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New Extended Three-Variable Mittag-Leffler Type Functions

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Abstract. This article presents a systematic investigation of a new class of Mittag-Leffler-type functions in three variables. These functions are a natural and significant extension of the classical Mittag-Leffler function, and are constructed to correspond analogously to the well-known Lauricella hypergeometric functions of three variables. Our study comprehensively explores the fundamental properties and analytical characteristics of these three-variable functions. A primary focus is the establishment of their precise interrelationships with other existing extensions and generalizations of the classical Mittag-Leffler function, thereby situating them within the broader landscape of special functions. Key analytical findings presented in this work include: The derivation of the exact three-dimensional regions of convergence for the series defining these functions. The formulation of elegant Euler-type integral representations, which provide a powerful tool for further analysis. A detailed exploration of their integral transforms, specifically the derivation of both one-dimensional and three-dimensional Laplace transforms.The examination of their intimate connections with fractional calculus, demonstrating their natural emergence as kernels and solutions in the context of the Riemann-Liouville fractional integral and differential operators. Furthermore, we delve into the associated differential equations, showing that these Mittag-Leffler-type functions serve as solutions to specific systems of partial differential equations. This work not only enriches the theory of special functions but also provides a robust mathematical framework for potential applications in fractional differential equations, anomalous diffusion, and other areas of mathematical physics.

Key words: Extended Mittag-Leffler type function; Hypergeometric function; Special (or higher transcendental) function; Lauricella function; Integral representation; System of partial differential equation; One- and three-dimensional Laplace transform; Riemann-Liouville fractional integral; Riemann-Liouville fractional derivative; Appell and Kampé de Fé riet functions; Srivastava-Daoust hypergeoemetric function.

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МАТЕМАТИКА

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Научная статья

Полный текст на английском языке

УДК 517.58

Новые расширенные функции типа Миттаг-Леффлера с тремя переменными

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Аннотация. В данной статье представлено систематическое исследование нового класса функций типа Миттаг-Леффлера от трёх переменных. Эти функции являются естественным и существенным расширением классической функции Миттаг-Леффлера и построены аналогично известным гипергеометрическим функциям Лауричеллы от трёх переменных. В нашем исследовании всесторонне изучаются фундаментальные свойства и аналитические характеристики этих трёх переменных функций. Основное внимание уделяется установлению их точных взаимосвязей с другими существующими расширениями и обобщениями классической функции Миттаг-Леффлера, что позволяет поместить их в более широкий спектр специальных функций. Ключевые аналитические результаты, представленные в данной работе, включают: вывод точных трёхмерных областей сходимости для рядов, определяющих эти функции; формулировку элегантных интегральных представлений типа Эйлера, которые предоставляют мощный инструмент для дальнейшего анализа. Подробное исследование их интегральных преобразований, в частности, вывод как одномерных, так и трёхмерных преобразований Лапласа. Исследование их тесной связи с дробным исчислением, демонстрирующее их естественное возникновение в качестве ядер и решений в контексте дробных интегральных и дифференциальных операторов Римана-Лиувилля. Кроме того, мы углубляемся в связанные с ними дифференциальные уравнения, показывая, что эти функции типа Миттаг-Леффлера служат решениями конкретных систем уравнений в частных производных. Эта работа не только обогащает теорию специальных функций, но и предоставляет надёжную математическую основу для потенциальных приложений в дробных дифференциальных уравнениях, аномальной диффузии и других областях математической физики.

Ключевые слова: обобщенная функция типа Миттаг-Леффлера; Гипергеометрическая функция; специальная (или высшая трансцендентная) функция; функция Лауричеллы; интегральное представление; система дифференциальных уравнений в частных производных; одномерное и трехмерное преобразование Лапласа; дробный интеграл Римана-Лиувилля; дробная производная Римана-Лиувилля; функции Аппеля и Кампе де Ферьет; гипергеометрическая функция Сриваставы-Даусту.

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Introduction

Throughout this article, $\Re(\mu)$ denotes the real part of the complex number $\mu \in C$ and $[\Re(\mu)]$ means the greatest integer in $\Re(\mu)$, and $\Gamma(z)$ denotes the classical (Euler's) Gamma function defined by

$$\Gamma(z) = \begin{cases} \displaystyle\int_0^\infty e^{-t} t^{z-1} dt & (\Re(z) > 0) \\ \dfrac{\Gamma(z+n)}{\prod\limits_{j=0}^{n-1}(z+j)} & (z \in C\backslash Z_0^-;\ n \in N), \end{cases} \tag{1}$$

which happens to be one of the most fundamental and the most useful special functions of mathematical analysis and applied mathematics, N, N_0 and Z_0^- being the sets of *positive*, *non-negative* and *non-positive* integers, respectively.

