



## Dirichlet Problem for a Third-Order Equation of Mixed Hyperbolic-Pseudoparabolic Type in a Rectangular Domain

*Zh. A. Balkizov\**

Institute of Applied Mathematics and Automation, Kabardino-Balkarian Scientific Center  
of the Russian Academy of Sciences, 89A Shortanova St., Nalchik 360005, Russia

**Abstract.** This paper formulates and investigates the Dirichlet problem for a third-order equation of mixed hyperbolic-pseudoparabolic type in a rectangular domain  $D = \{(x, y) : 0 < x < r, -\alpha < y < \beta\}$ , where one of the matching conditions on the line of type change  $y = 0$  is replaced by a Dirichlet condition on one of the domain boundaries. In the upper part of the mixed domain (relative to the line of type change  $y = 0$ ), the governing equation is a third-order pseudoparabolic Aller's equation, while in the remaining part, it becomes a non-homogeneous wave equation. Using Sabitov's spectral analysis method, we prove the well-posedness of the Dirichlet problem under consideration. We establish sufficient conditions on the given functions that ensure the well-posedness of the problem and the regularity of its solution.

*Key words:* Aller's equation, inhomogeneous wave equation, equations of mixed hyperbolic-pseudoparabolic type, Dirichlet problem, regular solution, class of regular functions, spectral analysis method, problem of small denominators.

Received: 08.07.2026; Revised: 25.07.2026; Accepted: 09.09.2026; First online: 14.09.2026

**For citation.** Balkizov Zh. A. Dirichlet Problem for a Third-Order Equation of Mixed Hyperbolic-Pseudoparabolic Type in a Rectangular Domain. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2026, 54: 2, 26-37. EDN: ABWTOF. <https://doi.org/10.26117/2079-6641-2026-54-2-26-37>.

**Funding.** The study was conducted without the support of foundations.

**Competing interests.** There are no conflicts of interest regarding authorship and publication.

**Contribution and Responsibility.** The author contributed to this article and is solely responsible for providing the final version of the article in print. The final version of the manuscript was approved by the author.

\*Correspondence:  E-mail: [Giraslan@yandex.ru](mailto:Giraslan@yandex.ru)

The content is published under the terms of the Creative Commons Attribution 4.0 International License

© Balkizov Zh. A., 2026

© Institute of Cosmophysical Research and Radio Wave Propagation, 2026 (original layout, design, compilation)



МАТЕМАТИКА

 <https://doi.org/10.26117/2079-6641-2026-54-2-26-37>

Научная статья

Полный текст на английском языке

УДК 517.95



## Задача Дирихле для уравнения третьего порядка смешанного гиперболо-псевдопараболического типа в прямоугольной области

*Ж. А. Балкизов\**

Институт прикладной математики и автоматизации КВНЦ РАН, Шортанова, д.89А,  
Нальчик, 360017, Россия

**Аннотация.** В данной работе формулируется и исследуется задача Дирихле для уравнения третьего порядка смешанного гиперболо-псевдопараболического типа в прямоугольной области  $D = \{(x, y) : 0 < x < r, -\alpha < y < \beta\}$ , в которой одно из условий сопряжения на линии смены типа  $y = 0$  заменено условием Дирихле на одной из границ области. В верхней части смешанной области (относительно линии смены типа  $y = 0$ ) определяющим уравнением является псевдопараболическое уравнение Аллера третьего порядка, тогда как в остальной части оно переходит в неоднородное волновое уравнение. С помощью метода спектрального анализа Сабитова доказывается корректность рассматриваемой задачи Дирихле. Устанавливаются достаточные условия на заданные функции, обеспечивающие корректность задачи и регулярность ее решения.

*Ключевые слова:* уравнение Аллера, неоднородное волновое уравнение, уравнения смешанного гиперболо-псевдопараболического типа, задача Дирихле, регулярное решение, класс регулярных функций, метод спектрального анализа, проблема малых знаменателей.

Получение: 08.07.2026; Исправление: 25.07.2026; Принятие: 09.09.2026; Публикация онлайн: 14.09.2026

**Для цитирования.** Balkizov Zh. A. Dirichlet problem for a third-order equation of mixed hyperbolic-pseudoparabolic type in a rectangular domain // Вестник КРАУНЦ. Физ.-мат. науки. 2026. Т. 54. № 2. С. 26-37. EDN: ABWTOF. <https://doi.org/10.26117/2079-6641-2026-54-2-26-37>.

**Финансирование.** Исследование было проведено без поддержки фондов

**Конкурирующие интересы.** Конфликтов интересов в отношении авторства и публикации нет.

**Авторский вклад и ответственность.** Автор участвовал в написании статьи и полностью несет ответственность за предоставление окончательной версии статьи в печать.

