



Global Solvability of the Inverse Problem of Determining the Kernel for a Pseudoparabolic Equation with Overdetermination at an Interior Point

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Abstract. This paper considers the inverse problem of determining the kernel of the integral term in an integro-differential pseudoparabolic equation with distributed data over a bounded domain. First, the direct problem is studied. The direct problem represents the first initial-boundary value problem for this integro-differential equation. For this direct problem, the inverse problem is studied, which consists of determining the kernel of the integral under a given additional condition regarding the solution of the direct problem. Using the semi-inversion method, this problem is replaced by an equivalent system of integral equations for the unknown functions, and, based on a contraction mapping, the global unique solvability of this inverse problem in the space of continuous functions with weighted norms is proved.

Key words: pseudoparabolic equation, integro-differential equation, inverse problem, semi-inversion method, contraction mapping principle.

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Глобальная разрешимость обратной задачи определения ядра для псевдопараболического уравнения с переопределением во внутренней точке

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Аннотация. Рассматривается обратная задача определения ядра интегрального члена в интегро-дифференциальном псевдопараболическом уравнении с распределенными данными в ограниченной области. Сначала изучается прямая задача. Прямая задача представляет собой первую начальную краевую задачу для этого интегро-дифференциального уравнения. Для этой прямой задачи изучаются обратная задача, заключающиеся в определении ядра интеграла при заданной дополнительной условии относительно решения прямой задачи. Используя метод полуобращения, эта задача заменяются эквивалентной системой интегральных уравнений относительно неизвестных функций и на основе сжимающего отображения доказывается глобальная однозначная разрешимость рассматриваемой обратной задачи в пространстве непрерывных функций с весовыми нормами.

Ключевые слова: псевдопараболическое уравнение, интегро-дифференциальное уравнение, обратная задача, метод полуобращения, принцип сжимающего отображения.

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Introduction

Inverse problems involve the parameters of differential equations (coefficients, right-hand sides, conditions, or boundaries) from additional data on the solution of a direct problem. Their importance stems from a wide range of applications—from geophysics and mineral exploration to biomedicine and industrial flaw detection—making this area one of the most relevant in modern mathematic.

The theory of inverse problems for second-order equations is described in detail in classical monographs [1–9].

Since the late 1980s, the study of inverse problems for integro-differential equations of hyperbolic type has been developing independently. The introduction of Volterra convolution operators into mathematical models makes it possible to take into account the "memory" or delay effects characteristic of the propagation of elastic and electromagnetic waves in real media. The development of this field is associated with the names of such researchers as A. L. Bukhgeim [10], Bukhgeim A.L, Kalinina N. I. [11], Lorenzi A. [12–14], Grasselli M., S. I. Kabanikhin [15], J. Yanno [16], Romanov V.G. [17, 18], Romanov V.G., Yamamoto M. [19], Durdiev D.K. [20] and others. Similar problems for heat equations under various overdetermination regimes are also considered in detail in [21, 22].

In a recently published monograph [20], a new class of inverse problems related to the determination of the convolution kernel in second-order hyperbolic integro-differential equations is investigated

Inverse problems of determining the kernel in the integro-differential equation of heat conductivity under various overdetermination conditions were considered in a number of works (see [21, 22]).

Inverse problems arise in a wide variety of scientific fields, such as seismology, mineral exploration, biology, medicine, industrial quality control, and so on, making them among the most pressing problems in modern mathematics.

Inverse problems for hyperbolic integro-differential equations are a relatively new field of inverse problem theory, emerging in the late 1980s. Under certain conditions, Volterra convolution operators are added to the equations of propagation of elastic and or electromagnetic waves, which describe the phenomena of "delay" or memory.

The study of inverse problems for pseudoparabolic equations began in the 1980s. The first significant result was obtained in the work of W. Rundel (see [23]), where the semigroup method was used to study two inverse source determination problems depending on space and time, respectively. Then, starting in the early 1982s, inverse problems for pseudoparabolic equations began to be intensively studied by Kyrgyz mathematicians in various formulations and with various overdetermination conditions. The results of these studies are reflected in the monographs [24–26] (see also the references therein).

Inverse problems for third-order equations are studied in the monographs of B.S. Ablabekov (see [25, 26]), A. Asanov, E.R. Atamanov [27]. Inverse problems for integro-differential pseudo parabolic equations are investigated in the works [24, 29–31].

Problem statement

Let us consider the initial-boundary value problem

$$u_t(x, t) - u_{xxt}(x, t) - u_{xx}(x, t) = \int_0^t k(\tau)u(x, t - \tau)d\tau + f(x, t), \quad (x, t) \in \Omega_T, \quad (1)$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l, \quad (2)$$

$$u_x(0, t) = \mu_1(t), \quad u_x(l, t) = \mu_2(t), \quad 0 \leq t \leq T, \quad (3)$$

where $f(x, t)$, $\mu_1(t)$, $\mu_2(t)$, $\varphi(x)$ are given functions.

Inverse problem: find pairs of functions $u(x, t)$ and $k(t)$, if relative to the solution of the direct problem (1) – (3), the overdetermination condition at the interior point is known

$$u(x_0, t) = \psi(t), \quad 0 < x_0 < l, \quad 0 \leq t \leq T, \quad (4)$$

where $\psi(t)$ is a known function, l and T are arbitrary positive numbers.

The problem of determining the function $u(x, t) \in C^{(2,1)}(\Omega_T)$ from conditions (1)–(3) for given numbers l, T and sufficiently smooth functions $f(x, t), k(t), \varphi(x), \mu_1(t), \mu_2(t)$ is called the direct problem.

Definition 1. We call the function $u(x, t) \in C^{(2,1)}(\Omega_T)$ a classical solution of the direct problem (1) – (3) if, in the classical sense, it satisfies equation (1), the initial condition (2) and the boundary conditions (3).

Definition 2. The solution of the inverse problem (1) – (4) is called a pair of functions $u(x, t), k(t)$ such that $k(t) \in C[0, T]$, $u(x, t) \in C^{(2,1)}(\Omega_T) \cap C(\bar{\Omega}_T)$, and the functions $u(x, t), k(t)$ satisfy conditions (1) – (4).

Investigation of the direct problem

Before moving on to the formulation of the inverse problem, let us formulate the properties of the solution of the direct problem (1)–(3).

Theorem 1. Let $\varphi(x) \in C^2[0, l]$, $k(t) \in C[0, T]$, $\mu_1(t), \mu_2(t) \in C^1[0, T]$, $f(x, t) \in C^{(0,1)}(\bar{\Omega}_T)$ and satisfy the following approval conditions: $\varphi(0) = \mu_1(0)$, $\varphi(l) = \mu_2(0)$ and $\varphi'(0) = \varphi'(l) = 0$. Then there exists a unique solution to problem (1)–(3).

