in this case, given by Eq. (30), reflects the impact of the deformation parameter α alone:

$$E_n = \left(2n + \frac{3}{2}\right) \sqrt{\alpha^2 + \omega^2} - \alpha \left(2n^2 + 3n + \frac{3}{2}\right). \quad (30)$$

On the other hand, when $\alpha \rightarrow 0$, the mass becomes constant and the model corresponds to an isotonic oscillator analyzed in [30], where the parameter β governs the strength of the singular interaction. The spectrum in this case, as shown by Eq. (31):

$$E_n = \left(2n + 1 + \frac{\epsilon}{2}\right) \omega. \quad (31)$$

Finally, when $\alpha \rightarrow 0$ and $\beta \rightarrow 0$, one recovers the well-known spectrum:

$$E_n = \left(2n + \frac{3}{2}\right) \omega, \quad (32)$$

which is the energy spectrum of a harmonic oscillator defined by the Hamiltonian H' in spherical coordinates with orbital quantum number $l = 0$ [10].

Cross-validation with reference [8]. The limit $\beta \rightarrow 0$, Eq. (30), reproduces *exactly*, term by term, the spectrum of the harmonic oscillator with position-dependent mass obtained by Tchoffo *et al.* [8]. This constitutes a non-trivial cross-validation of our NU calculation, independent of the presence of the isotonic term: the only difference between our result and that of [8] lies in the shift $n \rightarrow n + \frac{\epsilon}{2} - \frac{3}{4}$ induced by $\beta \neq 0$ in Eq. (26), which rigidly shifts each level without changing the asymptotic quadratic spacing $-2\alpha n^2$ set by α alone.

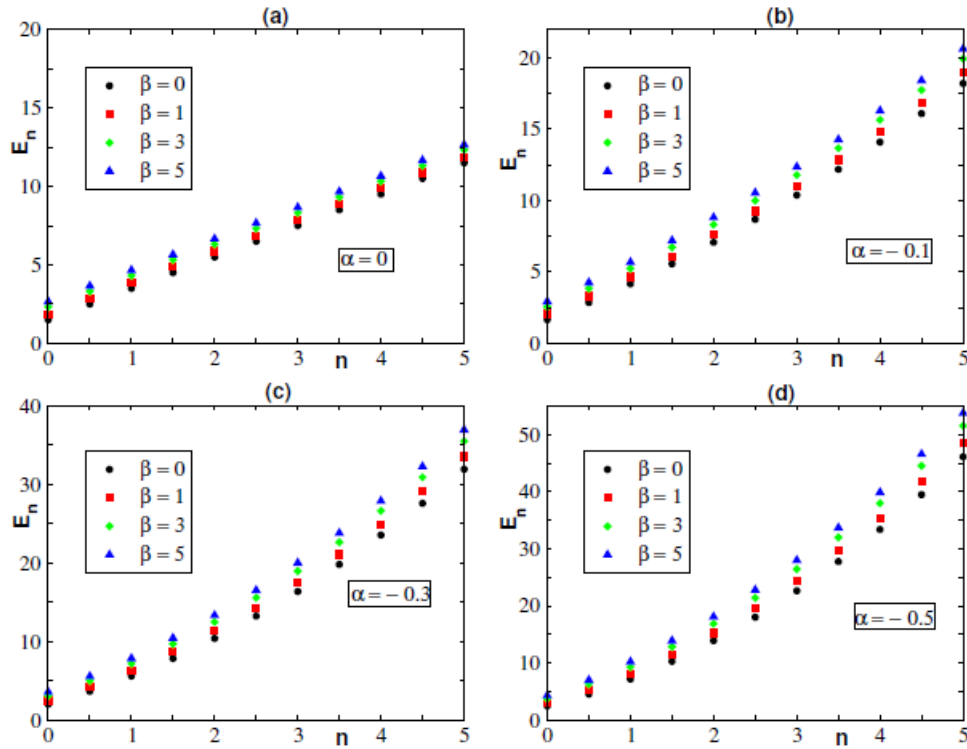


Fig. 1. Energy spectrum E_n as a function of n for different values of β : (a) $\alpha = 0$, (b) $\alpha = -0.1$, (c) $\alpha = -0.3$ and (d) $\alpha = -0.5$

Figure 1 illustrates the behavior of the energy spectrum E_n of Eq. (26) as a function of the quantum number n for a quantum isotonic oscillator with deformed, position-dependent effective mass, for various values of the deformation parameter α and the

isotonicity parameter β . It is clear that as α becomes more negative, the energy levels rise more rapidly with n , indicating that mass deformation strengthens the quantum confinement and significantly modifies the spectral structure – a behavior directly linked to the quadratic growth $+2|\alpha|n^2$ established above. This deformation introduces a nonlinear spacing between the energy levels, reflecting a stiffening of the effective potential. Moreover, at fixed α , increasing β also raises the energy levels, particularly at small quantum numbers, owing to the singular potential term centered at $x = 0$. As $\alpha \rightarrow 0$, the spectrum becomes more regular, resembling that of a harmonic oscillator with constant mass, confirming that the model generalizes both the classical and the deformed behaviors. This figure thus highlights the combined influence of α and β on the energy structure, underscoring the importance of accounting for the deformed effective mass in the quantum modeling of inhomogeneous systems, particularly in nanostructures and semiconductor devices.

3 Thermodynamic properties

To study the thermodynamic behavior of our system, we first compute the partition function defined by [6, 24]:

$$Z = \sum_{n=0}^k e^{-\sigma E_n}, \quad (33)$$

where k is the number of energy levels used to numerically compute the partition function, E_n the energy given by Eq. (26), and $\sigma = \frac{1}{k_B T}$, with k_B the Boltzmann constant.