\*Корреспонденция: ✉ E-mail: Giraslan@yandex.ru

Контент публикуется на условиях Creative Commons Attribution 4.0 International License

© Balkizov Zh. A., 2026

© ИКИР ДВО РАН, 2026 (оригинал-макет, дизайн, составление)



## Introduction. Problem Formulation

The theoretical foundation for the mathematical modeling of physical processes in real media is partial differential equations. When constructing models of real processes, the well-posedness of the formulated problem is of fundamental importance. Of the many diverse model problems, the Dirichlet boundary value problems occupy a prominent position. These problems arise in the theories of thin shells, aerospace engineering, transonic gas dynamics, and hydrodynamics. Thus, if we consider, for example, the problem of a thermal-to-critical transition in steady two-dimensional irrotational flows of an ideal gas in nozzles, where supersonic waves are adjacent to the nozzle walls near the throat, it reduces to the study of the Dirichlet problem for the Chaplygin equation. [1], [2].

Conversely, boundary value problems with data prescribed on the entire boundary (such as the Dirichlet problem) are classic examples of ill-posed problems in the theory of both hyperbolic and mixed partial differential equations. The first results on the study of boundary value problems with data given on the entire boundary for hyperbolic equations were obtained in [7]. Thus, [7] investigated the Dirichlet problem for the homogeneous wave equation in a rectangular domain. Here, it was first demonstrated that the well-posedness of the Dirichlet problem for the wave equation in a rectangle fundamentally depends on its aspect ratio, which is required to be an irrational number. In [4], the author demonstrated the ill-posedness of the Dirichlet problem for the Lavrent'ev–Bitsadze equation in a conventional mixed domain with an inward departure from the characteristics. Furthermore, the ill-posedness of the Dirichlet problem is independent of how small the measure of the domain enclosed between the hyperbolic boundary and the line of type change is.

These studies initiated the search for mixed equations and appropriate domains that ensure the well-posedness of the Dirichlet problem. Subsequently, the Dirichlet problem for various degenerate hyperbolic and mixed-type equations was extensively studied by many authors. We note, for instance, the works [5–17]. A more comprehensive bibliography on this topic is available in the doctoral theses [18], [19], including a number of monographs [20–22].

The paper studies the Dirichlet problem for a third-order equation of mixed hyperbolic-pseudoparabolic type, where one of the matching conditions on the line of type change is replaced by the Dirichlet condition on one of the domain boundaries. Using K. B. Sabitov's spectral analysis method, we find the conditions for the well-posedness of the Dirichlet problem under study in the formulation given below.

In a rectangular domain  $D = \{(x, y) : 0 < x < r, -\alpha < y < \beta\}$  of the Euclidean plane of points  $x$  and  $y$  we consider the equation

$$0 = \begin{cases} u_y - a u_{xx} - b u_{xxy}, & y > 0, \\ u_{xx} - u_{yy} + f, & y < 0, \end{cases} \quad (1)$$

where  $a$ ,  $b$ ,  $\alpha$ ,  $\beta$  and  $r$  are prescribed positive constants;  $f = f(x, y)$  is a given function of variables  $x$  and  $y$ ; and  $u = u(x, y)$  is the function to be determined.

Let us denote  $D_1 = \{(x, y) : 0 < x < r, -\alpha < y < 0\}$ , and  $D_2 = \{(x, y) : 0 < x < r, 0 < y < \beta\}$ . Then  $D = D_1 \cup D_2 \cup I$ , where  $I = \{(x, 0) : 0 < x < r\}$ .

**Problem 1.** Find the solution  $u = u(x, y)$  to equation (1), with the following properties:

$$u(x, y) \in C(\bar{D}) \cap C^2(D_1) \cap C_{x,y}^{2,1}(D_2); \quad (2)$$

$$u(x, -\alpha) = \psi_1(x), \quad u(x, \beta) = \psi_2(x), \quad 0 \leq x \leq r; \quad (3)$$

$$u(0, y) = 0, \quad u(r, y) = 0, \quad -\alpha \leq y \leq \beta, \quad (4)$$

where  $\psi_1(x)$ ,  $\psi_2(x)$  are given functions, moreover, the following matching conditions hold:  $\psi_1(0) = \psi_1(r) = 0$ ,  $\psi_2(0) = \psi_2(r) = 0$ .