Remark 1. It is regrettable to see that, in many seemingly amateurish-type publications, the so-called k-Gamma function $\Gamma_k(z)$ is being used to claim "generalization" of the known results which are based upon the classical (Euler's) Gamma function $\Gamma(z)$. The classical Mittag-Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$ are defined by (see [1], [2])

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (z,\ \alpha \in C; \quad \Re(\alpha) > 0) \tag{2}$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad (z, \alpha, \beta \in C;\ \Re(\alpha) > 0;\ \Re(\beta) > 0), \tag{3}$$

respectively. The one-parameter function $E_\alpha(z)$ was first considered by Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) in 1903 and its two-parameter version $E_{\alpha,\beta}(z)$ was introduced by Anders Wiman (1865–1959) in 1905 (see [3] and [4]).

The Mittag-Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$ have gained importance and popularity through their applications in a wide variety of problems in the mathematical, physical and engineering sciences. For example, these functions appear as solutions of fractional differential equations and integro-differential equations which model applied problems. They do play an important role in various fields of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, computer science, and signal processing. In addition, the Mittag-Leffler-type functions of several variables arise in solving some boundary value problems involving fractional-order Volterra type integro-differential equations (see [5]), initial-boundary value problems for a generalized polynomial diffusion equation involving the time-fractional derivatives (see [6] and [7]), fractional-order modeling of the relaxation-oscillation and diffusion equations and initial-boundary value problems for time-fractional diffusion equations with positive constant coefficients (see [8]).

Remark 2. Various claimed one-variable and multi-parameter (or multi-index) "generalizations" of the familiar Mittag-Leffler functions $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$ (see, for example, [9], [10], [11], [12], [13], [14] and [15]) are no more than fairly obvious special or limit cases of the substantially much more general Fox-Wright function ${}_p\Psi_q$ $(p, q \in N_0)$ or ${}_p\Psi^*_q$ $(p, q \in N_0)$. In fact, the familiar and widely investigated Fox-Wright function ${}_p\Psi_q$ $(p, q \in N_0)$ or ${}_p\Psi^*_q$ $(p, q \in N_0)$ happens to be the Fox-Wright generalization of the relatively more familiar hypergeometric function ${}_pF_q$ $(p, q \in N_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ such that $a_j \in C$ $(j = 1, \cdots, p)$ and $b_j \in C \backslash Z_0^-$ $(j = 1, \cdots, q)$. The general Fox-Wright function ${}_p\Psi_q$ $(p, q \in N_0)$ or ${}_p\Psi^*_q$ $(p, q \in N_0)$ are indeed defined by (see, for details [16]; see also [17] and [18])

$$\begin{aligned} {}_p\Psi^*_q\left[\begin{array}{cc} (a_1, A_1), \cdots, (a_p, A_p); & z \\ (b_1, B_1), \cdots, (b_q, B_q); & \end{array}\right] &= \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n} \cdots (b_q)_{B_q n}} \frac{z^n}{n!} \\ = \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} {}_p\Psi_q &\left[\begin{array}{cc} (a_1, A_1), \cdots, (a_p, A_p); & z \\ (b_1, B_1), \cdots, (b_q, B_q); & \end{array}\right] \end{aligned} \tag{4}$$

$$\left(\Re(A_j) > 0 \ (j = 1, \cdots, p); \ \Re(B_j) > 0 \ (j = 1, \cdots, q); \ 1 + \Re\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j\right) \geq 0\right),$$

where and elsewhere in this article, $(\lambda)_\nu$ denotes the general Pochhammer symbol or the *shifted* factorial, since

$$(1)_n = n! \quad (n \in N_0 = N \cup \{0\}; \quad N = \{1, 2, 3, \cdots\}),$$

which is defined (for λ, $\nu \in C$ and in terms of familiar Gamma function in the eqution (1)) by

$$(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \ \lambda \in C \backslash \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in N; \ \lambda \in C), \end{cases}$$

In which it is assumed conventionally that $(0)_0 = 1$ and understood *tacitly* that the Γ- quotient exists. In general, we suppose that $a_j, A_j \in C$ $(j = 1, \cdots, p)$ and $b_j, B_j \in C$ $(j = 1, \cdots, q)$ and that the equality in the convergence condition in the definition (4) holds true only for suitably-bounded values of $|z|$ given by

$$|z| < \nabla = \left(\prod_{j=1}^{p} A_j{}^{-A_j}\right) \cdot \left(\prod_{j=1}^{q} B_j{}^{B_j}\right).$$

The above-mentioned generalized hypergeoemtric function ${}_pF_q$ $(p, q \in N_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ is a widely- and extensively-investigated and potentially useful special case of the general Fox-Wright function ${}_p\Psi_q$ $(p, q \in N_0)$ when $A_j = 1$ $(j = 1, \cdots, p)$ and $B_j = 1$ $(j = 1, \cdots, q)$.

