Vestnik KRAUNC. Fiz.-Mat. Nauki. 2025. vol. 52. no. 3. P. 53 - 62. ISSN 2079-6641

MATHEMATICS

https://doi.org/10.26117/2079-6641-2025-52-3-53-62

Research Article

Full text in English

MSC 35R11

# Nonlocal Problem for the Time-Fractional Hyperbolic-Type Equation with the Prabhakar Fractional Derivative

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**Abstract.** In this paper, we investigate a nonlocal boundary value problem for a time-fractional hyperbolic-type partial differential equation involving fractional derivatives of regularized Prabhakar. The fractional differentiation is defined through the regularized Prabhakar operator, which provides a flexible framework for modeling memory effects with non-singular kernels. The equation is considered on a bounded rectangular domain in the plane with respect to two independent variables. The boundary conditions are nonlocal and are prescribed in the form of partial integral expressions of the unknown solution along each spatial variable, where the corresponding kernels are assumed to be continuous. Building upon previously obtained representation formulas for the solution of the associated Goursat problem in terms of Mittag-Leffler type functions, the original boundary value problem is transformed into a coupled system of Volterra integral equations of the second kind for the traces of the solution on a portion of the boundary. This reduction allows us to apply classical methods of integral equations to analyze the problem. By employing appropriate estimates for the regularized Prabhakar kernels and the properties of the resulting integral operators, we rigorously establish the existence and uniqueness of the solution to the nonlocal boundary value problem. Furthermore, an explicit representation of the solution is derived in terms of the solutions of the obtained system of integral equations. The results demonstrate that the regularized Prabhakar framework provides a robust and analytically tractable approach for treating time-fractional hyperbolic problems with nonlocal boundary interactions.

*Key words: Telegraph equation, nonlocal problem, integral equation, Goursat problem, Prabhakar fractional order derivative, Mittag-Leffler type function.*

Received: 10.11.2025; Revised: 11.11.2025; Accepted: 12.11.2025; First online: 14.11.2025

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**For citation.** Turdiev Kh. N. Nonlocal problem for the time-fractional hyperbolic-type equation with the Prabhakar fractional derivative. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2025, **52**: 3, 53-62. EDN: IBVSBJ. https://doi.org/10.26117/2079-6641-2025-52-3-53-62.

**Funding.**The study was conducted without the support of foundations.

**Competing interests.** There are no conflicts of interest regarding authorship and publication.

**Contribution and Responsibility.** The author participated in the writing of the article and is fully responsible for the submission of the final version of the article for publication.

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Вестник КРАУНЦ. Физ.-мат. науки. 2025. Т. 52. №3. С. 53 - 62. ISSN 2079-6641

МАТЕМАТИКА

https://doi.org/10.26117/2079-6641-2025-52-3-53-62

Научная статья

Полный текст на английском языке

УДК 517.958

# Нелокальная задача для дробно-временного уравнения гиперболического типа с дробной производной Прабхакара

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**Аннотация.** В данной работе исследуется нелокальная краевая задача для уравнения в частных производных гиперболического типа, дробно-временного, включающего дробные производные регуляризованного оператора Прабхакара. Дробное дифференцирование определяется через регуляризованный оператор Прабхакара, что обеспечивает гибкую структуру для моделирования эффектов памяти с невырожденными ядрами. Уравнение рассматривается в ограниченной прямоугольной области на плоскости относительно двух независимых переменных. Граничные условия нелокальны и задаются в виде выражений в частных интегральных функциях неизвестного решения по каждой пространственной переменной, где соответствующие ядра предполагаются непрерывными. Опираясь на ранее полученные формулы представления решения ассоциированной задачи Гурса в терминах функций типа Миттаг-Леффлера, исходная краевая задача преобразуется в связанную систему интегральных уравнений Вольтерра второго рода для следов решения на участке границы. Такое сведение позволяет применять классические методы интегральных уравнений для анализа задачи. Используя соответствующие оценки регуляризованных ядер Прабхакара и свойства полученных интегральных операторов, мы строго устанавливаем существование и единственность решения нелокальной краевой задачи. Более того, решение представлено в явном виде через решения полученной системы интегральных уравнений. Результаты показывают, что регуляризованный подход Прабхакара обеспечивает надежный и аналитически точный подход к решению дробно-временных гиперболических задач с нелокальными граничными взаимодействиями.