## Uniqueness of the Solution to Problem 1

**Theorem 1.** Let the given numbers  $\alpha$  and  $r$  be such that the ratio  $\alpha/r$  is irrational, and let the unknown function  $u = u(x, y)$  possess the same properties as in

$$\lim_{\varepsilon \rightarrow 0} u_x(\varepsilon, y) \sin(\mu_k \varepsilon) = 0; \quad \lim_{\varepsilon \rightarrow 0} u_x(r - \varepsilon, y) \sin(\mu_k(r - \varepsilon)) = 0, \quad -\alpha \leq y \leq \beta, \quad (5)$$

$$\lim_{\varepsilon \rightarrow 0} u_{xy}(r - \varepsilon, y) \sin \mu_k(r - \varepsilon) = \lim_{\varepsilon \rightarrow 0} u_{xy}(\varepsilon, y) \sin \mu_k \varepsilon = 0, \quad -\alpha \leq y \leq \beta, \quad (6)$$

$$\mu_k = \frac{\pi k}{r}, \quad k \in \mathbb{N}.$$

Then, if a solution to Problem 1 exists, it is unique.

**Proof.** Indeed, let us consider the homogeneous problem corresponding to Problem 1, that is, we assume that  $f(x, y) \equiv 0 \forall (x, y) \in \bar{D}_1$  and  $\psi_1(x) = \psi_2(x) \equiv 0 \forall x \in [0, r]$ . Particular solutions of equation (1) satisfying the zero boundary conditions (4) are sought in the form of the product  $u(x, y) = X(x)Y(y)$ . Then, under the assumption that  $X(x) \neq 0 \forall x \in (0, r)$ , equation (1) combined with (4) yields the following equations for the functions  $X(x)$  and  $Y(y)$  to be determined:

$$X''(x) + \mu^2 X(x) = 0, \quad 0 < x < r, \quad \mu = \text{const}, \quad (7)$$

$$X(0) = X(r) = 0, \quad (8)$$

$$Y'(y) + \frac{a\mu^2}{1 + b\mu^2} Y(y) = 0, \quad 0 < y < \beta, \quad (9)$$

$$Y''(y) + \mu^2 Y(y) = 0, \quad -\alpha < y < 0. \quad (10)$$

As is well known, problem (7)–(8) admits non-trivial solutions of the form

$$X_k(x) = \sqrt{\frac{2}{r}} \sin(\mu_k x) = \sqrt{\frac{2}{r}} \sin\left(\frac{\pi k}{r} x\right) \quad (11)$$

for the eigenvalues  $\mu_k = \frac{\pi k}{r}$ ,  $k \in \mathbb{N}$ . Furthermore, the eigenfunction system  $\{X_k(x)\}_{k=1}^{\infty}$  is complete and orthonormal, forming a basis in the space  $L_2[0, 1]$ . Then, the differential

equations (9) and (10) have general solutions

$$Y_k(y) = \begin{cases} a_k \exp\left(-\frac{a \mu_k^2 y}{1+b \mu_k^2}\right), & 0 < y < \beta, \\ b_k \cos\left(\frac{\pi k}{r} y\right) + c_k \sin\left(\frac{\pi k}{r} y\right), & -\alpha < y < 0, \end{cases} \quad (12)$$

respectively, where  $a_k$ ,  $b_k$  and  $c_k$  are arbitrary constants.

Since the solutions  $u_k(x, y) = X_k(x) Y_k(y)$  are required to satisfy condition (2), the constants  $a_k$ ,  $b_k$  and  $c_k$  are chosen to satisfy the equality:

$$Y_k(0+0) = Y_k(0-0). \quad (13)$$

Condition (13) applied to (12) yields  $a_k = b_k$ . Thus, (12) takes the following form:

$$Y_k(y) = \begin{cases} a_k \exp\left(-\frac{a \mu_k^2 y}{1+b \mu_k^2}\right), & 0 < y < \beta, \\ a_k \cos\left(\frac{\pi k}{r} y\right) + c_k \sin\left(\frac{\pi k}{r} y\right), & -\alpha < y < 0. \end{cases} \quad (14)$$

Suppose now that  $u(x, y)$  is a solution to Problem 1. Following [22][p. 58] we introduce the functions

$$u_k(y) = \sqrt{\frac{2}{r}} \int_0^r u(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}. \quad (15)$$

Based on (15), we additionally introduce the functions

$$v_\varepsilon(y) = u_k^\varepsilon(y) = \sqrt{\frac{2}{r}} \int_\varepsilon^{r-\varepsilon} u(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}, \quad (16)$$

where  $\varepsilon$  is a sufficiently small positive number.