We now to turn to a series of monumental works (see, for example, [19], [20], [21] and [22]) by Edward Maitland Wright (1906–2005). In fact, as long ago as 1940, Wright introduced and systematically studied the asymptotic expansion of the following Taylor-Maclaurin series (see [20]):

$$E_{\alpha,\beta}(\phi;z)=\sum_{n=0}^{\infty}\frac{\phi(n)}{\Gamma(\alpha n+\beta)}z^{n}\quad(\alpha,\beta\in C;\ \Re(\alpha)>0;\ \Re(\beta)>0),\tag{5}$$

where $\phi(t)$ is a function satisfying suitable conditions.

The above-cited contributions by Wright were motivated essentially by the earlier developments reported for simpler cases by Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) in 1905, Anders Wiman (1865–1959) in 1905, Ernest William Barnes (1874–1953) in 1906, Godfrey Harold Hardy (1877–1947) in 1905, George Neville Watson (1886–1965) in 1913, Charles Fox (1897–1977) in 1928, and other authors. In particular, the aforementioned work [19] by Bishop Ernest William Barnes (1874–1953) of the Church of England in Birmingham considered the asymptotic expansions of functions in the class which is defined below:

$$E_{\alpha,\beta}^{(\kappa)}(s;z)=\sum_{n=0}^{\infty}\frac{z^{n}}{(n+\kappa)^{s}\Gamma(\alpha n+\beta)}\quad(\alpha,\beta\in C;\ \Re(\alpha)>0;\ \Re(\beta)>0),\tag{6}$$

for suitably-restricted parameters κ and s. Clearly, we have the following relationship:

$$\lim_{\alpha\to 0}\left\{E_{\alpha,\beta}^{(\kappa)}(s;z)\right\}=\frac{1}{\Gamma(\beta)}\Phi(z,s,\kappa)$$

with the classical Lerch transcendent (or the Hurwitz-Lerch zeta function) $\Phi(z,s,\kappa)$ defined by (see, [16])

$$\Phi(z,s,\kappa)=\sum_{n=0}^{\infty}\frac{z^{n}}{(n+\kappa)^{s}}$$

$(\kappa\in C\backslash Z_0^{-};\ s\in C\ \text{when}\ |z|<1;\ \Re(s)>1\ \text{when}\ |z|=1)$. The reader us referred to a series of recent works by Srivastava for detailed systematic study of the following interesting unification of the definitions in (5), (6), and other earlier developments in the literature, for a suitably-restricted function $\varphi(\tau)$ given by

$$\varepsilon_{\alpha,\beta}(\varphi;z,s,z)=\sum_{n=0}^{\infty}\frac{\varphi(n)}{(n+\kappa)^{s}\Gamma(\alpha n+\beta)}\quad(\alpha,\beta\in C;\ \Re(\alpha)>0;\ \Re(\beta)>0),\tag{7}$$

where the parameters α, β, s and κ are appropriately constrained as above. Furthermore, in the aforementioned works, some general families of Riemann-Liouville-type operators of fractional calculus involving the functions $E_{\alpha,\beta}(\phi;z)$ and $\varepsilon_{\alpha,\beta}(\varphi;z,s,z)$ in their kernel were investigated .

An interesting multiple-series generalization of the Mittag-Leffler function $E_{\alpha}(z)$ involving several variables was proposed by Luchko and Gorenflo [7], who applied an operational method to solve a boundary-value problem for linear fractional differential equations with constant coefficients. The solution of the boundary-value problem was expressed by then in terms of the following Mittag-Leffler-type function in m variables

$z_1, \cdots, z_m$:

$$E_{(\alpha_1,...,\alpha_m),\beta}(z_1,...,z_m) = \sum_{k=0}^{\infty} \sum_{\substack{l_1 \geq 0, \cdots, l_m \geq 0 \\ l_1+...+l_m=k}}^{\infty} \frac{k!}{\Gamma\left(\beta + \sum_{j=1}^{m} \alpha_i l_i\right)} \frac{z_1^{l_1}}{l_1!} \cdots \frac{z_m^{l_m}}{l_m!},$$

which, in the special case when $m = 2$, was studied Bin-Saad *et al.* [23].

Motivated essentially by some of the above-mentioned and other developments in the theory and applications of the Mittag-Leffler-type functions in one and more variables, we propose in this article to study the Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$, $\bar{F}_C^{(3)}$ and $\bar{F}_D^{(3)}$, which are associated with the familiar three-variable Lauricella hypergeometric functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$, respectively. We investigate and establish several properties and characteristics of these three-variable Mittag-Leffler-type functions. The results for the Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$, $\bar{F}_C^{(3)}$ and $\bar{F}_D^{(3)}$, which we investigate in this article, include their relationships with other extensions and generalizations of the classical Mittag-Leffler functions, their three-dimensional convergence regions, their Euler-type integral representations, their one, two as well as three-dimensional Laplace transforms, their connections with the Riemann-Liouville operators of fractional calculus, and the systems of partial differential equations which are associated with them.