*Ключевые слова: Телеграфное уравнение, нелокальная задача, интегральное уравнение, задача Гурса, дробная производная Прабхакара, функция типа Миттаг-Леффлера.*

Получение: 10.11.2025; Исправление: 11.11.2025; Принятие: 12.11.2025; Публикация онлайн: 14.11.2025

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**Для цитирования.** Turdiev Kh. N. Nonlocal problem for the time-fractional hyperbolic-type equation with the Prabhakar fractional derivative // *Вестник КРАУНЦ. Физ.-мат. науки.* 2025. Т. 52. № 3. С. 53-62. EDN: IBVSBJ. https://doi.org/10.26117/2079-6641-2025-52-3-53-62.

**Финансирование.** Исследование было проведено без поддержки фондов

**Конкурирующие интересы.** Конфликтов интересов в отношении авторства и публикации нет.

**Авторский вклад и ответственность.** Автор участвовал в написании статьи и полностью несет ответственность за предоставление окончательной версии статьи в печать.

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## Introduction

Fractional differential equations have a strong theoretical framework and deep physical significance, and their relevance has increased notably in recent years. These equations incorporate fractional derivatives or fractional integrals and have attracted attention across multiple fields, such as physics, biology, and chemistry. Fractional differential equations are extensively employed in the analysis of dynamical systems characterized by chaotic or quasi-chaotic behavior, in the investigation of complex materials and porous media dynamics, as well as in modeling random walks that exhibit memory effects [1].

The derivation of the telegraph equation is attributed to Heaviside, as indicated in [2]. The telegraph equation is primarily applied in the transmission of signals and energy; however, it also serves as a fundamental framework for modeling and analyzing various phenomena associated with the propagation of electromagnetic waves in guided media. Numerous researchers have investigated this equation using classical spectral parameter and system equations, notably in works such as [3] and [4].

Lately, many researchers have become interested in studying nonlocal and inverse problems of the telegraph equation that include fractional-order operators.

Recent studies [5–7] have examined various nonlocal problems associated with fractional-order telegraph equations. In [5], R.A. Pshibikhova investigated a nonlocal boundary value problem for a generalized telegraph equation involving fractional derivatives. The solution was expressed in terms of Wright-type functions and reduced to a system of Volterra integral equations, through which the existence and uniqueness of the solution were established. In [6], R.R. Ashurov and Yu.E. Fayziyev considered initial–boundary value problems with time-nonlocal conditions for a subdiffusion equation containing Riemann–Liouville time-fractional derivatives. They proved the existence and uniqueness of the solution and addressed inverse problems aimed at identifying the right-hand side term and a function within the time-nonlocal condition using Fourier’s method, with possible extensions to more general elliptic operators.

Furthermore, Kh. Turdiev [7] studied two nonlocal boundary value problems for a system of coupled telegraph equations defined on a mixed pentagonal domain, derived the corresponding second-kind Volterra integral equation, and demonstrated the uniqueness of its solution based on the general theory of integral equations.

In addition, various aspects of fractional-order telegraph equations have been examined in prior works such as [8–13].

Nevertheless, problems concerning fractional-order hyperbolic-type equations involving the regularized Prabhakar operator remain largely unexplored. Accordingly, the present work investigates the unique solvability of a nonlocal boundary value problem for a time-fractional hyperbolic-type equation incorporating the regularized Prabhakar operator.

## Formulation of problem

We consider the following time-fractional hyperbolic-type equation

$$\frac{\partial}{\partial x}\,{}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}u\left(t,x\right)+\lambda u\left(t,x\right)=f\left(t,x\right), \tag{1}$$

in a domain $\Omega=\{(t,x):0<t<b,0<x<a\}$. Here $f\left(t,x\right)$ is a given function and

$${}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}y\left(t\right)={}^{P}I_{0t}^{\alpha,m-\beta,-\gamma,\delta}\frac{d^m}{dt^m}y\left(t\right)$$

represents regularized Prabhakar fractional derivative [14] and

$${}^{P}I_{0t}^{\alpha,\beta,\gamma,\delta}y\left(t\right)=\int\limits_0^t\left(t-\xi\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left[\delta\left(t-\xi\right)^{\alpha}\right]y\left(\xi\right)d\xi,\quad t>0$$

represents Prabhakar fractional integral [15]. We note that above-given definitions are valid for $\alpha$, $\beta$, $\gamma$, $\delta$, $a,b,q,p\in\mathbb{R}$ such that $\alpha>0$ and $m-1<\beta<m$.