Differentiating (16) twice with respect to  $y$  yields

$$v'_\varepsilon(y) = \sqrt{\frac{2}{r}} \int_\varepsilon^{r-\varepsilon} u_y(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}, \quad (17)$$

$$v''_\varepsilon(y) = \sqrt{\frac{2}{r}} \int_\varepsilon^{r-\varepsilon} u_{yy}(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}. \quad (18)$$

Integrating by parts twice in (16) and (17) yields:

$$v_\varepsilon(y) = \sqrt{\frac{2}{r}} \int_\varepsilon^{r-\varepsilon} u(x, y) \sin(\mu_k x) dx = \frac{1}{\mu_k^2} \sqrt{\frac{2}{r}} [u(\varepsilon, y) \cos \mu_k \varepsilon - u(r-\varepsilon, y) \cos \mu_k (r-\varepsilon) + \\ + u_x(r-\varepsilon, y) \sin \mu_k (r-\varepsilon) - u_x(\varepsilon, y) \sin \mu_k \varepsilon - \int_\varepsilon^{r-\varepsilon} u_{xx}(x, y) \sin(\mu_k x) dx]; \quad (19)$$

$$\begin{aligned}
v'_\varepsilon(y) &= \sqrt{\frac{2}{r}} \int_{\varepsilon}^{r-\varepsilon} u_y(x, y) \sin(\mu_k x) dx = \\
&= \frac{1}{\mu_k^2} \sqrt{\frac{2}{r}} [u_y(\varepsilon, y) \cos \mu_k \varepsilon - u_y(r - \varepsilon, y) \cos \mu_k (r - \varepsilon) + \\
&\quad + u_{xy}(r - \varepsilon, y) \sin \mu_k (r - \varepsilon) - u_{xy}(\varepsilon, y) \sin \mu_k \varepsilon - \int_{\varepsilon}^{r-\varepsilon} u_{xxy}(x, y) \sin(\mu_k x) dx], \quad (20)
\end{aligned}$$

while (18) yields

$$v''_\varepsilon(y) = \sqrt{\frac{2}{r}} \int_{\varepsilon}^{r-\varepsilon} u_{yy}(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}. \quad (21)$$

Taking the limit as  $\varepsilon \rightarrow 0$  in equalities (17), (19), (20), (21) and using conditions (5)-(6) of Theorem 1, we find

$$u_k(y) = -\frac{1}{\mu_k^2} \sqrt{\frac{2}{r}} \int_0^r u_{xx}(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}, \quad (22)$$

$$u'_k(y) = \sqrt{\frac{2}{r}} \int_0^r u_y(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}, \quad (23)$$

$$u'_k(y) = -\frac{1}{\mu_k^2} \sqrt{\frac{2}{r}} \int_0^r u_{xxy}(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}, \quad (24)$$

$$u''_k(y) = \sqrt{\frac{2}{r}} \int_0^r u_{yy}(x, y) \sin\left(\frac{\pi k}{r} x\right) dx, \quad k \in \mathbb{N}. \quad (25)$$

Given the form of equation (1) for  $0 < y < \beta$ , equations (22)–(24) yield:

$$u'_k(y) + a\mu_k^2 u_k(y) + b\mu_k^2 u'_k(y) = \sqrt{\frac{2}{r}} \int_0^r [u_y - au_{xx} - bu_{xxy}] \sin\left(\frac{\pi k}{r} x\right) dx = 0.$$

Now, considering equation (1) for  $-\alpha < y < 0$  and assuming that  $f(x, y) \equiv 0 \forall (x, y) \in \bar{D}_1$ , from (22) and (25) obtain

$$u''_k(y) + \mu_k^2 u_k(y) = \sqrt{\frac{2}{r}} \int_0^r [u_{yy} - u_{xx}] \sin\left(\frac{\pi k}{r} x\right) dx = 0.$$

Thus, to determine the unknown function  $u_k(y)$  for  $y > 0$  and  $y < 0$  arrive at the corresponding equations:

$$(1 + b\mu_k^2) u'_k(y) + a\mu_k^2 u_k(y) = 0, \quad 0 < y < \beta, \quad (26)$$

$$u_k''(y) + \mu_k^2 u_k(y) = 0, \quad -\alpha < y < 0. \quad (27)$$

As shown above, the general solution to equations (26) and (27) is given by

$$u_k(y) = \begin{cases} a_k \exp\left(-\frac{a\mu_k^2 y}{1+b\mu_k^2}\right), & 0 < y < \beta, \\ a_k \cos(\mu_k y) + c_k \sin(\mu_k y), & -\alpha < y < 0. \end{cases} \quad (28)$$

To find the constants  $a_k$  and  $c_k$  to be determined, we apply conditions (3). Since  $\psi_1(x) = \psi_2(x) \equiv 0 \quad \forall x \in [0, r]$  equation (28) combined with (3) yields the system

$$\begin{cases} u_k(-\alpha) = a_k \cos(\mu_k \alpha) - c_k \sin(\mu_k \alpha) = 0, \\ u_k(\beta) = a_k \exp\left(-\frac{a\mu_k^2 \beta}{1+b\mu_k^2}\right) = 0. \end{cases} \quad (29)$$