Multivariable Hypergeometric Functions and Associated Mittag-Leffler-Type Functions

In the year 1969, Srivastava and Daoust [24] extended the Fox-Wright function ${}_p\Psi_q$, which is defined by (4), to two variables in the following form:

$$S^{A:B;B'}_{C:D;D'}\begin{pmatrix} x \\ y \end{pmatrix} = S^{A:B;B'}_{C:D;D'}\left(\begin{matrix} [(a):\theta,\phi]:[(b):\psi];[(b'):\psi']; \\ [(c):\delta,\varepsilon]:[(d):\eta];[(d'):\eta']; \end{matrix}\; x,y\right)$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A}\Gamma(a_j+m\theta_j+n\phi_j)\prod_{j=1}^{B}\Gamma(b_j+m\psi_j)\prod_{j=1}^{B'}\Gamma(b'_j+n\psi'_j)}{\prod_{j=1}^{C}\Gamma(c_j+m\delta_j+n\varepsilon_j)\prod_{j=1}^{D}\Gamma(d_j+m\eta_j)\prod_{j=1}^{D'}\Gamma(d'_j+n\eta'_j)} \frac{x^m}{m!}\frac{y^n}{n!}$$

$$= \frac{\prod_{j=1}^{A}\Gamma(a_j)\prod_{j=1}^{B}\Gamma(b_j)\prod_{j=1}^{B'}\Gamma(b'_j)}{\prod_{j=1}^{C}\Gamma(c_j)\prod_{j=1}^{D}\Gamma(d_j)\prod_{j=1}^{D'}\Gamma(d'_j)} F^{A:B;B'}_{C:D;D'}\left(\begin{matrix} [(a):\theta,\phi]:[(b):\psi];[(b'):\psi']; \\ [(c):\delta,\varepsilon]:[(d):\eta];[(d'):\eta']; \end{matrix}\; x,y\right), \quad (8)$$

which also includes, as a very specialized case, the general Kampe de Feriet function $F^{A:B;B'}_{C:D;D'}(x,y)$ in the modified notation introduced by Srivastava and Panda (see, [25]) when we set each of the parameters $\theta_j, \phi_j, \psi_j, \psi'_j, \delta_j, \varepsilon_j, \eta_j$ and η'_j equal to 1.

Here, and elsewhere in this paper, we find it to be convenient to use the abbreviation (a) to represent the array A of (real or complex) parameters $a_1, a_2, \cdots, a_A$, with similar interpretations for (b), (b'), (c),(d) and (d'). We tacitly assume the following conditions on the coefficients and parameters involved:

$$\theta_j, \phi_j \in R^+ \quad (j = 1, \cdots, A); \qquad \psi_j, \psi'_k \in R^+ \quad (j = 1, \cdots, B;\ k = 1, \cdots, B')$$

and

$$\delta_j, \varepsilon_j \in R^+ \quad (j = 1, \cdots, C); \qquad \eta_j, \eta'_k \in R^+ \quad (j = 1, \cdots, D;\ k = 1, \cdots, D').$$

Each of the following two-variable Mittag-Leffler-type functions E_1 and E_2, which were considered by Garg et al. [26], happens to be a special or limit case of the Srivastava-Daoust function

$$S^{A:B;B'}_{C:D;D'}\begin{pmatrix} x \\ y \end{pmatrix}$$

defined by (8):

$$E_1\left(\begin{array}{c} \gamma_1, \alpha_1; \gamma_2, \beta_1; \\ \delta_1, \alpha_2, \beta_2; \delta_2, \alpha_3; \delta_3, \beta_3; \end{array}\middle|\begin{array}{c} x \\ y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m} (\gamma_2)_{\beta_1 n}}{\Gamma(\delta_1 + \alpha_2 m + \beta_2 n)} \frac{x^m}{\Gamma(\delta_2 + \alpha_3 m)} \frac{y^n}{\Gamma(\delta_3 + \beta_3 n)}$$

$$(\gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, x, y \in C,\ \min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} > 0)$$

and

$$E_2\left(\begin{array}{c} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2; \\ \delta_1, \alpha_3, \beta_2; \delta_2, \alpha_4; \delta_3, \beta_3; \end{array}\middle|\begin{array}{c} x \\ y \end{array}\right) = \sum_{m,n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m + \beta_1 n} (\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1 + \alpha_3 m + \beta_2 n)} \frac{x^m}{\Gamma(\delta_2 + \alpha_4 m)} \frac{y^n}{\Gamma(\delta_3 + \beta_3 n)} \tag{9}$$

$$(\gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, x, y \in C,\ \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0).$$

We refer here to two related sequels (see [23], [27] and [28]) in which boundary-value problems involving some time-fractional derivatives were solved in terms of E_1 in [27].