Proof. Let's make a substitution

$$u(x, t) = v(x, t) + \chi(x, t), \quad (5)$$

where $\chi(x, t) = x\mu_1(t) + \frac{x^2}{2l}(\mu_2(t) - \mu_1(t))$ is a given function, and $v(x, t)$ is a new unknown function.

Then the inverse problem (1) – (4) is reduced to the problem

$$v_t - v_{xxt} - v_{xx} = \int_0^t k(s)v(x, t - s)ds + \int_0^t k(s)\chi(x, t - s)ds + F(x, t), \quad (x, t) \in \Omega_T, \quad (6)$$

$$v(x, 0) = \bar{\varphi}(x), \quad 0 \leq x \leq l, \quad (7)$$

$$v_x(0, t) = v_x(l, t) = 0, \quad 0 \leq t \leq T, \quad (8)$$

$$v(x_0, t) = \tilde{\psi}(t), \quad 0 \leq t \leq T, \quad (9)$$

where

$$F(x, t) = f(x, t) - (\chi_t - \chi_{xxt} - \chi_{xx}),$$

$$\bar{\varphi}(x) = \varphi(x) - \chi(x, 0),$$

$$\tilde{\psi}(t) = \psi(t) - \chi(x_0, t).$$

Let us introduce the function $v(x, t) = e^{-t}w(x, t)$. Then problem (7) - (8) will take the form

$$w_t - w_{xxt} = w + \int_0^t e^{t-s}k(s)w(x, t-s)ds + e^t \int_0^t k(s)\chi(x, t-s)ds + e^t F(x, t), \quad (x, t) \in \Omega_T, \quad (10)$$

$$w(x, 0) = \bar{\varphi}(x), \quad 0 \leq x \leq l, \quad (11)$$

$$w_x(0, t) = w_x(l, t) = 0, \quad 0 \leq t \leq T, \quad (12)$$

We replace problem (10)-(12) with an equivalent integral equation. From (10)-(12), inverting the operator $-d^2/dx^2 + I$ and integrating the resulting relation over the variable t from 0 to t , we obtain

$$w(x, t) = F_1(x, t) + \int_0^t \int_0^l G(x, \xi) e^\tau w(\xi, \tau) d\xi d\tau + \int_0^t \int_0^l G(x, \xi) \int_0^\tau e^{\tau-s} k(s) w(\xi, \tau-s) ds d\xi d\tau, \quad (13)$$

where

$$G(x, \xi) = \frac{1}{\operatorname{sh} l} \begin{cases} \operatorname{ch} x \operatorname{ch}(l - \xi), & 0 \leq x \leq \xi, \\ \operatorname{ch}(l - x) \operatorname{ch} \xi, & \xi \leq x \leq l. \end{cases} \quad (14)$$

$$F_1(x, t) = \int_0^l G(x, \xi) \left[e^t \int_0^t k(s)\chi(\xi, t-s)ds \right] d\xi + \bar{\varphi}(x) + \int_0^t \int_0^l G(x, \xi) e^\tau F(\xi, \tau) d\xi d\tau,$$

Since $\varphi(x) \in C^2[0, l]$, $k(t) \in C[0, T]$, $\mu_1(t), \mu_2(t) \in C^1[0, T]$, $f(x, t) \in C^{(0,1)}(\bar{\Omega}_T)$, then the expression $F_1(x, t) \in C^{(2,1)}(\bar{\Omega}_T)$, therefore, the expression on the right side of formula (13) has derivatives of the second order with respect to x and of the first order with respect to t , i.e. $w(x, t) \in C^{(2,1)}(\bar{\Omega}_T)$.

Using the property of the Green's $G(x, \xi)$ function, it is easy to show that the function $w(x, t)$ satisfies conditions (10)-(12) (see [1]).

Let us prove the existence of a continuous solution to equation (13).

Let us define a sequence of functions $w_n(x, t)$, $n \geq 1$ by the equalities

$$w_n(x, t) = \int_0^t \int_0^l G(x, \xi) e^\tau w_{n-1}(\xi, \tau) d\xi d\tau + \int_0^t \int_0^l G(x, \xi) \int_0^\tau e^{\tau-s} k(s) w_{n-1}(\xi, \tau-s) ds d\xi d\tau, \quad (15)$$

$$w_0(x, t) = 0.$$

Let us introduce the notation

$$\varphi_0 = \max_{0 \leq x \leq l} |\varphi(x)|, \quad \mu_0 = \max \left\{ \max_{0 \leq t \leq T} |\mu_1(t)|, \max_{0 \leq t \leq T} |\mu_2(t)|, \max_{0 \leq t \leq T} |\mu'_1(t)|, \max_{0 \leq t \leq T} |\mu'_2(t)| \right\},$$

$$\|k\| = \max_{0 \leq t \leq T} |k(t)|, \quad f_0 = \max_{(x,t) \in \bar{\Omega}_T} |f(x,t)|.$$

To do this, we estimate the function $w_0(x, t)$.

Since

$$\int_0^l G(x, \xi) d\xi = 1, \quad (16)$$

we obtain

$$\begin{aligned} |F_1(x, t)| &\leq \left| \int_0^l G(x, \xi) \left[e^t \int_0^t k(s) \chi(\xi, t-s) ds \right] d\xi \right| + |\bar{\varphi}| + |f(x, t)| + \\ &+ |\chi_t - \chi_{xxt} - \chi_{xx}| \leq (l + l^2) \mu_0 \|k\| e^T T + f_0 + \bar{\varphi}_0 + \left| \frac{3}{2} + \frac{l^2-1}{2} + 2 \right| \mu_0 \equiv M. \end{aligned} \quad (17)$$

Here we used $e^t \leq e^T$ and equality (16).

Then

$$|w_1(x, t) - w_0(x, t)| \leq |F_1(x, t)| \equiv M,$$

$$\begin{aligned} |w_2(x, t) - w_1(x, t)| &\leq \int_0^t \int_0^l G(x, \xi) e^\tau |w_1(\xi, \tau) - w_0(\xi, \tau)| d\xi d\tau + \\ &+ \int_0^t \int_0^l G(x, \xi) \int_0^\tau e^{\tau-s} |k(s)| |w_1(\xi, \tau-s) - w_0(\xi, \tau-s)| ds d\xi d\tau \leq (e^T - 1)M + M \|k\| \frac{t^2}{2!}, \\ |w_3(x, t) - w_2(x, t)| &\leq \int_0^t \int_0^l G(x, \xi) e^\tau |w_2(\xi, \tau) - w_1(\xi, \tau)| d\xi d\tau + \\ &+ \int_0^t \int_0^l G(x, \xi) \int_0^\tau e^{\tau-s} |k(s)| |w_2(\xi, \tau-s) - w_1(\xi, \tau-s)| ds d\xi d\tau \leq \frac{(e^T - 1)^2}{2!} M + M \|k\|^2 \frac{t^4}{4!}. \end{aligned}$$

Then for an arbitrary n the following estimate holds:

$$|w_n(x, t) - w_{n-1}(x, t)| \leq \frac{(e^T - 1)^{n-1}}{(n-1)!} M + M \|k\|^2 \frac{t^{2(n-1)}}{2(n-1)!}. \quad (18)$$

From the above estimate (18) it follows that the series

$$\sum_{n=0}^{\infty} [w_n(x, t) - w_{n-1}(x, t)]$$

converges in region Ω_T .