Using the Poisson summation formula, as given in [24, 33–35]:

$$\sum_{n=0}^k f(n) = \frac{1}{2} [f(0) + f(k+1)] + \int_0^{k+1} f(x) dx, \quad (34)$$

where $f(n) = e^{-\sigma E_n}$, one obtains the following approximate expression for the partition function:

$$Z = \frac{1}{2} [e^{-\sigma \tilde{a}} + e^{-\sigma \tilde{b}}] - \rho, \quad (35)$$

with

$$\rho = \frac{1}{2\alpha} \sqrt{\frac{\pi}{2}} e^{-\frac{\sigma(4\omega^2 + \alpha^2 \epsilon^2)}{8\alpha}} \left[\frac{\tilde{c} \operatorname{erf}\left(\sqrt{\frac{-\sigma \tilde{c}^2}{8\alpha}}\right)}{\sqrt{\frac{-\sigma \tilde{c}^2}{\alpha}}} - \frac{\tilde{d} \operatorname{erf}\left(\sqrt{\frac{-\sigma \tilde{d}^2}{8\alpha}}\right)}{\sqrt{\frac{-\sigma \tilde{d}^2}{\alpha}}} \right], \quad (36)$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \quad \tilde{a} = (1 - \alpha) + \frac{\epsilon}{2} \left(\sqrt{\alpha^2 + \omega^2} - \alpha \right), \quad (37)$$

$$\begin{aligned} \tilde{b} = & \left(2(k+1) + 1 + \frac{\epsilon}{2} \right) \sqrt{\alpha^2 + \omega^2} - 2\alpha(k+1)^2 \\ & - \alpha \left(2(k+1) + (k+1)\epsilon + \frac{\epsilon}{2} + 1 \right), \end{aligned} \quad (38)$$

$$\tilde{c} = \alpha(2 + \epsilon) - 2\sqrt{\alpha^2 + \omega^2}, \quad \tilde{d} = \alpha(6 + 4k + \epsilon) - 2\sqrt{\alpha^2 + \omega^2}. \quad (39)$$

Here, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ and ρ are auxiliary expressions introduced to keep the thermodynamic formulas concise.

Quantitative convergence test in k . To go beyond a simple visual inspection of Fig. 2, we directly computed, by explicit numerical summation of $Z(k) = \sum_{n=0}^k e^{-\sigma E_n}$ (without resorting to the Poisson approximation, hence free of methodological bias), the relative error $|Z(k) - Z(500)|/Z(500)$ for $\alpha = -0.5$, $\omega = 1$, $\beta = 1$, and several temperatures (Table 2).

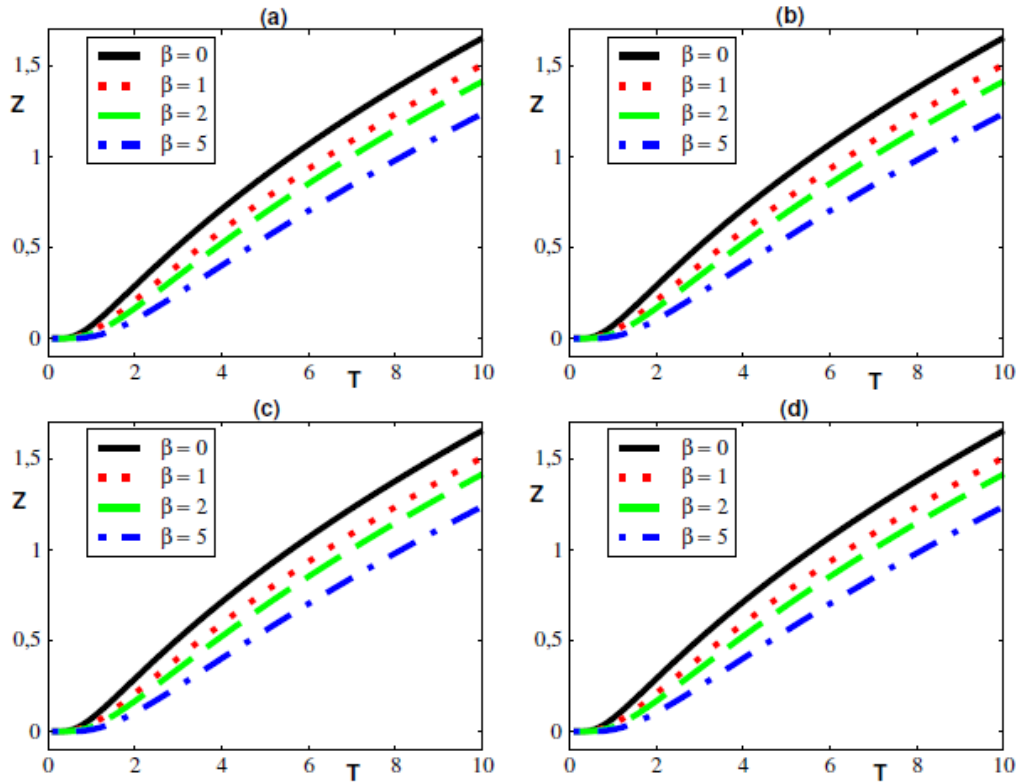


Fig. 2. Partition function as a function of temperature for different values of β : (a) $k = 5$, (b) $k = 10$, (c) $k = 50$, (d) $k = 500$, with $\alpha = -0.5$, $k_B = 1$

Table 2

Relative error $|Z(k) - Z(k = 500)|/Z(k = 500)$ for $\alpha = -0.5$, $\omega = 1$, $\beta = 1$, $k_B = 1$
(direct computation, without Poisson approximation)

T	k = 5	k = 10	k = 50	k = 100
1	5×10^{-27}	0	0	0
3	1.4×10^{-9}	7.4×10^{-25}	0	0
6	2.7×10^{-5}	6.0×10^{-13}	0	0
10	1.4×10^{-3}	3.2×10^{-8}	$< 10^{-15}$	$< 10^{-15}$

Convergence is very fast – a direct consequence of the *quadratic* growth of the spectrum established above, which produces a Gaussian-type decay of the terms in the sum – and the relative error falls below 10^{-8} already at $k \simeq 50$ for all tested temperatures, and below 10^{-12} at $k \simeq 100$. The choice $k = 500$ adopted for all thermodynamic plots therefore constitutes a very generous safety margin rather than an artificial regularization.