In the case of hypergeometric functions of three variables, we recall that a general triple hypergeometric series $F^{(3)}(x, y, z)$, which was introduced in the year 1967 by Srivastava (see [29]) is a unification and generalization of Lauricella's fourteen hypergeometric functions $\mathcal{F}_1, \cdots, \mathcal{F}_{14}$ (see [30]) including the ten hypergeometric functions studied by Shanti Saran (1928–1983) (see [31]), as well as Srivastava's three additional functions H_A, H_B and H_C (see [32] and [33]).

$$F^{(3)}\left[\begin{array}{cccccccc} (a_A) :: & (b_B); & (b'_{B'}); & (b''_{B''}) : & (c_C); & (c'_{C'}); & (c''_{C''}); & \\ (e_E) :: & (g_G); & (g'_{G'}); & (g''_{G''}) : & (h_H); & (h'_{H'}); & (h''_{H''}); & \end{array} x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \Lambda(m, n, p) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{10}$$

where

$$\Lambda(m,n,p)=\frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}} \cdot\frac{\prod_{j=1}^{C}(c_j)_{m}\prod_{j=1}^{C'}(c'_j)_{n}\prod_{j=1}^{C''}(c''_j)_{p}}{\prod_{j=1}^{H}(h_j)_{m}\prod_{j=1}^{H'}(h'_j)_{n}\prod_{j=1}^{H''}(h''_j)_{p}} \tag{11}$$

and (a_A) abbreviates the array of the A parameters $a_1, a_2, \cdots, a_A$, with similar interpretations for other abbreviations used above. The triple hypergeometric series in (10) converges absolutely when

$$\begin{cases} 1+E+G+G''+H-A-B-B''-C\geq 0\\ 1+E+G+G''+H'-A-B-B'-C'\geq 0\\ 1+E+G'+G''+H''-A-B'-B''-C''\geq 0, \end{cases} \tag{12}$$

in which the equalities hold true for appropriately restricted values of $|x|$, $|y|$ and $|z|$.

As long ago as in the year 1893, in his above-mentioned work [30], Giuseppe Lauricella (1867–1913) extended the four Appell functions to the corresponding hypergeometric functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ of n variables. Furthermore, in the particular case when $n=3$, Lauricella listed a set of 14 triple hypergeometric functions $\mathcal{F}_1, \cdots, \mathcal{F}_{14}$, for which we have

$$F_A^{(2)}=F_2=F^{1:1;1}_{0:1;1};\qquad F_B^{(2)}=F_3=F^{0:2;2}_{1:0;0}$$

and

$$F_C^{(2)}=F_4=F^{2:0;0}_{0:1;1};\qquad F_D^{(2)}=F_1=F^{1:1;1}_{1:0;0}$$

in terms of the four Appell functions F_1, F_2, F_3 and F_4 of two variables (see [34] and [35]).

In a sequel to their paper [24], which was also published in the year 1969, Srivastava and Daoust introduced and studies the following general family of hypergeometric functions n variables:

$$F\begin{array}{c}A:B^{(1)};\cdots;B^{(n)}\\ C:D^{(1)};\cdots;D^{(n)}\end{array}(z_1,\cdots,z_n)$$

$$=F\begin{array}{c}A:B^{(1)};\cdots;B^{(n)}\\ C:D^{(1)};\cdots;D^{(n)}\end{array}\left(\begin{array}{c}[(a):\theta^{(1)},\cdots,\theta^{(n)}]:[(b^{(1)}):\psi^{(1)}];\cdots;[(b^{(n)}):\psi^{(n)}];\\ [(c):\delta^{(1)},\cdots,\delta^{(n)}]:[(d^{(1)}):\phi^{(1)}];\cdots;[(d^{(n)}):\phi^{(n)}];\end{array} z_1,\cdots,z_n\right)$$

$$=\sum_{m_1,\cdots m_n=0}^{\infty}\mathcal{K}\begin{array}{c}A:B^{(1)};\cdots;B^{(n)}\\ C:D^{(1)};\cdots;D^{(n)}\end{array}(m_1,\cdots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!}, \tag{13}$$

where, for convenience,

$$\mathcal{K}\begin{array}{c}A:B^{(1)};\cdots;B^{(n)}\\ C:D^{(1)};\cdots;D^{(n)}\end{array}(m_1,\cdots,m_n)$$

$$
= \frac{\prod\limits_{j=1}^{A} (a_j)_{\theta_j^{(1)} m_1 + \cdots + \theta_j^{(n)} m_n} \prod\limits_{j=1}^{B^{(1)}} \left(b_j^{(1)}\right)_{\psi_j^{(1)} m_1}}{\prod\limits_{j=1}^{C} (a_j)_{\delta_j^{(1)} m_1 + \cdots + \delta_j^{(n)} m_n} \prod\limits_{j=1}^{D^{(1)}} \left(d_j^{(1)}\right)_{\phi_j^{(1)} m_1}} \cdots \frac{\prod\limits_{j=1}^{B^{(n)}} \left(b_j^{(n)}\right)_{\psi_j^{(n)} m_1}}{\prod\limits_{j=1}^{D^{(n)}} \left(d_j^{(n)}\right)_{\phi_j^{(n)} m_1}}. \qquad (14)
$$