System (29) implies that  $a_k = 0 \quad \forall k \in \mathbb{N}$ ,  $c_k \sin(\mu_k \alpha) = c_k \sin\left(\frac{\pi k \alpha}{r}\right) = 0$ ,  $k \in \mathbb{N}$ . Since the ratio  $\alpha/r$  is irrational, we have  $\sin\left(\frac{\pi k \alpha}{r}\right) \neq 0$  for all  $k \in \mathbb{N}$ , which implies  $c_k = 0 \quad \forall k \in \mathbb{N}$ . Then  $u_k(y) \equiv 0 \quad \forall k \in \mathbb{N}$ , and for all  $y \in [-\alpha, \beta]$ . From the latter, it follows for any  $y \in [-\alpha, \beta]$  that

$$\int_0^r u(x, y) \sin(\mu_k x) dx = 0, \quad k \in \mathbb{N}.$$

Hence, by virtue of the completeness of the eigenfunction system

$$\{X_k(x)\}_{k \in \mathbb{N}} = \left\{ \sqrt{\frac{2}{r}} \sin(\mu_k x) \right\}_{k \in \mathbb{N}} = \left\{ \sqrt{\frac{2}{r}} \sin\left(\frac{\pi k}{r} x\right) \right\}_{k \in \mathbb{N}}$$

in the space  $L_2[0, r]$  it follows that  $u(x, y) = 0$  almost everywhere on  $[0, r]$  for any  $y \in [-\alpha, \beta]$ . Since  $u(x, y) \in C(\bar{D})$ , by virtue of (2), this means that  $u(x, y) \equiv 0$  в  $\bar{D}$ . Theorem 1 is proved.  $\square$

## Existence of a Solution to Problem 1

Returning now to the initial inhomogeneous problem (1)–(4). We assume that the conditions listed in the above uniqueness theorem hold. Then, a solution to problem (1)–(4) is sought in the form

$$u(x, y) = \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} u_k(y) \sin\left(\frac{\pi k}{r} x\right). \quad (30)$$

As before, finding the coefficients  $u_k(y)$  to be determined for the inhomogeneous problem (1)–(3) reduces to the following system of ordinary differential equations:

$$\begin{cases} u_k''(y) + \mu_k^2 u_k(y) = f_k(y), & y < 0, \\ (1 + b\mu_k^2) u_k'(y) + a\mu_k^2 u_k(y) = 0, & y > 0, \end{cases} \quad (31)$$

where

$$f_k(y) = \sqrt{\frac{2}{r}} \int_0^r f(x, y) \sin(\mu_k x) dx. \quad (32)$$

The general solution to system (31) is given by

$$u_k(y) = \begin{cases} \frac{1}{\mu_k} \int_0^y f_k(t) \sin[\mu_k(y-t)] dt + A_k \cos(\mu_k y) + B_k \sin(\mu_k y), & y < 0, \\ C_k \exp\left(-\frac{a\mu_k^2}{1+b\mu_k^2} y\right), & y > 0, \end{cases} \quad (33)$$

where  $A_k, B_k, C_k$  are arbitrary constants to be determined,  $\mu_k = \frac{\pi k}{r}$ ,  $k \in \mathbb{N}$ .

To determine the unknown coefficients  $A_k, B_k, C_k$  in (33) we use the matching condition (2), as well as the boundary conditions (3). (3). Now we have:

$$\begin{cases} A_k = C_k, \\ \frac{1}{\mu_k} \int_{-\alpha}^0 f_k(t) \sin[\mu_k(\alpha+t)] dt + A_k \cos(\alpha\mu_k) - B_k \sin(\alpha\mu_k) = \psi_{1k}, \\ C_k \exp\left(-\frac{a\mu_k^2}{1+b\mu_k^2} \beta\right) = \psi_{2k}, \end{cases} \quad (34)$$

where

$$\psi_{1k} = \sqrt{\frac{2}{r}} \int_0^r \psi_1(x) \sin(\mu_k x) dx, \quad (35)$$

$$\psi_{2k} = \sqrt{\frac{2}{r}} \int_0^r \psi_2(x) \sin(\mu_k x) dx. \quad (36)$$

Solving system (34), we obtain

$$A_k = C_k = \psi_{2k} \exp\left(\frac{a\mu_k^2}{1+b\mu_k^2} \beta\right),$$

$$B_k = \frac{1}{\mu_k \sin(\alpha\mu_k)} \left[ \int_{-\alpha}^0 f_k(t) \sin[\mu_k(\alpha+t)] dt - \mu_k \psi_{1k} + \mu_k \exp\left(\frac{a\mu_k^2}{1+b\mu_k^2} \beta\right) \cos(\alpha\mu_k) \psi_{2k} \right].$$