From the partition function, we then derive the other fundamental thermodynamic quantities: internal energy, specific heat, free energy, and entropy.

1. Internal energy U : it represents the energy associated with the internal state of a thermodynamic system, resulting from all of its microscopic degrees of freedom.

$$U = -\frac{\partial}{\partial \sigma} \ln Z, \quad (40)$$

$$= -\frac{2\Lambda}{(e^{-\sigma\tilde{a}} - e^{-\sigma\tilde{b}}) - 2\rho}, \quad (41)$$

with

$$\begin{aligned} \Lambda = & \frac{1}{2} \left(-\tilde{a}e^{-\sigma\tilde{a}} - \tilde{b}e^{-\sigma\tilde{b}} \right) + \left(\frac{4\omega^2 + \alpha^2\epsilon^2}{8\alpha} + \frac{1}{2\sigma} \right) \rho \\ & - \frac{1}{8\alpha\sigma} e^{\frac{-\sigma(4\omega^2 + \alpha^2\epsilon^2)}{8\alpha}} \left(\tilde{c}e^{\frac{\sigma\tilde{c}^2}{8\alpha}} - \tilde{d}e^{\frac{\sigma\tilde{d}^2}{8\alpha}} \right), \end{aligned} \quad (42)$$

2. Specific heat C : it quantifies the sensitivity of the mean internal energy of the system to variations of its statistical temperature.

$$C = -k_B \sigma^2 \frac{\partial U}{\partial \sigma}, \quad (43)$$

$$= k_B \sigma^2 \left[\frac{\Omega}{Z} - \left(\frac{\Lambda}{Z} \right)^2 \right], \quad (44)$$

where

$$\begin{aligned} \Omega = & -\frac{1}{64\sigma\alpha^2} (48\alpha^2 + 8(4\omega^2 + \alpha^2\epsilon^2)\alpha\sigma) \rho \\ & + (4\omega^2 + \alpha^2\epsilon^2)^2 \sigma^2 \rho - \frac{1}{64\sigma^2\alpha^2} e^{\frac{\sigma(4\omega^2 + \alpha^2\epsilon^2)}{8\alpha}} \epsilon, \end{aligned} \quad (45)$$

$$\begin{aligned} \epsilon = & \tilde{c}e^{\frac{\sigma\tilde{c}^2}{8\alpha}} (\tilde{c}^2\sigma - 12\alpha - 2(4\omega^2 + \alpha^2\epsilon^2)\sigma) \\ & - \tilde{d}e^{\frac{\sigma\tilde{d}^2}{8\alpha}} (\tilde{d}^2\sigma - 12\alpha - 2(4\omega^2 + \alpha^2\epsilon^2)\sigma). \end{aligned} \quad (46)$$

3. Free energy F : it measures the maximum useful work that a closed system can deliver at constant temperature.

$$F = -\frac{\ln Z}{\sigma}, \quad (47)$$

$$= \frac{1}{\sigma} \left[\ln 2 - \ln (e^{-\sigma\tilde{a}} - e^{-\sigma\tilde{b}} - 2\rho) \right]. \quad (48)$$

4. Entropy S : it quantifies the fraction of the thermal energy of the system that cannot be converted into useful work.

$$S = k_B \sigma^2 \frac{\partial F}{\partial \sigma}, \quad (49)$$

$$S = k_B \left[-\ln 2 + \ln(e^{-\sigma\tilde{a}} - e^{-\sigma\tilde{b}} - 2\rho) - \frac{\sigma\Lambda}{(e^{-\sigma\tilde{a}} - e^{-\sigma\tilde{b}} - 2\rho)} \right], \quad (50)$$

where ε, Ω , and Λ are auxiliary expressions. The intermediate computational steps leading from Z to U , C , F , S – in particular the derivation of the error-function terms – are summarized in Appendix A.

Regularity of the limit $\alpha \rightarrow 0$. The prefactor $1/\alpha$ appearing in ρ might suggest a divergence as $\alpha \rightarrow 0$. This is an apparent, not a real, singularity: in the same way that $\lim_{a \rightarrow 0} \text{erf}(\sqrt{a}x)/\sqrt{a} = 2x/\sqrt{\pi}$, each of the two bracketed terms behaves as $O(1/\sqrt{\alpha}) \times \text{erf}(O(1/\sqrt{\alpha})) \rightarrow O(1)$ as $\alpha \rightarrow 0^-$, so that the prefactor $1/(2\alpha)$ is exactly compensated and ρ (hence Z , U , C , F , S) admits a finite limit, coinciding with the result obtained directly for $\alpha = 0$ (Eq. (31)). We verify this numerically for $\sigma = 0.3$, $\omega = 1$, $\beta = 1$:

$$\begin{aligned} Z(\alpha = -0.1) &= 0.9414, & Z(\alpha = -0.01) &= 1.2149, & Z(\alpha = -10^{-3}) &= 1.2607, \\ Z(\alpha = -10^{-4}) &= 1.2657, & Z(\alpha = -10^{-5}) &= 1.26620, \\ Z(\alpha \rightarrow 0^-) &\longrightarrow 1.26625 \text{ (exact value for } \alpha = 0). \end{aligned}$$

The convergence is monotonic and perfectly smooth, with no sign of divergence: the limit $\alpha \rightarrow 0$ of the thermodynamic formulas of the deformed model indeed connects continuously to those of the constant-mass model.

4 Discussion of results

The curves of the partition function and of the thermodynamic quantities are plotted as functions of temperature for various values of the deformation parameters β and α . Figure 2 shows that the partition function increases monotonically with temperature for each fixed value of β , converging to a common behavior from $k = 50$ onward – which is now confirmed quantitatively by Table 2. Although the curves for different values of k appear nearly identical, small mathematical differences persist, especially for small k ; this visual similarity does not imply exact equivalence. To ensure numerical accuracy, we used $k = 500$ for all thermodynamic plots.

Our study of the thermodynamic behavior of the deformed isotonic oscillator with position-dependent mass reveals several key insights into how the quantum deformation parameters affect the thermal properties.