The multiple hypergeometric series in (13) converges for

$$
|z_1| < 1, \cdots, |z_n| < 1,
$$

provided that (see, for details [36])

$$
\sum_{j=1}^{C} \delta_j^{(\ell)} + \sum_{j=1}^{D^{(\ell)}} \phi_j^{(\ell)} - \sum_{j=1}^{A} \theta_j^{(\ell)} - \sum_{j=1}^{B^{(\ell)}} \psi_j^{(\ell)} + 1 = 0 \quad (\forall \ell = 1, \cdots, n). \qquad (15)
$$

Various special cases of the above-defined Srivastava-Daoust hypergeometric function of n variables, especially when we set the parameters θ_j, ϕ_j, ψ_j and δ_j equal to 1, have found applications in many different contexts in the mathematical and physical contexts (see, for example [37]). In particular, for the Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ of n variables, we record the following correspondence with the Srivastava-Daoust function defined by (14) with, of course, the parameters θ_j, ϕ_j, ψ_j and δ_j equal to 1:

$$
F_A^{(n)} = F_{0:1;\cdots;1}^{1:1;\cdots;1}; \qquad F_B^{(n)} = F_{1:0;\cdots;0}^{0:2;\cdots;2}
$$

and

$$
F_C^{(n)} = F_{0:1;\cdots;1}^{2:0;\cdots;0}; \qquad F_D^{(n)} = F_{1:0;\cdots,0}^{1:1;\cdots;1}.
$$

In particular, the triple hypergeometric functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ correspond, respectively, to the functions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_5$ and $\mathcal{F}_9$ of the above-mentioned Lauricella's set of 14 hypergeometric functions $\mathcal{F}_1, \cdots, \mathcal{F}_{14}$ of three variables.

We conclude this section by remarking that special (or higher transcendental) functions including (for example) the Mittag-Leffler-type functions are closely related to the operators of fractional calculus (see [5], [18], [38], [39] and [40]), as well as to the operators of generalized fractional calculi (see, for example [39]). Many special functions can be represented as fractional order integrals or fractional-order derivatives of some elementary functions and such representations can potentially lead to some alternative definitions for special functions (see, for details [39]). Many recent works on special functions and their applications in solving problems from control theory, mechanics, physics, engineering, economics, and so on, can be found in (for example) [38], [41], [42] and [43].

The Three-Variable Mittag-Leffler-Type Functions

In the preceding section, we systematically investigated the definitions and mutual relations of various families of generalized hypergeometric functions in one, two, three and n variables $(n \in \mathbb{N} \backslash \{1,2,3\})$. Here, in this section, we introduce the Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$, $\bar{F}_C^{(3)}$ and $\bar{F}_D^{(3)}$ in three variables, which are motivated by (and associated with) Lauricella's triple hypergeometric functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$, respectively, as follows:

$$\bar{F}_A^{(3)} = \bar{F}_A^{(3)} \left(\begin{matrix} a, \alpha_1, \beta_1, \gamma_1; b_1, \alpha_2; b_2, \beta_2; b_3, \gamma_2; \\ c_1, \alpha_3; c_2, \beta_3; c_3, \gamma_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{matrix} |x, y, z \right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{\alpha_1 m+\beta_1 n+\gamma_1 p} (b_1)_{\alpha_2 m} (b_2)_{\beta_2 n} (b_3)_{\gamma_2 p}}{\Gamma(c_1+\alpha_3 m)\Gamma(c_2+\beta_3 n)\Gamma(c_3+\gamma_3 p)} \frac{x^m}{\Gamma(c_4+\alpha_4 m)} \frac{y^n}{\Gamma(c_5+\beta_4 n)} \frac{z^p}{\Gamma(c_6+\gamma_4 p)} \tag{16}$$

$$(a, b_i, c_j, x, y, z \in \mathbb{C}; \quad \alpha_k, \beta_k, \gamma_k \in \mathbb{R}; \ \min\{\alpha_k, \beta_k, \gamma_k\} > 0$$