Substituting the obtained values of  $A_k, B_k$ , and  $C_k$  into (33), yields

$$u_k(y) = \begin{cases} \frac{1}{\mu_k} \int_0^y f_k(t) \sin[\mu_k(y-t)] dt + \exp\left(\frac{a\mu_k^2}{1+b\mu_k^2} \beta\right) \cos(\mu_k y) \psi_{2k} - \frac{\sin(\mu_k y)}{\sin(\alpha\mu_k)} \psi_{1k} + \\ + \frac{\sin(\mu_k y)}{\mu_k \sin(\alpha\mu_k)} \left[ \int_{-\alpha}^0 f_k(t) \sin[\mu_k(\alpha+t)] dt + \mu_k \exp\left(\frac{a\mu_k^2}{1+b\mu_k^2} \beta\right) \cos(\alpha\mu_k) \psi_{2k} \right], & y < 0, \\ \exp\left(\frac{a\mu_k^2}{1+b\mu_k^2} (\beta-y)\right) \psi_{2k}, & y > 0. \end{cases}$$

Once the functions  $u_k(y)$  are found, the solution to the initial problem (1)-(3) can be formally expressed by (30).

Let us show that under appropriate conditions on the prescribed functions  $\psi_1(x), \psi_2(x)$  and  $f(x, y)$  the solution  $u = u(x, y)$ , defined by series (30) with the coefficients  $u_k(y)$  constitutes a regular solution to Problem 1.

Term-by-term differentiation of representation (30), yields

$$u_x(x, y) = \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} \frac{\pi k}{r} u_k(y) \cos\left(\frac{\pi k}{r} x\right),$$

$$\begin{aligned}
u_y(x, y) &= \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} u'_k(y) \sin\left(\frac{\pi k}{r} x\right), \\
u_{xx}(x, y) &= -\sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} u_k(y) \left(\frac{\pi k}{r}\right)^2 \sin\left(\frac{\pi k}{r} x\right), \\
u_{yy}(x, y) &= \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} u''_k(y) \sin\left(\frac{\pi k}{r} x\right), \\
u_{xy}(x, y) &= -\sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} u'_k(y) \left(\frac{\pi k}{r}\right)^2 \sin\left(\frac{\pi k}{r} x\right).
\end{aligned}$$

The following estimates hold for the function  $u = u(x, y)$  given by (30) and its aforementioned derivatives  $u_x$ ,  $u_y$ ,  $u_{xx}$ ,  $u_{yy}$ ,  $u_{xy}$ :

$$\begin{aligned}
|u(x, y)| &\leq \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} |u_k(y)|, \\
|u_x(x, y)| &\leq \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} \frac{\pi k}{r} |u_k(y)|, \\
|u_y(x, y)| &\leq \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} |u'_k(y)|, \\
|u_{xx}(x, y)| &\leq \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} \left(\frac{\pi k}{r}\right)^2 |u_k(y)|, \\
|u_{yy}(x, y)| &\leq \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} |u''_k(y)|, \\
|u_{xy}(x, y)| &\leq \sqrt{\frac{2}{r}} \sum_{k=1}^{\infty} \left(\frac{\pi k}{r}\right)^2 |u'_k(y)|.
\end{aligned}$$

The representation for the function  $u_k(y)$  obtained above implies the following estimates, given that the conditions of the aforementioned uniqueness theorem hold:

$$|u_k(y)| \leq \frac{C_1}{k} |f_k(y)| + C_2 |\psi_{1k}| + C_3 |\psi_{2k}|, \quad (37)$$

$$|u'_k(y)| \leq C_4 |f_k(y)| + C_5 k |\psi_{1k}| + C_6 k |\psi_{2k}|, \quad (38)$$

$$|u''_k(y)| \leq C_7 k |f_k(y)| + C_8 k^2 |\psi_{1k}| + C_9 k^2 |\psi_{2k}|, \quad (39)$$

where  $C_j$ ,  $j = \overline{1;9}$  – some constants independent of both the unknown function  $u = u(x, y)$  and the index  $k$ .