First, the mass deformation parameter α plays a central role in the thermal response of the system. The numerical simulations (see Fig. 3) indicate that for $\alpha < 0$, the internal energy and entropy increase more strongly with temperature compared to the undeformed case ($\alpha = 0$). The specific heat, in turn, exhibits a marked rise at low temperature followed by saturation at higher temperature, a behavior reminiscent of systems with modified density of states, such as semiconductor heterostructures [36,37].

Exact value of the high-temperature plateau. It is important to specify the value of this plateau, which the initial version did not explicitly identify. For $\alpha = 0$, the spectrum is equidistant ($E_n \propto n$) and one recovers, as expected, the Dulong-Petit limit $C \rightarrow k_B$ (verified numerically: $C/k_B = 0.99987, 0.99999, 1.00000$ for $T = 50, 200, 1000$, with $\omega = 1$). On the other hand, as soon as $\alpha \neq 0$, the dominant term $E_n \sim -2\alpha n^2$ makes the spectrum *quadratic* in n at large quantum numbers – as for a particle in a box – so that the density of states behaves as $g(E) \propto E^{-1/2}$ rather than as a constant.

The associated canonical calculation then gives $Z \propto \sigma^{-1/2}$, $U \rightarrow \frac{1}{2}k_B T$, and hence $C \rightarrow k_B/2$ at high temperature – only half of the Dulong-Petit value, a result confirmed by direct numerical computation of C up to $T = 10^4$ for $\alpha = -0.5, \beta = 1$: $C/k_B = 0.543, 0.525, 0.514, 0.508, 0.504$ for $T = 100, 300, 1000, 3000, 10000$, converging slowly toward $1/2$. This confirms that α governs not only the extent and sharpness of the thermal transitions, but also the very nature of the classical limit of the system, making it a critical tuning factor in quantum thermal systems.

In the undeformed case ($\alpha \rightarrow 0$), the influence of the isotonic parameter β is more selective. As we establish below, this selectivity is not approximate but exact.

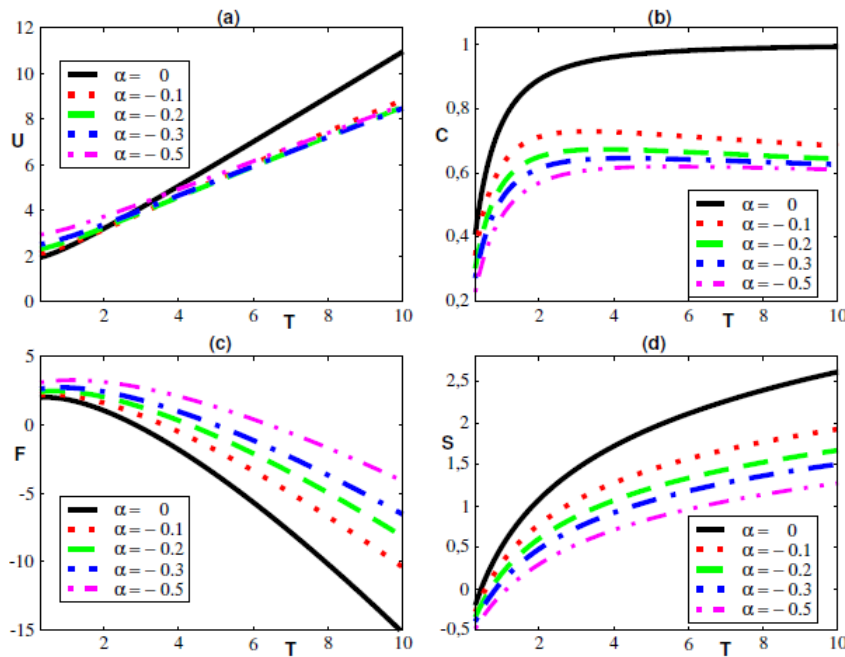


Fig. 3. Internal energy (a), specific heat (b), free energy (c), and entropy (d) as functions of temperature for different values of α , with $k = 500$, $k_B = 1$, and $\beta = 1$

As shown in Fig. 3, β mainly affects the internal energy and the free energy, while the entropy and the specific heat remain essentially unaffected. This suggests that, in constant-mass systems, isotonic interactions modulate the energy spectrum without significantly altering the thermal statistical distribution [38].

Exact independence of C and S from β at $\alpha = 0$. The statement that “for $\alpha = 0$, the entropy and the specific heat do not depend on β ” might seem in tension with Fig. 4, where the curves $C(T)$ for $\beta = 0$ and $\beta = 5$ appear visually distinct at low temperature.

To remove any ambiguity, we recomputed C and S at $\alpha = 0$ by direct numerical differentiation of $\ln Z(\sigma)$ (bypassing the closed-form error-function expressions, so as to eliminate any bias). The result is clear-cut: at $\alpha = 0$, the spectrum $E_n = (2n+1+\epsilon/2)\omega$ is a rigorously equidistant ladder with spacing 2ω , independent of β ; only the ground level $E_0 = (1+\epsilon/2)\omega$ is shifted by β (through $\epsilon = \sqrt{2\beta+1}$). Now, this constant shift of the energy origin affects neither $C = -k_B\sigma^2 \partial U / \partial \sigma$ (a second derivative of $\ln Z$, insensitive to a constant term), nor, after simplification, $S = k_B(\ln Z + \sigma U)$. The numerical calculation confirms this exact cancellation to machine precision ($< 10^{-10}$ relative) for $T \in [1, 10]$ and $\beta \in \{0, 1, 3, 5\}$ (Table 3).