Here $E_{\alpha,\beta}^{\gamma}\left(z\right)$ is the generalized Mittag-Leffler function [15]:

$$E_{\alpha,\beta}^{\gamma}\left(z\right)=\sum_{m=0}^{\infty}\frac{\left(\gamma\right)_m}{\Gamma\left(\alpha m+\beta\right)}\frac{z^m}{m!}.$$

**Problem N.** We are interested to find a regular solution of the equation (1) with $0<\beta<1$ in $\Omega$, satisfying initial condition

$$u\left(0,x\right)=\varphi\left(x\right),\quad 0\le x\le a. \tag{2}$$

and nonlocal condition

$$u(t,0)+\int\limits_0^a M(x)u(t,x)dx=\tau(t),\quad 0\le t\le b, \tag{3}$$

where $\varphi\left(x\right)$, $\tau(t)$ and $M(x)$ are given functions ($M(x)>0$, $0\le x\le a$), such that $\varphi(0)+\int\limits_0^a M(x)\varphi(x)dx=\tau(0)$.

**Definition.** We call a function $u\left(t,x\right)$ as a regular solution of problem (1), (2)-(3), if $u(t,x)\in C(\overline{\Omega})$, $u_x(t,x)\in C(\Omega)$, ${}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}u(t,x)\in C(\Omega)$.

We note that at $\beta=1$, $\delta=0$, Eq. (1) becomes classical hyperbolic-type equation:

$$u_{tx}(t,x)+\lambda u(t,x)=f(t,x).$$

## Main results

Let us introduce a notation

$$u(t,0)=\psi(t),\quad 0\le t\le b. \tag{4}$$

**Theorem 1.** *If* $\varphi(0)=\psi(0)$, $\varphi(x)\in C[0,a]\cap C^1(0,a)$, $\psi(t)\in C[0,b]\cap C^1(0,b)$, *and* $f(t,x)=t^{-\varepsilon_1}x^{-\varepsilon_2}\widetilde{f}_1(t,x)$, *where* $\widetilde{f}_1(t,x)\in C\left(\overline{\Omega}\right)$ *and* $0\le\varepsilon_1<\beta$, $0\le\varepsilon_2<1$, *then the solution of Goursat problem hyperbolic-type equation* (1), *that satisfies boundary condition* (4) *and the initial condition* (2) *will be represented as follows:*

$$u(t,x)=\varphi(x)+\psi(t)-\varphi(0)-$$

$$-\lambda t^{\beta}\int_0^x\varphi(\xi)\,E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1;\end{matrix}\middle|\begin{matrix}-\lambda(x-\xi)t^{\beta}\\ \delta t^{\alpha}\end{matrix}\right)d\xi-$$

$$-\lambda x\int_0^t(\psi(\eta)-\varphi(0))(t-\eta)^{\beta-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1;\end{matrix}\middle|\begin{matrix}-\lambda x(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)d\eta+$$

$$+\int_0^x\int_0^t(t-\eta)^{\beta-1}f(\xi,\eta)\,E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,1;1,1;\end{matrix}\middle|\begin{matrix}-\lambda(x-\xi)(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)d\xi d\eta. \quad (5)$$

*The solution to the Goursat problem exists and is unique (see for details [16]).*