From estimate (18), it follows that the sequence of functions $w_n(x, t)$, continuous in Ω_T , uniformly converges to a function $w(x, t)$, continuous in Ω_T . Passing to the limit in equality (15) at $n \rightarrow \infty$, we find that $w(x, t)$ is a solution to equation (13). Thus, the existence of a continuous solution to equation (13) is proven.

Let us prove the uniqueness of the solution to equation (13). Let the functions $w_1(x, t), w_2(x, t)$ be continuous in Ω_T and satisfy equation (13). Then for their difference $Z(x, t) = w_1(x, t) - w_2(x, t)$ the estimate holds

$$|Z(x, t)| \leq \int_0^t \int_0^l G(x, \xi) e^\tau |Z(\xi, \tau)| d\xi d\tau + \int_0^t \int_0^l G(x, \xi) \int_0^\tau e^{\tau-s} |k(s)| |Z(\xi, \tau-s)| ds d\xi d\tau.$$

We introduce the function $V(t) = \max_{0 \leq x \leq l} |Z(x, t)|$, $0 \leq t \leq T$. From the estimate follows the inequality

$$|V(t)| \leq \int_0^t (e^\tau + k_0(1 - e^\tau)) |V(\tau)| d\tau, \quad 0 \leq t \leq T,$$

and, therefore, $V(t) = 0$ for $0 \leq t \leq T$. Consequently, $w_1(x, t) = w_2(x, t)$ in Ω_T . Now, sequentially, we find the function $v(x, t)$, then, using formula (5), the function $u(x, t)$. Theorem 1 is proven. $\square$

Study of the inverse problem

Let us consider instead of the inverse problem (1)–(4) the inverse problem (6)–(9). Let us rewrite equation (6) in the form

$$v_t - v_{xxt} - v_{xx} = \int_0^t k(s) \chi(x, t-s) ds + \int_0^t k(s) v(x, t-s) ds + F(x, t), \quad (x, t) \in \Omega_T. \quad (19)$$

Applying the operator $\partial/\partial t$ to both parts of equation (11), and introducing the function $\vartheta = v_t$, from (11) – (14) we obtain

$$\begin{aligned} \vartheta_t - \vartheta_{xxt} - \vartheta_{xx} &= k(t)(\bar{\varphi}(x) + \chi(x, 0)) + \int_0^t k(\tau) \chi_t(x, t-\tau) d\tau + \\ &+ \int_0^t k(s) \vartheta(x, t-s) ds + F_t(x, t), \end{aligned} \quad (20)$$

$$\vartheta(x, 0) = \vartheta_0(x), \quad 0 \leq x \leq l, \quad (21)$$

$$\vartheta(0, t) = \vartheta(l, t) = 0, \quad 0 \leq t \leq T, \quad (22)$$

$$\vartheta(x_0, t) = \bar{\psi}'(t), \quad 0 \leq t \leq T. \quad (23)$$

Here $\vartheta_0(x)$ is the solution to the boundary value problem

$$-\vartheta_{xx}(x, 0) + \vartheta(x, 0) = \bar{\varphi}(x) + F(x, 0), \quad \vartheta_x(0, 0) = \vartheta_x(l, 0) = 0,$$

which has the form

$$\vartheta_0(x) = \int_0^l G(x, \xi) [\bar{\varphi}(\xi) + F(\xi, 0)] d\xi,$$

where $G(x, \xi)$ is determined by formula (14). Let us introduce the function $\vartheta(x, t) = e^{-t}\theta(x, t)$. Then the inverse problem (20) – (23) takes the form

$$\begin{aligned} (-\theta_{xx} + \theta)_t &= \theta + e^t k(t) \varphi(x) + \int_0^t k(\tau) e^{(t-\tau)} \theta(x, t-\tau) d\tau + \\ &+ e^t \int_0^t k(s) \chi_t(x, t-s) ds + e^t F_t(x, t), \end{aligned} \quad (24)$$

$$\theta(x, 0) = \vartheta_0(x), \quad (25)$$

$$\theta_x(0, t) = \theta_x(l, t) = 0, \quad 0 \leq t \leq T, \quad (26)$$

$$\theta(x_0, t) = e^t \bar{\psi}'(t), \quad (27)$$

We replace the inverse problem (24)–(27) with an equivalent system of nonlinear integral equations. From (24), (26), inverting the operator $-d^2/dx^2 + I$, we obtain

$$\begin{aligned} \theta_t(x, t) = & \int_0^l G(x, \xi) (\theta(\xi, t) + e^t k(t) \varphi(\xi)) d\xi + \\ & + \int_0^l G(x, \xi) \int_0^t k(\tau) e^{(t-\tau)} \theta(\xi, t - \tau) d\tau d\xi + \\ & + \int_0^l G(x, \xi) e^t \int_0^t k(s) \chi_t(\xi, t - \tau) d\tau d\xi + e^t \int_0^l G(x, \xi) F_t(\xi, t) d\xi, \end{aligned} \quad (28)$$

To obtain an additional condition, we set $x = x_0$ in equation (28) and use condition (27). Then, after simple transformations, we obtain

$$\begin{aligned} (\bar{\psi}''(t) + \bar{\psi}'(t)) = & \eta(x_0) k(t) + e^{-t} \int_0^l G(x_0, \xi) \theta(\xi, t) d\xi + \\ & + \int_0^l G(x_0, \xi) \int_0^t k(\tau) e^{-\tau} \theta(\xi, t - \tau) d\tau d\xi + \\ & + \int_0^l G(x_0, \xi) \int_0^t k(s) \chi_t(x_0, t - s) ds d\xi + \int_0^l G(x_0, \xi) F_t(\xi, t) d\xi, \end{aligned} \quad (29)$$

where

$$\eta(x_0) = \int_0^l G(x_0, \xi) \varphi(\xi) d\xi.$$

Since $\varphi(x_0) \neq 0$, then according to the theorem on the mean value of the integral, and also considering (17), we have $\eta(x_0) \neq 0$. From relation (29) we obtain

$$\begin{aligned} k(t) = & k_0(t) - \frac{1}{\eta(x_0)} \int_0^l G(x_0, \xi) \theta(\xi, t) d\xi - \\ & - \frac{1}{\eta(x_0)} \int_0^l G(x_0, \xi) \int_0^t k(\tau) e^{-\tau} \theta(\xi, t - \tau) d\tau d\xi - \\ & - \frac{1}{\eta(x_0)} \int_0^l G(x_0, \xi) \int_0^t k(s) \chi_t(x_0, t - s) ds d\xi, \end{aligned} \quad (30)$$

where

$$k_0(t) = \frac{1}{\eta(x_0)} (\bar{\psi}''(t) + \bar{\psi}'(t)) - \frac{1}{\eta(x_0)} \int_0^l G(x_0, \xi) F_t(\xi, t) d\xi.$$