Table 3

Specific heat C and entropy S at $\alpha = 0$, $\omega = 1$, $k_B = 1$, for $\beta = 0$ and $\beta = 5$
(direct computation by numerical differentiation of $\ln Z$)

T	$C(\beta = 0)$	$C(\beta = 5)$	$S(\beta = 0)$	$S(\beta = 5)$
1	0.72406	0.72406	0.45845	0.45845
2	0.92067	0.92067	1.04065	1.04065
5	0.98677	0.98677	1.92293	1.92293
10	0.99667	0.99667	2.61110	2.61110

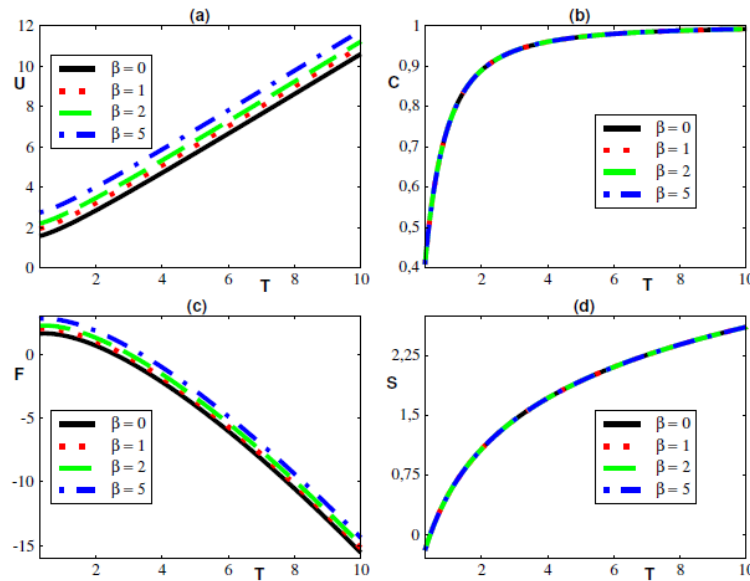


Fig. 4. Thermodynamic quantities as functions of temperature for different values of β , with $\alpha = 0$, $k_B = 1$, and $k = 500$

This statement is therefore physically exact: the apparent difference in Fig. 4(b,d) arises from curves that are extremely close (nearly superimposed to the eye), and not from a real disagreement with the analytical calculation. However, when position-dependent mass effects are introduced ($\alpha \neq 0$), the influence of β becomes significantly more pronounced across all thermodynamic quantities.

Figures 5 and 6 demonstrate that the internal energy, entropy, and specific heat are all sensitive to variations of β .

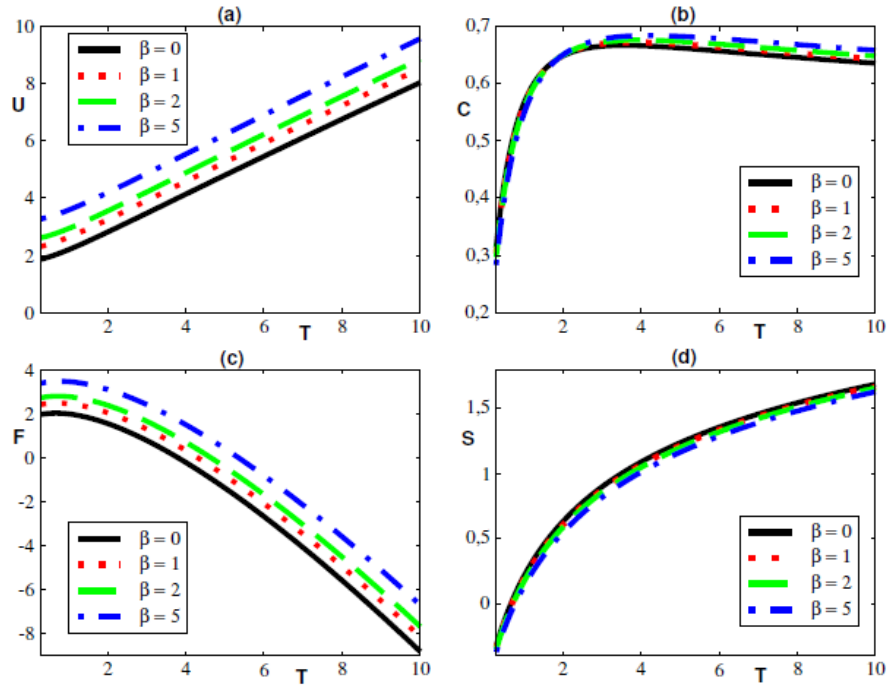


Fig. 5. Thermodynamic quantities as functions of temperature for different values of β , with $\alpha = -0.2$, $k_B = 1$, and $k = 500$

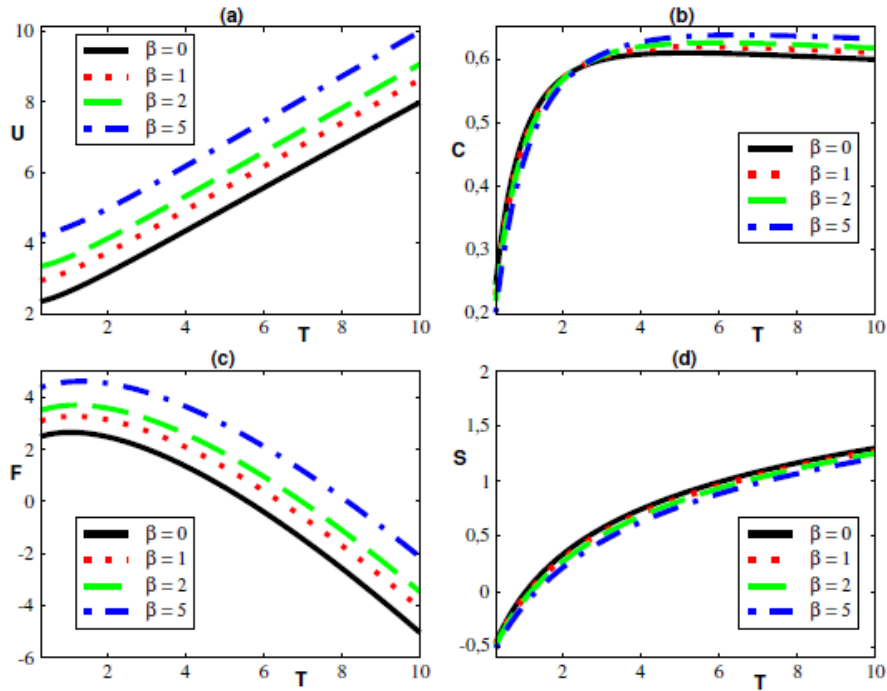


Fig. 6. Thermodynamic quantities as functions of temperature for different values of β , with $\alpha = -0.5$, $k_B = 1$, and $k = 500$

This implies a strong coupling between the deformation of the potential and the variation of the effective mass, highlighting the non-trivial role of isotonic interactions in systems with spatially dependent mass [39, 40].