If the given functions  $\psi_1(x)$ ,  $\psi_2(x)$  and  $f(x, y)$  are such that

$$\psi_1(x) \in C^4[0, r], \quad \psi_1^{(i)}(0) = \psi_1^{(i)}(r) = 0, \quad i = \{0; 2\}; \quad (40)$$

$$\psi_2(x) \in C^4[0, r], \quad \psi_2^{(i)}(0) = \psi_2^{(i)}(r) = 0, \quad i = \{0; 2\}; \quad (41)$$

$$f(x, y) \in C^1(\overline{D_1}), f_{xxx} \in L_2(D_1), f(0, y) = f_{xx}(0, y) = f(r, y) = f_{xx}(r, y) = 0, \quad (42)$$

then, for the components  $|f_k(y)|$ ,  $|\psi_{1k}|$  and  $|\psi_{2k}|$  the following equalities hold by virtue of estimates (37)–(39):

$$|\psi_{1k}| = \frac{1}{\mu_k^4} |\psi_{1k}^{(4)}|, \quad |\psi_{2k}| = \frac{1}{\mu_k^4} |\psi_{2k}^{(4)}|, \quad (43)$$

$$|f_k(y)| = \frac{1}{\mu_k^3} |f_k^{(3)}(y)|, \quad (44)$$

where

$$\psi_{ik}^{(4)} = \sqrt{\frac{2}{r}} \int_0^r \psi^{(4)}(x) \sin(\mu_k x) dx, \quad i = \{1; 2\},$$

$$|f_k^{(3)}(y)| = \sqrt{\frac{2}{r}} \int_0^r f_{xxx}(x, y) \cos(\mu_k x) dx.$$

Applying integration by parts [23, p. 481] to (32), (35), and (36) under conditions (40)–(42) yields formulas (43)–(44).

Equalities (43) and (44) imply the absolute and uniform convergence of the majorizing series for the unknown function  $u = u(x, y)$  and its derivatives  $u_x$ ,  $u_y$ ,  $u_{xx}$ ,  $u_{yy}$ ,  $u_{xy}$  under the aforementioned conditions (40)–(42) on the given functions  $\psi_1(x)$ ,  $\psi_2(x)$ , and  $f(x, y)$ . By the Weierstrass M-test [23, p. 304], this absolute and uniform convergence ensures their regularity in the respective domains.

Hence, the following theorem holds.

**Theorem 2.** *Suppose that the functions  $\psi_1(x)$ ,  $\psi_2(x)$  and  $f(x, y)$  satisfy properties (40)–(42) and the conditions of Theorem 1. Then there exists a unique regular solution to Problem 1 in the domain  $D$ .*

## Conclusion

Within the framework of this study, we formulate and study the Dirichlet problem for a third-order mixed hyperbolic-pseudoparabolic equation in the rectangular domain  $D = \{(x, y) : 0 < x < r, -\alpha < y < \beta, \alpha, \beta > 0\}$ . Above the line of type change  $y = 0$ , the equation under consideration is a third-order pseudoparabolic equation (Aller's equation), whereas in the remaining part of the mixed domain, it becomes an inhomogeneous wave equation. As is well known, the boundary value problem with data given on the entire boundary (the Dirichlet problem) is a prime example of an ill-posed problem in the theory of mixed-type partial differential equations, specifically for the Lavrent'ev–Bitsadze equation [4]. Replacing one of the transmission conditions on the line of type change  $y = 0$  with a Dirichlet condition on a part of the domain boundary ensures the well-posedness of the Dirichlet problem for the equation under consideration. To study the posed problem, we used the spectral analysis method developed by K.B. Sabitov. Sufficient conditions have been found on the functions specified in the problem

formulation, ensuring the well-posedness of the investigated problem and the regularity of its solution.