On the other hand, integrating relation (28) with respect to the variable t from 0 to t , we obtain

$$\begin{aligned} \theta(x, t) = & \theta_0(x, t) + \int_0^t \int_0^l G(x, \xi) (\theta(\xi, \tau) + e^\tau k(\tau) \varphi(\xi)) d\xi d\tau + \\ & + \int_0^t \int_0^l G(x, \xi) \int_0^\tau k(s) e^{(\tau-s)} \theta(\xi, \tau - s) ds d\xi d\tau + \\ & + \int_0^t \int_0^l e^\tau G(x, \xi) \int_0^\tau k(s) \chi_t(\xi, t - s) ds d\xi d\tau, \end{aligned} \quad (31)$$

where

$$\theta_0(x, t) = \int_0^t e^\tau \int_0^l F_\tau(\xi, \tau) d\xi d\tau + \vartheta_0(x).$$

System (30), (31) is a system of nonlinear integral equations of the second kind with respect to functions $\theta(x, t), k(t)$.

To do this, we write this system of equations as an operator equation

$$z = Az, \quad (32)$$

where z is a vector function of variables x, t with components z_1, z_2 , and

$$z_1(x, t) = \theta(x, t), \quad z_2(x, t) = z_2(t) = k(t),$$

and the operator A is defined on the set of functions $z \in C(\bar{\Omega}_T)$ and, in accordance with equalities (30), (31), has the form $A = (A_1, A_2)$:

$$\begin{aligned} A_1 z = & \theta_0(x, t) + \int_0^t \int_0^l G(x, \xi) (z_1(\xi, \tau) + e^\tau z_2(\tau) \bar{\varphi}(\xi)) d\xi d\tau + \\ & + \int_0^t \int_0^l G(x, \xi) \int_0^\tau z_2(s) e^{-s} z_1(\xi, \tau - s) ds d\xi d\tau + \\ & + \int_0^t \int_0^l e^\tau G(x, \xi) \int_0^\tau z_2(s) \chi_t(\xi, t - s) ds d\xi d\tau; \end{aligned} \quad (33)$$

$$\begin{aligned} A_2 z(t) = & k_0(t) - \frac{1}{\eta(x_0)} \int_0^l G(x_0, \xi) z_1(\xi, t) d\xi - \\ & - \frac{1}{\eta(x_0)} \int_0^l G(x_0, \xi) \int_0^t z_2(\tau) e^{-\tau} z_1(\xi, t - \tau) d\tau d\xi - \\ & - \frac{1}{\eta(x_0)} \int_0^l G(x_0, \xi) \int_0^t \chi_t(\xi, t - s) z_2(s) ds d\xi. \end{aligned} \quad (34)$$

Let C_σ be the family of Banach spaces of continuous functions equipped with a family of weighted norms

$$\|z\|_\sigma = \max \left\{ \sup_{(x,t) \in \bar{\Omega}_T} |z_1(x, t) e^{-\sigma t}|, \sup_{t \in [0, T]} |z_2(t) e^{-\sigma t}| \right\} \equiv \max \{ \|z_1\|_\sigma, \|z_2\|_\sigma \}, \quad \sigma \geq 0.$$

It is obvious that at $\sigma = 0$ this space coincides with the space of continuous functions with the usual norm $\|z\|$.

Consider in the space C_σ the set $S_\sigma(z_0, r) = \{z : \|z - z_0\|_\sigma \leq r\}$ – a ball of radius r with center at the point z_0 of the weight space $C_\sigma(\Omega_T)$.

Note that for $z(x, t) \in S_\sigma(z_0, r)$ the following estimate is valid:

$$\|z\|_\sigma \leq \|z_0\|_\sigma + r \leq 2r.$$

Note that for each $t \in (0, T]$ the weighted norm $\|\cdot\|_\sigma$ is equivalent to the ordinary norm $\|\cdot\|$:

$$e^{-\sigma t} \|z\| \leq \|z\|_\sigma \leq \|z\|, \quad \sigma \geq 0.$$

The number σ will be chosen later.

Note that the convolution operator is commutative and invariant under multiplication by $e^{-\sigma t}$:

$$\begin{aligned}(g_1 * g_2)(t) &= (g_2 * g_1)(t), \\ e^{-\sigma t}(g_1 * g_2)(t) &= (e^{-\sigma t}g_1(t)) * (e^{-\sigma t}g_2(t)) \\ \|g_1 * g_2\|_\sigma &\leq \|g_1\|_\sigma \|g_2\|_\sigma.\end{aligned}$$

Let $z(x, t) \in S_\sigma(z_0, r)$ and we will show that with the appropriate choice the operator A maps the ball $S_\sigma(z_0, r)$ into itself, i.e. $Az \in S_\sigma(z_0, r)$.

Indeed, from (33), (34), composing the norm of differences for $(x, t) \in \Omega_T$, we find