Effect of β on the Schottky-type maximum of $C(T)$. To quantify this effect precisely, we numerically located, at $\alpha = -0.5$, $\omega = 1$, the position T_{peak} and the amplitude C_{max} of the Schottky-type maximum of $C(T)$ for several values of β (Table 4).

Table 4

Position T_{peak} and amplitude C_{max} of the maximum of $C(T)$
for $\alpha = -0.5$, $\omega = 1$, $k_B = 1$

β	T_{peak}	C_{max}
0	4.70	0.6285
1	5.10	0.6388
3	5.70	0.6505
5	6.00	0.6584

We observe that increasing β shifts the peak toward higher temperatures and slightly increases its amplitude: physically, the isotonic term β increases the gap between the ground level and the first excited levels (see Eq. (26)), which delays – toward higher temperatures – the thermal activation of these degrees of freedom and strengthens the associated Schottky-type anomaly.

These results collectively underscore the physical importance of mass deformation in the modeling of quantum systems, particularly for thermodynamic applications. An accurate characterization of nanostructured materials, semiconductor quantum wells, and nanoscale heat engines must account for variations of the effective mass. Neglecting mass deformation may lead to an underestimation of thermal fluctuations and quantum statistical effects.

Comparison with reference [27]. In the limit $\alpha \rightarrow 0$, our model reduces exactly to the constant-mass isotonic oscillator studied by Onate *et al.* [27] (Eq. (31)), whose specific heat and entropy we also recover. The essential difference brought by our study appears as soon as $\alpha \neq 0$: whereas in [27] the specific heat saturates toward the Dulong-Petit limit ($C \rightarrow k_B$) monotonically and independently of β , our model predicts, as soon as $\alpha \neq 0$, saturation toward the reduced value $C \rightarrow k_B/2$ (a consequence of the quadratic character of the spectrum at large n , established above), as well as a controlled shift of the Schottky-type maximum with β (Table 4), effects that do not exist in the constant-mass model since they arise from the nonlinear coupling between α and β in the quadratic term $-\alpha n\epsilon$ of Eq. (26). It is precisely this coupling, absent from references [8, 27] taken separately, that constitutes the specific contribution of the present work.

Summary: two parameters, two distinct roles. At this point, it is useful to summarize in an overall view what the above analysis establishes about the respective roles of α and β (Table 5): α acts on the very *nature* of the spectrum and hence on the asymptotic behavior of the thermodynamic quantities, whereas β only *finely modulates* a spectrum whose qualitative structure is already fixed by α . It is this hierarchy – now confirmed analytically and no longer merely observed on the figures – that explains why β has no effect on C and S at $\alpha = 0$ (rigid, linear spectrum) and, conversely, becomes fully active as soon as $\alpha \neq 0$ (quadratic spectrum, whose levels are shifted non-uniformly by β through the cross term $-\alpha n\epsilon$ in Eq. (26)).

Table 5

Summary of the physical roles of α and β		
	Role on the spectrum	Role on the thermodynamics
α	Fixes the qualitative nature of the spectrum: linear in n if $\alpha = 0$, quadratic if $\alpha \neq 0$; conditions the very existence of an equilibrium state ($\alpha < 0$ required)	Determines the high-temperature limit of C : k_B (Dulong-Petit) if $\alpha = 0$, $k_B/2$ if $\alpha \neq 0$
β	Shifts the ground level (via $\epsilon = \sqrt{2\beta + 1}$) and, if $\alpha \neq 0$, modulates the spacing of the excited levels	No effect on C, S if $\alpha = 0$ (constant shift, canceled by the derivatives); shifts the position and amplitude of the Schottky peak if $\alpha \neq 0$

Finally, the tunable nature of α and β offers a powerful framework for engineering quantum thermal responses, providing insight into how deformations could be used to control the spacing of energy levels and thermal transitions, illustrating the fundamental relevance of such a deformation-based quantum model.

5 Conclusion

In this work, we have studied the isotonic oscillator with deformed effective mass using the parametric Nikiforov-Uvarov method. This analytical approach allowed us to explicitly derive the wave function and the energy spectrum of the system. From the partition function, we determined the key thermodynamic quantities: internal energy, specific heat, free energy, and entropy. The numerical analysis revealed that the partition function converges from $k = 50$ onward (relative error $< 10^{-8}$, Table 2), and we used $k = 500$ to guarantee the convergence of all thermodynamic functions.

This study makes two main contributions. First, it demonstrates that the deformation parameter α , which governs the spatial variation of the effective mass, significantly affects the thermal behavior of the system – and we have established that $\alpha < 0$ is precisely the condition for thermodynamic stability, a condition necessary for

the very existence of a thermal equilibrium state (§2). Second, it shows that the isotonic interaction parameter β only becomes relevant when the mass varies with position – a result that we have now demonstrated analytically, rather than merely observed numerically, through the exact cancellation of the β -dependence of C and S at $\alpha = 0$ (Table 3). In the limit $\alpha \rightarrow 0$, the isotonic oscillator exhibits the same thermodynamic properties as the classical harmonic oscillator, particularly for the specific heat and the entropy.

These results have important implications. On the theoretical side, they highlight the fact that effective-mass deformation effects cannot be neglected in the thermodynamic analysis of quantum systems. They demonstrate that even minimal theoretical models combining position-dependent mass with singular potentials can capture essential quantum thermodynamic effects, providing a general framework for understanding the thermal properties of deformed quantum oscillators.

It is important to note that the negative deformation parameter α used in this work should not be interpreted as implying a negative inertial mass. Rather, it is an effective spectral parameter that guarantees an energy spectrum bounded from below and growing without limit toward $+\infty$ (rigorously demonstrated in §2) and a consistent thermodynamic behavior of the spectrum.