Here, $E_{12}(\cdot)$ is the bivariate Mittag-Leffler type function [16]:

$$E_{12}\left(\begin{matrix}\alpha_1,\beta_1,\delta_1;\\ \alpha_2,\beta_2,\delta_2;\alpha_3,\delta_3;\alpha_4,\delta_4;\beta_3,\delta_5;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=$$

$$=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\Gamma(\alpha_1 n+\beta_1 m+\delta_1)\,x^n y^m}{\Gamma(\alpha_2 n+\beta_2 m+\delta_2)\,\Gamma(\alpha_3 n+\delta_3)\,\Gamma(\alpha_4 n+\delta_4)\,\Gamma(\beta_3 m+\delta_5)},$$

$$\left(x,y,\alpha_l,\beta_i,\delta_j\in\mathbb{R};\ \min\{\alpha_l,\beta_i,\delta_j\}>0;\ \left(l=\overline{1,4},\ i=\overline{1,3},\ j=\overline{1,5}\right)\right)$$

in which the double series converges for $x,y\in\mathbb{R}$, if $\Delta_1>0$ and $\Delta_2>0$. Here

$$\Delta_1=\alpha_2+\alpha_3+\alpha_4-\alpha_1,\ \ \Delta_2=\beta_2+\beta_3-\beta_1.$$

Previously, in [17], 11 similar functions were introduced and studied.

To determine the function $\psi(t)$, we will use condition (3). Following the representation (5), and considering the nonlocal condition (3), we obtain

$$\psi(t)+\int_0^a M(x)\varphi(x)dx+(\psi(t)-\varphi(0))\int_0^a M(x)dx-$$

$$-\lambda t^{\beta}\int_0^a M(x)\int_0^x\varphi(\xi)\,E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1;\end{matrix}\middle|\begin{matrix}-\lambda(x-\xi)t^{\beta}\\ \delta t^{\alpha}\end{matrix}\right)d\xi dx-$$

$$-\lambda\int_0^a xM(x)\int_0^t(\psi(\eta)-\varphi(0))(t-\eta)^{\beta-1}\times$$

$$\times E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1;\end{matrix}\middle|\begin{matrix}-\lambda x(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)d\eta dx+$$

$$+\int\limits_0^a M(x)\int\limits_0^x\int\limits_0^t (t-\eta)^{\beta-1} f(\xi,\eta)\times$$

$$\times E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,1;1,1;\end{matrix}\left|\begin{matrix}-\lambda(x-\xi)(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right.\right)d\xi d\eta dx=\tau(t). \qquad (6)$$

Here, since $0<\beta<1$, it follows that $(t-\eta)^{\beta-1}\in L_1(0,t)$.

Given that $\varphi(x)\in C[0,a]\cap C^1(0,a)$, $\psi(t)\in C[0,b]\cap C^1(0,b)$, $M(x)\in C[0,a]$, and that $E_{12}(\cdot)$, the bivariate Mittag-Leffler type function, is continuous, the order of integration in the integrals in equation (6) can be interchanged as follows:

$$\psi(t)+\int\limits_0^a M(x)\varphi(x)dx+\psi(t)\int\limits_0^a M(x)dx-\varphi(0)\int\limits_0^a M(x)dx$$

$$-\lambda t^{\beta}\int\limits_0^a\int\limits_{\xi}^a M(x)\varphi(\xi)E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1;\end{matrix}\left|\begin{matrix}-\lambda(x-\xi)t^{\beta}\\ \delta t^{\alpha}\end{matrix}\right.\right)dxd\xi-$$

$$-\lambda\int\limits_0^t(t-\eta)^{\beta-1}\psi(\eta)\int\limits_0^a xM(x)E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1;\end{matrix}\left|\begin{matrix}-\lambda x(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right.\right)dxd\eta+$$

$$+\lambda\varphi(0)\int\limits_0^t\int\limits_0^a xM(x)(t-\eta)^{\beta-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1;\end{matrix}\left|\begin{matrix}-\lambda x(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right.\right)dxd\eta+$$

$$+\int\limits_0^a M(x)\int\limits_0^x\int\limits_0^t (t-\eta)^{\beta-1} f(\xi,\eta)\times$$

$$\times E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,1;1,1;\end{matrix}\left|\begin{matrix}-\lambda(x-\xi)(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right.\right)d\xi d\eta dx=\tau(t). \qquad (7)$$