$$\begin{aligned}\|A_1 z - z_{01}\|_\sigma &= \sup_{(x,t) \in \bar{\Omega}_T} |(A_1 z - z_{01})e^{-\sigma t}| \leq \\ &\leq \sup_{(x,t) \in \bar{\Omega}_T} \left| e^{-\sigma t} \int_0^t \int_0^l G(x, \xi) (z_1(\xi, \tau) + e^\tau z_2(\tau) \bar{\varphi}(\xi)) d\xi d\tau \right| + \\ &+ \sup_{(x,t) \in \bar{\Omega}_T} \left| e^{-\sigma t} \int_0^t \int_0^l G(x, \xi) \int_0^\tau e^{(\tau-s)} z_2(s) z_1(\xi, \tau-s) ds d\xi d\tau \right| + \\ &+ \sup_{(x,t) \in \bar{\Omega}_T} \left| e^{-\sigma t} \int_0^t \int_0^l e^\tau G(x, \xi) \int_0^\tau z_2(s) \chi_t(\xi, \tau-s) ds d\xi d\tau \right| \leq \\ &\leq \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^t \int_0^l G(x, \xi) [z_1(\xi, \tau) + e^\tau z_2(\tau) \bar{\varphi}(\xi)] e^{-\sigma \tau} e^{-\sigma(t-\tau)} d\xi d\tau \right| + \\ &+ \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^t e^{-\sigma(t-\tau)} \int_0^l G(x, \xi) \left(\int_0^\tau e^{(\tau-s)} e^{-\sigma s} z_2(s) z_1(\xi, \tau-s) e^{-\sigma(\tau-s)} ds \right) d\xi d\tau \right| + \\ &+ \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^t \int_0^l e^{-\sigma(t-\tau)} e^\tau G(x, \xi) \int_0^\tau e^{-\sigma s} z_2(s) e^{-\sigma(\tau-s)} \chi_t(\xi, \tau-s) ds d\xi d\tau \right|; \\ I_{11} &= \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^t \int_0^l G(x, \xi) [z_1(\xi, \tau) + e^\tau z_2(\tau) \bar{\varphi}(\xi)] e^{-\sigma \tau} e^{-\sigma(t-\tau)} d\xi d\tau \right| \leq \\ &\leq \frac{1}{\sigma} [(\|z_0\|_\sigma + r) + (\|z_0\|_\sigma + r)e^T |\varphi|] \leq \frac{(\|z_0\|_\sigma + r)}{\sigma} (1 + e^T \varphi_0); \\ I_{12} &= \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^t e^{-\sigma(t-\tau)} \int_0^l G(x, \xi) \left(\int_0^\tau e^{(\tau-s)} e^{-\sigma s} z_2(s) z_1(\xi, \tau-s) e^{-\sigma(\tau-s)} ds \right) d\xi d\tau \right| \leq \\ &\leq \sup_{0 \leq t \leq T} \left| \int_0^t e^{-\sigma(t-\tau)} (e^\tau - 1) \|z_2\|_\sigma \|z_1\|_\sigma d\tau \right| \leq \frac{(e^T - 1)}{\sigma} (\|z_0\|_\sigma + r)^2 \leq \frac{(\|z_0\|_\sigma + r)^2 e^T}{\sigma}; \\ I_{13} &= \sup_{0 \leq t \leq T} \left| \int_0^t \int_0^l e^\tau G(x, \xi) \int_0^\tau e^{-\sigma s} z_2(s) e^{-\sigma(\tau-s)} \chi_t(\xi, \tau-s) ds d\xi d\tau \right| \leq \\ &\leq \left(\frac{\|z_0\|_\sigma + r}{\sigma} \right) (1 + l^2) (e^T - 1) \mu_0 \leq \frac{\|z_0\|_\sigma + r}{\sigma} (1 + l^2) e^T \mu_0.\end{aligned}$$

This means,

$$\|A_1 z - z_{01}\|_\sigma \leq \frac{(\|z_0\|_\sigma + r)}{\sigma} (1 + e^T \varphi_0) + \frac{(\|z_0\|_\sigma + r)^2 e^T}{\sigma} + \frac{\|z_0\|_\sigma + r}{\sigma} (1 + l^2) e^T \mu_0;$$

If we choose σ as follows

$$\sigma \geq \sigma_1 = \frac{r}{(\|z_0\|_\sigma + r) (1 + e^T \varphi_0 + (\|z_0\|_\sigma + r) e^T + (1 + l^2) e^T \mu_0)},$$

Then $\|A_1 z - z_{01}\|_\sigma \leq r$, i.e., the first condition of contractive mapping for A_1 is satisfied. Now let's evaluate $\|A_2 z - z_{02}\|_\sigma$:

$$\begin{aligned} \|A_2 z - z_{02}\|_\sigma &= \sup_{(x,t) \in \bar{\Omega}_T} |(A_2 z - z_{02})e^{-\sigma t}| \leq \frac{1}{\eta(x_0)} \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^l G(x_0, \xi) e^{-\sigma t} z_1(\xi, t) d\xi \right| + \\ &+ \frac{1}{\eta(x_0)} \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\sigma t} z_2(\tau) e^{-\tau} z_1(\xi, t - \tau) d\tau d\xi \right| + \\ &+ \frac{1}{\eta(x_0)} \sup_{0 \leq t \leq T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\sigma t} \chi_t(\xi, t - s) z_2(s) ds d\xi \right| \leq I_{21} + I_{22} + I_{23} \end{aligned}$$

$$I_{21} = \frac{1}{\eta(x_0)} \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^l G(x_0, \xi) e^{-\sigma t} z_1(\xi, t) d\xi \right| \leq \frac{1}{\eta(x_0)} \|z_1\|_\sigma \leq \frac{r + \|z_0\|_\sigma}{\eta(x_0)};$$

$$\begin{aligned} I_{22} &= \frac{1}{\eta(x_0)} \sup_{0 \leq t \leq T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\sigma \tau} z_2(\tau) e^{-\tau} e^{-\sigma(t-\tau)} z_1(\xi, t - \tau) d\tau d\xi \right| \leq \frac{1}{\eta(x_0)} \|z_2\|_\sigma \|z_1\|_\sigma \leq \\ &\leq \frac{(r + \|z_0\|_\sigma)^2}{\eta(x_0)}; \end{aligned}$$

$$I_{23} = \frac{1}{\eta(x_0)} \sup_{0 \leq t \leq T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\sigma t} \chi_t(\xi, t - s) z_2(s) ds d\xi \right| \leq \frac{(r + \|z_0\|_\sigma)}{\sigma \eta(x_0)} (1 + l^2) \mu_0.$$

Accordingly, we obtain

$$\|A_2 z - z_{02}\|_\sigma \leq \frac{r + \|z_0\|_\sigma}{\eta(x_0)} + \frac{(r + \|z_0\|_\sigma)^2}{\eta(x_0)} + \frac{(r + \|z_0\|_\sigma)}{\sigma \eta(x_0)} (1 + l^2) \mu_0.$$

Now we can choose r, σ such that there hold the inequalities:

$$\begin{cases} \frac{r + \|z_0\|_\sigma}{\eta(x_0)} < \frac{1}{3}, \\ \frac{(r + \|z_0\|_\sigma)^2}{\eta(x_0)} < \frac{1}{3}, \\ \frac{(r + \|z_0\|_\sigma)}{\sigma \eta(x_0)} (1 + l^2) \mu_0 < \frac{1}{3}. \end{cases}$$

It follows that if

$$\begin{cases} r < \frac{\eta(x_0)}{6} = r_1, \\ r < \sqrt{\frac{\eta(x_0)}{6}} = r_2, \\ \sigma_2 = \frac{6r(1 + l^2)\mu_0}{\eta(x_0)} < \sigma, \end{cases}$$

then $A_2 \in S(z_0, r)$.