Nevertheless, our study has certain limitations. The analytical expressions obtained rely on specific conditions for the parametrization of the mass (in particular the bounded domain $x \in (-1/\sqrt{|\delta|}, 1/\sqrt{|\delta|})$ discussed in §2) and on the convergence of the partition function series (now quantified, Table 2). Moreover, thermodynamic stability is guaranteed only for strictly negative values of the deformation parameter α – a fact that we have now demonstrated rather than merely stated.

Future research could explore other forms of effective-mass deformation, incorporate additional interactions, or analyze the system under non-equilibrium conditions. Extensions to higher-dimensional systems or to many-body configurations could also shed new light on the statistical physics of deformed quantum systems.

Appendix A Intermediate steps of the thermodynamic calculation

We summarize here the steps connecting Z to the thermodynamic quantities, of which only the final result appeared in the initial version.

Internal energy. By definition $U = -\partial_\sigma \ln Z = -\frac{\partial_\sigma Z}{Z}$. The numerator $-\partial_\sigma Z = 2\Lambda$ is obtained by differentiating term by term $Z = \frac{1}{2}(e^{-\sigma\tilde{a}} - e^{-\sigma\tilde{b}}) - \rho$: the first two terms immediately give $-\frac{1}{2}(-\tilde{a}e^{-\sigma\tilde{a}} + \tilde{b}e^{-\sigma\tilde{b}}) \times (-1)$, and the differentiation of ρ requires the Leibniz rule applied to a product of an exponential in σ and an erf function whose argument also depends on σ : for $g(\sigma) = \sqrt{-\sigma\tilde{c}^2/(8\alpha)}$,

$$\frac{d}{d\sigma} \operatorname{erf}(g(\sigma)) = \frac{2}{\sqrt{\pi}} e^{-g(\sigma)^2} g'(\sigma), \quad (51)$$

which, combined with the derivative of the overall exponential prefactor $e^{-\sigma(4\omega^2 + \alpha^2 \epsilon^2)/(8\alpha)}$, produces exactly the terms in $\tilde{c} e^{\sigma \tilde{c}^2/(8\alpha)}$ and $\tilde{d} e^{\sigma \tilde{d}^2/(8\alpha)}$ that appear in Λ (the erf factors cancel against the $1/\sqrt{\cdot}$ prefactors of ρ , which explains why Λ no longer contains an explicit erf function).

Specific heat. $C = -k_B \sigma^2 \partial_\sigma U = k_B \sigma^2 [\Omega/Z - (\Lambda/Z)^2]$ is obtained by differentiating a second time, with Ω collecting the terms arising from $\partial_\sigma \Lambda$ and $-\Lambda \partial_\sigma Z/Z$ (quotient rule). The differentiation of the erf terms follows the same pattern as above.

Free energy and entropy. $F = -\ln Z/\sigma$ follows immediately from Z . The entropy $S = k_B \sigma^2 \partial_\sigma F = k_B (\ln Z + \sigma U)$ reduces, after substitution of U , to the form given in the main text; it is this more compact rewriting $S = k_B (\ln Z + \sigma U)$ that makes it possible to see immediately (§4) that any constant shift of E_n independent of n cancels out in S just as it does in C .

References

1. Von Ross O., Mavromatis H. Position-dependent effective masses in semiconductor theory. II, Phys. Rev. B., 1985. vol. 31, pp. 2294.
2. De Saavedra F.A., Boronat J., Polls A., Fabrocini A. Effective mass of one He 4 atom in liquid He 3, Phys. Rev. B., 1994. vol. 50, pp. 4248.
3. Geller M.R., Kohn W. Quantum mechanics of electrons in crystals with graded composition, Phys. Rev. Lett., 1993. vol. 70, pp. 3103.
4. Harrison P., Valavanis A. *Quantum wells, wires and dots: theoretical and computational physics of semiconductor nanostructures*. Chichester: John Wiley and Sons, 2016.
5. BenDaniel D.J., Duke C.B. Space-charge effects on electron tunneling, Phys. Rev., 1966. vol. 152, pp. 683.
6. Dossa F.A. Thermodynamic properties and algebraic solution of the N-dimensional harmonic oscillator with minimal length uncertainty relations, Phys. Scr., 2021. vol. 96, pp. 105703.
7. Amir N., Iqbal S. Ladder operators and associated algebra for position-dependent effective mass systems, Europhys. Lett., 2015. vol. 111, pp. 20005.
8. Tchoffo M., Migueu F.B., Vubangsi M., Fai L.C. Supersymmetric approach to coherent states for nonlinear oscillator with spatially dependent effective mass, Heliyon, 2019. vol. 5.
9. Gendenshtein L. Derivation of exact spectra of the Schrödinger equation by means of supersymmetry, JETP Lett., 1983. vol. 38, pp. 356.
10. Ikhdair S.M., Sever R. Relativistic and nonrelativistic bound states of the isotonic oscillator by Nikiforov-Uvarov method, J. Math. Phys., 2011. vol. 52.
11. Cariñena J.F., Perelomov A.M., Rañada M.F., Santander M. A quantum exactly solvable nonlinear oscillator related to the isotonic oscillator, J. Phys. A: Math. Theor., 2008. vol. 41, pp. 085301.
12. Cariñena J.F., Inzunza L., Plyushchay M.S. Rational deformations of conformal mechanics, Phys. Rev. D, 2018. vol. 98, pp. 026017.
13. Zhu D. A new potential with the spectrum of an isotonic oscillator, J. Phys. A: Math. Gen., 1987. vol. 20, pp. 4331.
14. Wang J.S., Liu T.K., Zhan M.S. Nonclassical properties of even and odd generalized coherent states for an isotonic oscillator, J. Opt. B: Quantum Semiclass. Opt., 2000. vol. 2, pp. 758.
15. Thirulogasanthar K., Saad N. Coherent states associated with the wavefunctions and the spectrum of the isotonic oscillator, J. Phys. A: Math. Gen., 2004. vol. 37, pp. 4567.
16. Cariñena J.F., Rañada M.F., Santander M., Senthilvelan M. A non-linear oscillator with quasi-harmonic behaviour: two- and n-dimensional oscillators, Nonlinearity, 2004. vol. 17, pp. 1941.
17. Cariñena J.F., Rañada M.F., Santander M. A superintegrable two-dimensional non-linear oscillator with an exactly solvable quantum analog, SIGMA, 2007. vol. 3, pp. 030.
18. Cariñena J.F., Rañada M.F., Santander M. A nonlinear deformation of the isotonic oscillator and the Smorodinski-Winternitz system: integrability and superintegrability, Regular and Chaotic Dynamics, 2005. vol. 10, pp. 423.