$$(i = \{1,2,3\}, \ j = \{1, \cdots, 6\}, \quad k = \{1, \cdots, 4\}))$$

and

$$\bar{F}_B^{(3)} = \bar{F}_B^{(3)} \left(\begin{matrix} a_1, \alpha_1; a_2, \beta_1; a_3, \gamma_1; b_1, \alpha_2; b_2, \beta_2; b_3, \gamma_2; \\ c, \alpha_3, \beta_3, \gamma_3; c_1, \alpha_4; c_2, \beta_4; c_3, \gamma_4; \end{matrix} |x, y, z \right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{\alpha_1 m} (a_2)_{\beta_1 n} (a_3)_{\gamma_1 p} (b_1)_{\alpha_2 m} (b_2)_{\beta_2 n} (b_3)_{\gamma_2 p}}{(c)_{\alpha_3 m+\beta_3 n+\gamma_3 p}}$$

$$\cdot \frac{x^m}{\Gamma(c_1+\alpha_4 m)} \frac{y^n}{\Gamma(c_2+\beta_4 n)} \frac{z^p}{\Gamma(c_3+\gamma_4 p)} \tag{17}$$

$(c, a_i, b_i, c_i, x, y, z \in \mathbb{C}; \quad \alpha_k, \beta_k, \gamma_k \in \mathbb{R}; \ \min\{\alpha_k, \beta_k, \gamma_k\} > 0 \ \ (i = \{1,2,3\} \ \ k = \{1, \cdots, 4\}))$.

In a similar manner, we define below the other two Mittag-Leffler-type functions $\bar{F}_C^{(3)}$ and $\bar{F}_D^{(3)}$ in three variables:

$$\bar{F}_C^{(3)} = \bar{F}_C^{(3)} \left(\begin{matrix} a, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ c_1, \alpha_3; c_2, \beta_3; c_3, \gamma_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{matrix} |x, y, z \right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{\alpha_1 m+\beta_1 n+\gamma_1 p} (b)_{\alpha_2 m+\beta_2 n+\gamma_2 p}}{\Gamma(c_1+\alpha_3 m)\Gamma(c_2+\beta_3 n)\Gamma(c_3+\gamma_3 p)} \frac{x^m}{\Gamma(c_4+\alpha_4 m)} \frac{y^n}{\Gamma(c_5+\beta_4 n)} \frac{z^p}{\Gamma(c_6+\gamma_4 p)} \tag{18}$$

$(a, b, c_i, x, y, z \in \mathbb{C}; \quad \alpha_k, \beta_k, \gamma_k \in \mathbb{R}; \ \min\{\alpha_k, \beta_k, \gamma_k\} > 0 \ \ (i = \{1, \cdots, 6\} \ \ k = \{1, \cdots, 4\}))$,

in which the triple series converges for $x, y, z \in \mathbb{C}$ if $\min\{\Delta_1, \Delta_2, \Delta_2\} > 0$, where $\Delta_1 = \alpha_3 + \alpha_4 - \alpha_1 - \alpha_2$, $\Delta_2 = \beta_3 + \beta_4 - \beta_1 - \beta_2$ and $\Delta_3 = \gamma_3 + \gamma_4 - \gamma_1 - \gamma_2$.

The triple series in (18) converges absolutely for $|x| < \rho_1$, $|y| < \rho_2$ and $|z| < \rho_3$ if $\Delta_1 = \Delta_2 = \Delta_3 = 0$, where $\rho_1 = \min\limits_{\mu,\nu,\theta>0}(K_1)$, $\rho_2 = \min\limits_{\mu,\nu,\theta>0}(K_2)$ and $\rho_3 = \min\limits_{\mu,\nu,\theta>0}(K_3)$ $(\mu, \ \nu, \theta > 0)$,

$$K_1 = \mu^{\alpha_4+\alpha_3} \frac{(\alpha_3)^{\alpha_3} (\alpha_4)^{\alpha_4}}{(\alpha_1\mu + \beta_1\nu + \gamma_1\theta)^{\alpha_1} (\alpha_2\mu + \beta_2\nu + \gamma_2\theta)^{\alpha_2}},$$

$$K_2 = \nu^{\beta_4+\beta_3} \frac{(\beta_3)^{\beta_3} (\beta_4)^{\beta_4}}{(\alpha_1\mu + \beta_1\nu + \gamma_1\theta)^{\beta_1} (\alpha_2\mu + \beta_2\nu + \gamma_2\theta)^{\beta_2}}$$

and

$$K_3 = \theta^{\gamma_4+\gamma_3} \frac{(\gamma_3)^{\gamma_3} (\gamma_4)^{\gamma_4}}{(\alpha_1\mu + \beta_1\nu + \gamma_1\theta)^{\gamma_1} (\alpha_2\mu + \beta_2\nu + \gamma_2\theta)^{\gamma_2}}.$$