So, if the inequality

$$r > r_1 = \max\{\alpha_1, \alpha_2\} \tag{35}$$

and $\sigma > \alpha_1$ hold, then the operator A_2 maps $S(z_0, r)$ into itself, i.e., $A_2 \in S(z_0, r)$.

As a result, we conclude that if σ, ρ satisfy the conditions $\sigma > \max\{\sigma_1, \sigma_2\}$, $\rho > \rho_1 = \max\{\alpha_1, \alpha_2\}$, then operator A maps $S(z_0, r)$ into itself, i.e., $A \in S(z_0, r)$. Now let z, z_2 be any two elements from the set $S(z_0, r)$. Then, using auxiliary inequalities of the form

$$\begin{aligned} \|z_2^1 z_1^1 - z_2^2 z_1^2\|_\sigma &= \|(z_2^1 - z_2^2)z_1^1 + (z_1^1 - z_1^2)z_2^2\|_\sigma + \\ &+ 2\|z_1^1 - z_1^2\|_\sigma \max(\|z_1^1\|_\sigma, \|z_2^2\|_\sigma) \leq 2(\|z_0\|_\sigma + r)\|z_1^1 - z_1^2\|_\sigma. \end{aligned} \quad (36)$$

we will receive

$$\begin{aligned} \|(Az^1 - Az^2)_1\|_\sigma &\leq \sup_{(x,t) \in \bar{\Omega}_T} \left| e^{-\sigma t} \int_0^t \int_0^l G(x, \xi) ((z_1^1 - z_1^2)(\xi, \tau) + e^\tau (z_2^1 - z_2^2)(\tau) \bar{\varphi}(\xi)) d\xi d\tau \right| + \\ &+ \sup_{(x,t) \in \bar{\Omega}_T} \left| e^{-\sigma t} \int_0^t \int_0^l G(x, \xi) \int_0^\tau e^{(\tau-s)} (z_2^1(s) z_1^1(\xi, \tau-s) - z_2^2(s) z_1^2(\xi, \tau-s)) ds d\xi d\tau \right| + \\ &+ \sup_{(x,t) \in \bar{\Omega}_T} \left| e^{-\sigma t} \int_0^t \int_0^l e^\tau G(x, \xi) \int_0^\tau (z_2^1(s) - z_2^2(s)) \chi_t(\xi, t-s) ds d\xi d\tau \right| \leq \bar{I}_{11} + \bar{I}_{12} + \bar{I}_{13}. \end{aligned}$$

$$\begin{aligned} \bar{I}_{11} &= \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^t \int_0^l G(x, \xi) e^{-\sigma(t-\tau)} e^{-\sigma\tau} ((z_1^1 - z_1^2)(\xi, \tau) + e^\tau (z_2^1 - z_2^2)(\tau) \bar{\varphi}(\xi)) d\xi d\tau \right| \leq \\ &\leq \frac{1}{\sigma} \|z_1^1 - z_1^2\|_\sigma T + \frac{1}{\sigma} \|z_2^1 - z_2^2\|_\sigma e^T \|\bar{\varphi}\| \leq \frac{(T + e^T \|\bar{\varphi}\|)}{\sigma} \|z_1 - z_2\|_\sigma; \end{aligned}$$

To estimate the second integral, we use the estimate (36).

Then

$$\begin{aligned} \bar{I}_{12} &= \sup_{(x,t) \in \bar{\Omega}_T} \left| e^{-\sigma t} \int_0^t \int_0^l G(x, \xi) \int_0^\tau e^{(\tau-s)} (z_2^1(s) z_1^1(\xi, \tau-s) - z_2^2(s) z_1^2(\xi, \tau-s)) ds d\xi d\tau \right| = \\ &= \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^t \int_0^l e^{-\sigma(t-\tau)} G(x, \xi) \int_0^\tau e^{-\sigma(\tau-s)} e^{(\tau-s)} e^{-\sigma s} (z_2^1(s) z_1^1(\xi, \tau-s) - z_2^2(s) z_1^2(\xi, \tau-s)) ds d\xi d\tau \right| \leq \\ &\leq \frac{2(\|z_0\|_\sigma + r)(e^T - 1)}{\sigma} \|z_1^1 - z_1^2\|_\sigma; \end{aligned}$$

Analogically

$$\begin{aligned} \bar{I}_{13} &= \sup_{(x,t) \in \bar{\Omega}_T} \left| e^{-\sigma t} \int_0^t \int_0^l e^\tau G(x, \xi) \int_0^\tau (z_2^1(s) - z_2^2(s)) \chi_\tau(\xi, t-s) ds d\xi d\tau \right| = \\ &= \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^t \int_0^l e^\tau G(x, \xi) \int_0^\tau e^{-\sigma(t-s)} e^{-\sigma s} (z_2^1(s) - z_2^2(s)) \chi_\tau(\xi, t-s) ds d\xi d\tau \right| \leq \\ &\leq \frac{(e^T - 1)(3l + l^2)\mu_0}{\sigma} \|z_1 - z_2\|_\sigma. \end{aligned}$$

Therefore

$$\|(Az^1 - Az^2)_1\|_\sigma \leq \left(\frac{(T + e^T \|\bar{\varphi}\|)}{\sigma} + \frac{2(\|z_0\|_\sigma + r)(e^T - 1)}{\sigma} + \frac{(e^T - 1)(3l + l^2)\mu_0}{\sigma} \right) \|z_1 - z_2\|_\sigma.$$

If we choose σ as follows

$$\sigma \geq \sigma_3 = \frac{r}{(T + e^T \|\bar{\varphi}\|) + 2(\|z_0\|_\sigma + r)e^T + e^T(3l + l^2)\mu_0},$$

then

$$\|(Az^1 - Az^2)_1\|_\sigma \leq \frac{\sigma_3}{\sigma} \|z_1 - z_2\|_\sigma$$

i.e., the second condition of contractive mapping for A_1 is satisfied.