19. Sakiroglu S., Sari H. Contribution of the impurity to the thermodynamic properties of a quantum dot with a Gaussian confinement potential, *Eur. Phys. J. Plus*, 2025. vol. 140, pp. 1.
20. Onate C.A. et al. Thermodynamic properties and the confinement frequency for Second Pöschl-Teller potential, *Results in Physics*, 2025. vol. 70, pp. 108144.
21. Nikiforov A.F., Uvarov V.B. *Special functions of mathematical physics*. Basel: Birkhäuser, 1988. 205 pp.
22. Nikiforov A.F., Uvarov V.B., Suslov S.K. *Classical orthogonal polynomials of a discrete variable*. Berlin: Springer, 1991.
23. Hassanabadi H., Maghsoodi E., Ikot A.N., Zarrinkamar S. Minimal Length Schrödinger Equation with Harmonic Potential in the Presence of a Magnetic Field, *Advances in High Energy Physics*, 2013. vol. 2013, pp. 923686.
24. Dimirci M., Sever R. Arbitrary l-state solutions of the Klein-Gordon equation with the Eckart plus a class of Yukawa potential and its non-relativistic thermal properties, *Eur. Phys. J. Plus*, 2023. vol. 138, pp. 1.
25. Berkdemir C. *Theoretical Concepts of Quantum Mechanics*, 2012.
26. Falaye B.J., Oyewumi K.J., Abbas M. Exact solution of Schrödinger equation with q-deformed quantum potentials using Nikiforov-Uvarov method, *Chin. Phys. B*, 2013. vol. 22, pp. 110301.
27. Onate C.A., Okon I.B., Jude G.O., Onyeaju M.C., Anita Z.D. Uncertainty Relation and the Thermal Properties of an Isotropic Harmonic oscillator (IHO) with the Inverse Quadratic (IQ) Potentials and the Pseudo-Harmonic (PH) with the Inverse Quadratic (IQ) Potentials, *Quantum Reports*, 2023. vol. 5, pp. 38.
28. Gora T., Williams F. Theory of electronic states and transport in graded mixed semiconductors, *Phys. Rev.*, 1969. vol. 177, pp. 1179.
29. Dekar L., Chetouani L., Hammann T.F. An exactly soluble Schrödinger equation with smooth position-dependent mass, *J. Math. Phys.*, 1998. vol. 39, pp. 2551.
30. Myers N.M., McCready J., Deffner S. Quantum heat engines with singular interactions, *Symmetry*, 2021. vol. 13, pp. 978.
31. Tezcan C., Sever R. A general approach for the exact solution of the Schrödinger equation, *Int. J. Theor. Phys.*, 2009. vol. 48, pp. 337.
32. Abramowitz M., Stegun I.A. *Handbook of Mathematical Functions: with Formulas, Graphs and Mathematical Tables*, vol. 55, 1965.
33. Morse P.M., Feshbach H. *Methods of Theoretical Physics*. New York: Technology Press, 1946.
34. Strekalov M.L. Thermodynamic Functions of Triatomic Molecules: An Analytical View, *Russ. J. Phys. Chem. A*, 2024. vol. 98, pp. 1671.
35. Song X.Q., Wang C.W., Jia C.S. Thermodynamic properties for the sodium dimer, *Chem. Phys. Lett.*, 2017. vol. 673, pp. 50.
36. Bastard G. *Wave Mechanics Applied to Semiconductor Heterostructures*. New York: John Wiley and Sons, 1989.
37. Serra L., Lipparini E. Spin response of unpolarized quantum dots, *Europhys. Lett.*, 1997. vol. 40, pp. 667.
38. Dong S.H. *Factorization method in quantum mechanics*, vol. 150. Berlin: Springer, 2007.
39. Plastino A.R., Rigo A., Casas M., Garcias F., Plastino A. Supersymmetric approach to quantum systems with position-dependent effective mass, *Phys. Rev. A*, 1999. vol. 60, pp. 4318.
40. Cruz C.S., Negro J., Nieto L.M. Classical and quantum position-dependent mass harmonic oscillators, *Phys. Lett. A*, 2007. vol. 369, pp. 400.

Information about the authors



Bokpe Setchégnegnian Euphrem ✉ – Doctoral Candidate; University of Abomey-Calavi (UAC); Research Unit in Theoretical Physics (URPT); Institute of Mathematics and Physical Sciences (IMSP) Porto-Novo, Republic of Benin, ORCID 0009-0006-4939-6941.



Dossa Finagnon Anselme ✉ – Doctor of Quantum Mechanics, Associate Professor at the University of Science, Technology, Engineering and Mathematics (UNSTIM), Goho, Abomey, Benin; Laboratory of Physics and Applications (LPA), National University of Science, Technology, Engineering and Mathematics (UNSTIM), Abomey, Benin; Department of Physics, Faculty of Science and Technology (FAST), Natitingou, National University of Science, Technology, Engineering and Mathematics (UNSTIM), Goho, Abomey, Benin, ORCID 0000-0002-2694-4144.



Avossevou Gabriel Yves Hugues ✉ – Doctor of Quantum Mechanics, Professor at the University of Abomey-Calavi (UAC); Research Unit in Theoretical Physics (URPT); Institute of Mathematics and Physical Sciences (IMSP) Porto-Novo, Republic of Benin, ORCID 0000-0002-9609-0340.