Let us introduce the following notation:

$$A=1+\int\limits_0^a M(x)dx,$$

$$M_1(\eta,t)=\int\limits_0^a xM(x)E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1;\end{matrix}\left|\begin{matrix}-\lambda x(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right.\right)dx,$$

$$g(x)=\tau(t)-\int\limits_0^a M(x)\varphi(x)dx+\varphi(0)\int\limits_0^a M(x)dx+$$

$$+\lambda t^{\beta}\int\limits_0^a\int\limits_{\xi}^a M(x)\varphi(\xi)E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1;\end{matrix}\left|\begin{matrix}-\lambda(x-\xi)t^{\beta}\\ \delta t^{\alpha}\end{matrix}\right.\right)dxd\xi-$$

$$-\lambda\varphi\left(0\right)\int\limits_0^t\int\limits_0^a xM(x)(t-\eta)^{\beta-1}E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,2;1,1;\end{matrix}\middle|\begin{matrix}-\lambda x(t-\eta)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)dxd\eta-$$

$$-\int\limits_0^a M(x)\int\limits_0^x\int\limits_0^t (t-\eta)^{\beta-1}f\left(\xi,\eta\right)$$

$$\times E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta;\gamma,\gamma;1,1;1,1;\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)\left(t-\eta\right)^{\beta}\\ \delta(t-\eta)^{\alpha}\end{matrix}\right)d\xi d\eta dx.$$

Considering the introduced notations, from equation (7) we deduce

$$A\psi(t)-\lambda\int\limits_0^t(t-\eta)^{\beta-1}\psi(\eta)M_1(\eta,t)d\eta=g(t).$$

Let $A\neq 0$. Then, denoting

$$M_2(\eta,t)=\frac{M_1(\eta,t)}{A},\quad G(t)=\frac{g(t)}{A},$$

from the last relation, we deduce

$$\psi(t)-\lambda\int\limits_0^t(t-\eta)^{\beta-1}\psi(\eta)M_2(\eta,t)d\eta=G(t). \tag{8}$$

**Theorem 2.** *Let* $\alpha>0$*,* $0<\beta<1$*,* $\gamma>0$*,* $\delta<0$*,* $\lambda>0$*. If* $\varphi(0)=G(0)$*,* $\tau(t)\in C[0,b]\cap C^1(0,b)$*,* $\varphi(x)\in C[0,a]\cap C^1(0,a)$*,* $M(x)\in C[0,a]$*,* $f\left(t,x\right)=t^{-\varepsilon_1}x^{-\varepsilon_2}\widetilde{f_1}\left(t,x\right)$*,* $\widetilde{f_1}\left(t,x\right)\in C\left(\overline{\Omega}\right)$ *and* $0\leq\varepsilon_1<\beta$*,* $0\leq\varepsilon_2<1$*, then the solution to the problem* N *exists and is unique.*

*Here,* $G(0)=g(0)/A$*,*

$$g(0)=\varphi(0)\left(1+\int\limits_0^a M(x)dx\right).$$

**Proof.** First, let us consider the following lemma:

**Lemma 1.** *If* $\alpha>0$*,* $0<\beta<1$*,* $\gamma>0$*,* $\delta<0$*,* $\lambda>0$*, then the following holds [16]:*

$$\left|E_{12}\left(\begin{matrix}\gamma,1,\gamma;\\ \beta,\alpha,\beta+1;\gamma,\gamma;1,1;1,1\end{matrix}\middle|\begin{matrix}-\lambda\left(x-\xi\right)t^{\beta}\\ \delta t^{\alpha}\end{matrix}\right)\right|\leq C,$$

*where* C *is any positive real constant.*

According to the theory of integral equations, a Volterra integral equation of the second kind has a unique solution if its kernel, $(t-\eta)^{\beta-1}M_2(\eta,t)$, has a weak singularity and the right-hand side $G(t)$ is continuous.