The second component of A can be estimated in the following form:

$$\begin{aligned} \|(Az^1 - Az^2)_2\|_\sigma &\leq \frac{1}{\eta(x_0)} \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^l G(x_0, \xi) e^{-\sigma t} (z_1^1(\xi, t) - z_1^2(\xi, t)) d\xi \right| + \\ &+ \frac{1}{\eta(x_0)} \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\tau} e^{-\sigma t} (z_2^1(s) z_1^1(\xi, \tau - s) - z_2^2(s) z_1^2(\xi, \tau - s)) d\tau d\xi \right| + \\ &+ \frac{1}{\eta(x_0)} \sup_{0 \leq t \leq T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\sigma t} \chi_t(\xi, t - s) (z_2^1(s) - z_2^2(s)) ds d\xi \right| = \bar{I}_{21} + \bar{I}_{22} + \bar{I}_{23}; \end{aligned}$$

Let's perform estimates separately for each $\bar{I}_{21}$, $\bar{I}_{22}$, $\bar{I}_{23}$. The estimate for $\bar{I}_{21}$ is:

$$\bar{I}_{21} = \frac{1}{\eta(x_0)} \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^l G(x_0, \xi) e^{-\sigma t} (z_1^1(\xi, t) - z_1^2(\xi, t)) d\xi \right| \leq \frac{1}{\eta(x_0)} \|z_1^1 - z_1^2\|_\sigma = \frac{1}{\eta(x_0)} \|z_1 - z_2\|_\sigma;$$

Again applying equality (36), we obtain

$$\begin{aligned} \bar{I}_{22} &= \frac{1}{\eta(x_0)} \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\tau} e^{-\sigma t} (z_2^1(s) z_1^1(\xi, \tau - s) - z_2^2(s) z_1^2(\xi, \tau - s)) d\tau d\xi \right| = \\ &= \frac{1}{\eta(x_0)} \sup_{(x,t) \in \bar{\Omega}_T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\tau} e^{-\sigma(t-\tau)} e^{-\sigma \tau} (z_2^1(s) z_1^1(\xi, \tau - s) - z_2^2(s) z_1^2(\xi, \tau - s)) d\tau d\xi \right| \leq \\ &\leq \frac{2}{\sigma \eta(x_0)} (\|z_0\|_\sigma + r) \|z^1 - z^2\|_\sigma; \end{aligned}$$

$$\begin{aligned} \bar{I}_{23} &= \frac{1}{\eta(x_0)} \sup_{0 \leq t \leq T} \left| \int_0^l G(x_0, \xi) \int_0^t e^{-\sigma t} \chi_t(\xi, t - s) (z_2^1(s) - z_2^2(s)) ds d\xi \right| \leq \\ &\leq \frac{e^T (1 + l^2) \mu_0}{\sigma \eta(x_0)} \|z_2^1 - z_2^2\|_\sigma = \frac{e^T (1 + l^2) \mu_0}{\sigma \eta(x_0)} \|z_1 - z_2\|_\sigma. \end{aligned}$$

Having summed up the obtained estimates for $\bar{I}_{21}$, $\bar{I}_{22}$, $\bar{I}_{23}$, we obtain

$$\|(Az^1 - Az^2)_2\|_\sigma \leq \bar{I}_{21} + \bar{I}_{22} + \bar{I}_{23} \leq \frac{[\sigma + 2(\|z_0\|_\sigma + r) + (e^T - 1)(1 + l^2)\mu_0]}{\sigma \eta(x_0)} \|z_1 - z_2\|_\sigma.$$

Now we choose numbers σ , ρ so that the expression at $\|z_1 - z_2\|_\sigma$ becomes less than 1, i.e., the inequality

$$\left| \frac{[\sigma + 2(\|z_0\|_\sigma + r) + e^T(3 + l^2)\mu_0]}{\sigma \eta(x_0)} \right| < 1$$

is fulfilled. This inequality is valid if numbers σ , r will be chosen from conditions

$$\begin{cases} \frac{1}{\eta(x_0)} < \frac{1}{3}, \\ \frac{2}{\sigma \eta(x_0)} (\|z_0\|_\sigma + r) < \frac{1}{3}, \\ \frac{e^T (1 + l^2) \mu_0}{\sigma \eta(x_0)} < \frac{1}{3}. \end{cases}$$

Solving these inequalities with respect to σ, r we obtain

$$\begin{cases} 3 < \eta(x_0), \\ \sigma_3 = \frac{12r}{\eta(x_0)} < \sigma, \\ \sigma_4 = \frac{3e^T(l+l^2)\mu_0}{\eta(x_0)} < \sigma. \end{cases}$$

From these estimates it is clear that if σ, r are chosen from the condition

$$\sigma = \max\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\},$$

then the operator A in the set S is a contraction mapping.

According to the contraction principle, the operator A , therefore, if σ, r are taken from the condition $\sigma > \max\{\sigma_1, \sigma_2\}$, $r < r_1$, then the operator A in the set $S(z_0, r)$ has a unique fixed point.

The following theorem holds

Theorem 2. *Let $\varphi(x) \in C^2[0, l]$, $(f(x, t), f_t(x, t)) \in C(\Omega_T)$, $\psi(t) \in C^2[0, T]$, $\mu_i(t), \mu'_i(t) \in C^1[0, T]$, $i = 1, 2$, and satisfy the following compatibility conditions: $\varphi(0) = \mu_1(0)$, $\varphi(l) = \mu_2(0)$ and $\varphi(x_0) = \psi(0) \neq 0$. Then the operator equation (32) has a unique solution in domain $C(\Omega_T)$ for arbitrary $l > 0$ and $T > 0$ has a unique solution that continuously depends on the data.*

Consequently, the inverse problem (1)–(4) for an arbitrary and has a unique solution that continuously depends on the data.

It follows that

$$\|Az^1 - Az^2\|_\sigma \leq \frac{\sigma}{\max(\sigma^*)} \|z^1 - z^2\|_\sigma$$

and the operator A performs a contracted mapping of the ball $S(z_0, r)$ onto itself. Then, according to S. Banach's theorem, equation (32) has a unique solution belonging to this ball.

Consequently, by solving the system of equations (33), (34) by the method of successive approximations, we uniquely construct in the domain $\bar{\Omega}_T$ for arbitrary $l > 0$ and $T > 0$ the functions $w(x, t), k(t)$ in the class $C^{(2,1)}(\bar{\Omega}_T) \times C([0, T])$. Thus, we uniquely determine the function $k(t)$ on the interval $[0, T]$. Next, we sequentially find $v(x, t)$, and then $u(x, t)$. Theorem 2 is proven.

Theorem 3. *Let $f_i(x, t)$, $\varphi_i(x)$, $\mu_i(t)$, $\mu'_i(t)$, $\psi_i(t)$, $i = 1, 2$ be the data of the inverse problem (1)–(4). Then, for the corresponding solutions $k_1(t), k_2(t)$, the estimate holds:*

$$\|k_1(t) - k_2(t)\|_{C([0, T])} \leq C [\|\psi_1 - \psi_2\|_{C^2([0, T])} + \|f_1 - f_2\|_{C^{0,1}(\bar{\Omega}_T)} + \|\varphi_1 - \varphi_2\|_{C^2([0, l])} + \|\mu_1^1 - \mu_1^2\|_{C^1([0, T])} + \|\mu_2^1 - \mu_2^2\|_{C^1([0, T])}],$$

where the constant C depends only on the norms of the functions $u_1, u_2, f_1, f_2, \varphi_1, \varphi_2, k_1, k_2$, and the parameters T, l .

The proof of Theorem 3 can be obtained directly from the system of integral equations (33), (34).