## References

1. Frankl F. I. On the Problems of Chaplygin for Mixed Sub- and Supersonic Flows, *Mathematics of the USSR-Izvestiya*, 1945. vol. 9, no. 2, pp. 121–142 (In Russian).
2. Frankl F. I. *Osnovi Gazovoi Dinamiki [Selected Works on Gas Dynamics]*. M: Nauka, 1973. 771 pp. (In Russian)
3. Bourgin P. G., Duffin R. The Dirichlet problem the vibrating string equation, *Bulleten Amer. Math. Soc.*, 2016. vol. 45, no. 12, pp. 851–858.
4. Bitsadze A. V. On the Ill-Posedness of the Dirichlet Problem for Mixed-Type Equations, *Doklady Akademii Nauk SSSR*, 1953. vol. 122, no. 2, pp. 561–564 (In Russian).
5. Cannon J. R. Dirichlet problem for an equation of mixed type with a discontinuous coefficient, *Ann. Math. pura ed Appl.*, 2016. vol. 62, no. 4, pp. 371–377.
6. Nakhushev A. M. A Uniqueness Criterion for the Dirichlet Problem for a Mixed-Type Equation in a Cylindrical Domain, *Differents. Uravneniia*, 1970. vol. 6, no. 1, pp. 190–191 (In Russian).
7. Khachev M. M. The Dirichlet problem for the Tricomi equation in a rectangle, *Differents. Uravneniia*, 1975. vol. 11, no. 1, pp. 151–160 (In Russian).
8. Sokhadze P. S. On the first boundary value problem for a mixed-type equation in a rectangle, *Differents. Uravneniia*, 1983. vol. 19, no. 1, pp. 127–133 (In Russian).
9. Zhegalov V. I. A nonlocal Dirichlet problem for a mixed-type equation / *In Nonclassical Equations of Mathematical Physics*, IM SO AN SSSR, 1985, pp. 168–172 (In Russian).
10. Soldatov A. P. A Dirichlet-type problem for the Lavrent'ev–Bitsadze equation. II. Existence theorems., *Proceedings of the Russian Academy of Sciences*, 1993. vol. 333, no. 1, pp. 16–18 (In Russian).
11. Soldatov A. P. A Dirichlet-type problem for the Lavrent'ev–Bitsadze equation. I. Uniqueness theorems, *Proceedings of the Russian Academy of Sciences*, 1993. vol. 332, no. 6, pp. 696–698 (In Russian).
12. Soldatov A. P. A Dirichlet-type problem for the Lavrent'ev–Bitsadze equation, *Differents. Uravneniia*, 1994. vol. 30, no. 11, pp. 2001–2009 (In Russian).
13. Khachev M. M. Boundary value problems for mixed-type equations in canonical domains / *Proceedings of the All-Russian scientific and practical conference "Current problems and priority innovative technologies for the development of the regional agro-industrial complex"*, December 19–20, 2014.. Nalchik, KBSU named after V. M. Kokov, 2015, pp. 40–49 (In Russian).
14. Sabitov K. B. The Dirichlet problem for mixed-type equations with the Lavrent'ev–Bitsadze operator, *Vestnik Oshskogo gosudarstvennogo universiteta. Seriya: Matematika, fizika, tekhnika*, 2023. no. 1, pp. 194–196 (In Russian).
15. Sabitov K. B. The Dirichlet problem for equations of mixed type in a rectangular domain, *Doklady RAN*, 2007. vol. 413, no. 1, pp. 23–26 (In Russian).
16. Sabitov K. B., Melisheva, E. P. The Dirichlet problem for a loaded mixed-type equation in a rectangular domain, *Izvestiya Vysshikh Uchebnykh Zavedeniy. Matematika*, 2013. no. 7, pp. 62–76 (In Russian).
17. Martem'anova N. V. The Dirichlet problem for an equation of mixed elliptic-hyperbolic type with variable potential, *Izvestiya Vysshikh Uchebnykh Zavedeniy. Matematika*, 2015. no. 11, pp. 44–53 (In Russian).
18. Khachev M. M. *Zadacha Dirikhle dlya odnogo klassa uravneniy smeshannogo tipa v pryamougol'noy i v tsilindricheskoy oblastiakh [The Dirichlet problem for a class of mixed-type equations in rectangular and cylindrical domains]*. Nalchik, 1976. 75 pp. (In Russian)
19. Melisheva E. P. *Zadacha Dirikhle dlya nagruzhennykh uravneniy smeshannogo elliptiko-giperbolicheskogo tipa v pryamougol'noy oblasti [The Dirichlet problem for loaded equations of mixed elliptic-hyperbolic type in a rectangular domain]*. Kazan, 2013. 93 pp. (In Russian)
20. Khachev M. M. *Pervaya krayevaya zadacha dlya lineynykh uravneniy smeshannogo tipa v kanonicheskikh oblastiakh [The first boundary value problem for linear mixed-type equations in canonical domains]*. Nalchik: Elbrus, 1998. 170 pp. (In Russian)

21. Sabitov K. B. *K teorii uravneniy smeshannogo tipa [On the theory of mixed-type equations]*. M.: Fizmatlit, 2014. 304 pp. (In Russian)
22. Sabitov K. B. *Pryamyye i obratnyye zadachi dlya uravneniy smeshannogo parabologiperbolicheskogo tipa [Direct and inverse problems for mixed parabolic-hyperbolic equations]*. M.: Nauka, 2016. 272 pp. (In Russian)
23. Budak B. M., Fomin S. V. *Kratnyye integraly i ryady [Multiple Integrals and Series]*. M.: Nauka, 1967. 608 pp. (In Russian)

#### Information about the author



*Balkizov Zhiraslan Anatolyevich* ✉ – PhD. (Phys.& Math.),  
Leading Researcher, Department of Mixed-Type Equations,  
Institute of Applied Mathematics and Automation, Kabardino-  
Balkarian Scientific Center, Russian Academy of Sciences, Russia,  
 ORCID 0000-0001-5329-7766.