Since $0<\beta<1$, the term $(t-\eta)^{\beta-1}$ exhibits weak singularity. If we can show that the functions $M_2(\eta,t)$ and $G(t)$ are continuous, then, according to the above theory, the integral equation (8) admits a solution, which is unique. We will prove this below:

For the integral equation (8) to have a continuous $M_2(\eta, t)$ kernel, the $M_1(\eta, t)$ kernel must be continuous. If $M(x) \in C[0, a]$, we will prove the continuity of the $M_1(\eta, t)$ kernel using Lemma 1:

$$M_1(\eta, t) = \int_0^a xM(x)E_{12}\left(\begin{matrix}\gamma, 1, \gamma; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 2; 1, 1;\end{matrix}\middle|\begin{matrix}-\lambda x(t-\eta)^{\beta} \\ \delta(t-\eta)^{\alpha}\end{matrix}\right) dx$$

$$\leq C_1 \int_0^a xM(x)dx \leq C_2.$$

If $g(t)$ is continuous, then $G(t)$ will be also continuous. If $\tau(t) \in C[0, b]$, $\varphi(x) \in C[0, a]$, $M(x) \in C[0, a]$ and $f(t, x) = t^{-\varepsilon_1}x^{-\varepsilon_2}\widetilde{f}_1(t, x)$, $\widetilde{f}_1(t, x) \in C\left(\overline{\Omega}\right)$ and $0 \leq \varepsilon_1 < \beta$, $0 \leq \varepsilon_2 < 1$, then we could prove the continuity of the function $g(x)$ using Lemma 1:

$$g(x) = \tau(t) - \int_0^a M(x)\varphi(x)dx + \varphi(0)\int_0^a M(x)dx+$$

$$+\lambda t^{\beta}\int_0^a\int_{\xi}^a M(x)\varphi(\xi) E_{12}\left(\begin{matrix}\gamma, 1, \gamma; \\ \beta, \alpha, \beta+1; \gamma, \gamma; 1, 1; 1, 1;\end{matrix}\middle|\begin{matrix}-\lambda(x-\xi)t^{\beta} \\ \delta t^{\alpha}\end{matrix}\right) dxd\xi-$$

$$-\lambda\varphi(0)\int_0^t\int_0^a xM(x)(t-\eta)^{\beta-1}E_{12}\left(\begin{matrix}\gamma, 1, \gamma; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 2; 1, 1;\end{matrix}\middle|\begin{matrix}-\lambda x(t-\eta)^{\beta} \\ \delta(t-\eta)^{\alpha}\end{matrix}\right) dxd\eta-$$

$$-\int_0^a M(x)\int_0^x\int_0^t (t-\eta)^{\beta-1}f(\xi, \eta)$$

$$\times E_{12}\left(\begin{matrix}\gamma, 1, \gamma; \\ \beta, \alpha, \beta; \gamma, \gamma; 1, 1; 1, 1;\end{matrix}\middle|\begin{matrix}-\lambda(x-\xi)(t-\eta)^{\beta} \\ \delta(t-\eta)^{\alpha}\end{matrix}\right) d\xi d\eta dx \leq C_3.$$

□

## Conclusion

In this paper, a nonlocal boundary value problem for a time-fractional hyperbolic-type equation involving the regularized Prabhakar fractional derivative was formulated and rigorously analyzed. By employing the representation of the solution to the corresponding Goursat problem in terms of the Mittag-Leffler type bivariate function, the original problem was successfully reduced to a system of second-kind Volterra integral equations for the unknown boundary traces. The continuity of the kernel functions and the right-hand sides of these integral equations was established under appropriate smoothness assumptions on the given data. Consequently, the existence and uniqueness of the regular solution to the investigated nonlocal problem were proved by applying the classical theory of Volterra integral equations.

The obtained results extend previously known findings for fractional telegraph and hyperbolic-type equations with Caputo and Riemann-Liouville operators to the broader class of problems involving the regularized Prabhakar derivative. The methods and representations developed herein can be further adapted to study more complex fractional models, including systems of coupled equations, problems with variable coefficients, and inverse problems associated with Prabhakar-type operators.

Although the present study is mainly theoretical, the analytical framework and results can find potential applications in modeling physical and engineering processes exhibiting memory and hereditary effects-such as wave propagation in viscoelastic or complex media, anomalous diffusion, and heat transfer with fractional dynamics. These connections highlight the practical relevance and prospective interdisciplinary impact of the theoretical developments obtained in this work.

**Acknowledgement.** The authors express their gratitude to Professor E.T. Karimov for discussing the results of this article.

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