Conclusion

This study analyzes the inverse problem of identifying the temporal kernel $K(t)$ in a third-order linear integro-differential pseudoparabolic equation. The mathematical formulation incorporates an initial distribution, Neumann boundary conditions, and an additional overdetermination condition at an interior point of the spatial domain. Sufficient conditions on the input data are established to ensure the existence and uniqueness of a global solution to the original problem over the time interval. Additionally, a theorem is proved regarding the stability of the solution with respect to small perturbations of the initial and boundary data.

References

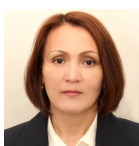
1. Romanov V.G. *Inverse problems of mathematical physics*. Utrecht, The Netherlands: VSP, 1987.
2. Romanov V. G. *Investigation Methods for Inverse Problems, Inverse and Ill-Posed Problems Series*. Utrecht: VSP, 2002. xii+280 pp.
3. Hasanov A., Romanov V. G. *Introduction to Inverse Problems for Differential Equations*. Cham: Springer, 2017.
4. Prilepko A. I., Orlovsky D. G., and Vasin U. A. *Methods for Solving Inverse Problems in Mathematical Physics*. New York: Marcel Dekker, 1999.
5. Kabanikhin S. I. *Inverse and Ill-posed Problems: Theory and Applications, Theory and Applications*. 55. Berlin: De Gruyter Academic Publishing, 2011.
6. Isakov V. *Inverse Problems for Partial Differential Equations*. New York: Springer Science+Business Media, Inc, 2006.
7. Kirsch A. *Introduction to the Mathematical Theory of Inverse Problems, Third edition*. Switzerland: Springer Nature, 2021.
8. Lesnic D. *Inverse Problems with Applications in Science and Engineering*. Leeds Univ., Leeds, UK, 2022.
9. Denisov A. M. *Elements of the Theory of Inverse Problems, Inverse and Ill-Posed Problems Series*. Utrecht: VSP, 1999. iv+272 pp.
10. Bukhgeim, A. L. Inverse Problems of Memory Reconstruction, *Journal of Inverse and Ill-posed Problems*, 1993. vol. 1, no. 3, pp. 193–205 DOI:10.1515/jiip.1993.1.3.193.
11. Bukhgeim A.L., Kalinina N.I. Global convergence of the Newton method in inverse problems of memory reconstruction, *Siberian Math. J.*, 1997. vol. 38, no. 5, pp. 881–895 (In Russian).
12. Lorenzi A., Sinestrari E. Stability results for a partial integrodifferential inverse problem, *Pitman Res. Notes Math.*, 1989. vol. 190, pp. 271–294.
13. Lorenzi A., Paparoni E. Direct and inverse problems in the theory of materials with memory, *Ren. Sem. Math. Univ.*, 1992. vol. 87, pp. 105–138.
14. Lorenzi A. An identification problem related to a nonlinear hyperbolic integro-differential equation, *Nonlin. Anal.: Theory, Methods Appl.*, 1994. vol. 22, no. 1, pp. 21–44.
15. Grasselli M., Kabanikhin S. I., Lorentsi A. An inverse problem for an integro-differential equation, *Siberian Math. J.*, 1992. vol. 33, no. 3, pp. 415–426 (In Russian).
16. Janno J. and von Wolfersdorf L. Inverse problems for identification of memory kernels in viscoelasticity, *Math. Methods Appl. Sci.*, 1997. vol. 20, pp. 291–314.
17. Romanov V. G. On the determination of the coefficients in the viscoelasticity equations, *Sib. Math. J.*, 2014. vol. 55, pp. 503–510 DOI:10.1134/S0037446614030124.
18. Romanov V.G. Inverse problems for differential equations with memory, *Eurasian J. of Mathematical and Computer Applications*, 2014. vol. 2, no. 4, pp. 51–80.
19. Romanov V.G., Yamamoto M. Recovering a Lamé kernel in a viscoelastic equation by a single boundary measurement, *Applicable Analysis*, 2010. vol. 89, no. 3, pp. 377–390.
20. Durdiev D.K., Totieva Zh.D. *Kernel determination problems in hyperbolic integro-differential equations*. Singapore: Springer Nature, 2023.
21. Durdiev D. K., Jumaev J. Ja. One-dimensional inverse problems of determining the kernel of the integro-differential heat equation in a bounded domain, *Nonautonomous Dynamical Systems*, 2023. vol. 10, no. 1 DOI:10.1515/msds-2022-0163.
22. Durdiev D.K., Zhumaev Zh.Zh. Problem of determining the thermal memory of a conducting medium, *Differential Equations*, 2020. vol. 56, no. 6, pp. 785–796 DOI:10.1134/S0012266120060117 (In Russian).

23. Rundell W. Determination of an unknown non-homogeneous term in a linear partial differential equation from overspecified boundary data, *Appl. Anal.*, 1980. vol. 10, no. 3, pp. 231–242 DOI:10.1080/00036818008839304.
24. Asanov, A., Atamanov E.R. Reconstruction of kernels in a pseudoparabolic integrodifferential operator equation, *J. of Inverse and Ill-Posed Problems*, 1995. vol. 3, no. 3, pp. 213–230.
25. Ablabekov B.S. *Inverse problems for pseudoparabolic equations*. Bishkek: Ilim, 2001. 180 pp. (In Russian)
26. Ablabekov B.S. *Inverse problems for third-order differential equations*. LAP Lambert Academic Publishing, 2013. 291 pp. (In Russian)
27. Asanov, A., Atamanov E.R. *Nonclassical and inverse problems for pseudo parabolic equations*. Tokyo, 1997. 152 pp.
28. Ablabekov B.S., Kurmanbaeva A.K. On the asymptotics of solutions to the inverse problem for a pseudo parabolic equation with a small parameter at the highest derivative, *Journal of Mathematical Sciences*, 2025 DOI:10.1007/s10958-025-08032-4.
29. Ablabekov B.S., Kurmanbaeva A.K., Kerimbekova A.N. The Inverse Problem of Determining the Kernel in an Integro-Differential Pseudo-Parabolic Equation, *Eurasian Scientific Association*, 2021. vol. 1, no. 4 (74) (In Russian).
30. Durdiev D. K., Elmuradova H. B., Rakhmonov A. A. Inverse kernel determination problem for a class of pseudo-parabolic integro-differential equations, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.]*, 2025. vol. 29, no. 1, pp. 7–20 DOI:10.14498/vsgtu2095.
31. Safarov, J. Sh., Kalandarov, U. N., Safarova, M. J. Inverse Problem of Determining a Kernel of the Viscoelasticity Equation with Distributed Data in a Limited Domain, *Lobachevskii Journal of Mathematics*, 2024. vol. 45, no. 7, pp. 3380–3390 DOI:10.1134/S1995080224604077.

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