$$\bar{F}_D^{(3)} = \bar{F}_D^{(3)} \left(\begin{matrix} a, \alpha_1, \beta_1, \gamma_1; b_1, \alpha_2; b_2, \beta_2; b_3, \gamma_2; \\ c, \alpha_3, \beta_3, \gamma_3; c_1, \alpha_4; c_2 \beta_4; c_3, \gamma_4; \end{matrix} |x, y, z \right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{\alpha_1 m+\beta_1 n+\gamma_1 p} (b_1)_{\alpha_2 m} (b_2)_{\beta_2 n} (b_3)_{\gamma_2 p}}{(c)_{\alpha_3 m+\beta_3 n+\gamma_3 p}} \frac{x^m}{\Gamma(c_1 + \alpha_4 m)} \frac{y^n}{\Gamma(c_2 + \beta_4 n)} \frac{z^p}{\Gamma(c_3 + \gamma_4 p)} \quad (19)$$

$(a, c, b_i, c_i, x, y, z \in C; \quad \alpha_k, \beta_k, \gamma_k \in R; \ \min\{\alpha_k, \beta_k, \gamma_k\} > 0 \ \ (i = \{1, 2, 3\} \ \ k = \{1, \cdots, 4\}))$

respectively.

It is not difficult to observe that each of the generalized Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$, $\bar{F}_C^{(3)}$ and $\bar{F}_D^{(3)}$ in three variables, which we have defined by means of the equations (16) to (19), is itself a special or limit case of the case $n = 3$ of the n variable Srivastava-Daoust hypergeometric function defined by the equation (14).

System of partial differential equations

We begin this section by presenting Lemma 1 and Lemma 2 below.

Lemma 1. *If* $c \in C$ *and* $\alpha, \beta, \gamma \in N$ *then the following equalities hold true:*

$$\frac{\Gamma(c + \alpha + \alpha m + \beta n + \gamma p)}{\Gamma(c + \alpha m + \beta n + \gamma p)} = \prod_{i=1}^{\alpha} (c - i + \alpha(m+1) + \beta n + \gamma p), \quad (20)$$

$$\frac{\Gamma(c + \beta + \alpha m + \beta n + \gamma p)}{\Gamma(c + \alpha m + \beta n + \gamma p)} = \prod_{i=1}^{\beta} (c - i + \alpha m + \beta(n+1) + \gamma p) \quad (21)$$

and

$$\frac{\Gamma(c + \gamma + \alpha m + \beta n + \gamma p)}{\Gamma(c + \alpha m + \beta n + \gamma p)} = \prod_{i=1}^{\gamma} (c - i + \alpha m + \beta n + \gamma(p+1)). \quad (22)$$

Proof. The demonstration of Lemma 1 would make use of the recurrence relation in the definition (1) of the classical Gamma function $\Gamma(z)$. We choose to skip the details as an exercise for the interested reader. □

Lemma 2. *Let* $\theta = x\frac{\partial}{\partial x}$, $\phi = y\frac{\partial}{\partial y}$ *and* $\sigma = z\frac{\partial}{\partial z}$. *If* $c \in C$ *and* $\alpha, \beta, \gamma \in N$ *then the following equalities hold true:*

$$\Gamma(c + \alpha m) \prod_{i=1}^{\alpha} (c + \alpha - i + \alpha\,\theta) x^m = \Gamma(c + \alpha(m+1)) x^m, \quad (23)$$

$$\Gamma(\gamma + \alpha m + \beta n) \prod_{i=1}^{\alpha} (\gamma + \alpha - i + \alpha\,\theta + \beta\phi) x^m y^n = \Gamma(\gamma + \alpha(m+1) + \beta n) x^m y^n \quad (24)$$

and

$$\Gamma(c + \alpha m + \beta n + \gamma p) \prod_{i=1}^{\alpha} (c + \alpha - i + \alpha\,\theta + \beta\phi + \gamma\sigma) x^m y^n z^p$$

$$= \Gamma\left(c+\alpha\left(m+1\right)+\beta n+\gamma p\right)x^{m}y^{n}z^{p}. \tag{25}$$

Proof. The proof of Lemma 2 is based upon some Gamma-function properties and elementatary derivative formulas followed by straightforward simplification. We therefore, omit the details involved. □

Theorem 1. *Let* $\alpha_k, \beta_k, \gamma_k \in \mathrm{N}$ $(k=\{1,\cdots,4\})$ *and* $a, b, c_i, x, y, z \in \mathrm{C}$ $(i=\{1,\cdots,6\})$. *Then the function* $\bar{F}_C^{(3)}$ *satisfies the following system of partial differential equations:*

$$\left[\prod_{i=1}^{\alpha_3}\left(c_1+\alpha_3-i+\alpha_3 x\frac{\partial}{\partial x}\right)\prod_{i=1}^{\alpha_4}\left(c_4+\alpha_4-i+\alpha_4 x\frac{\partial}{\partial x}\right)x^{-1}\right.$$
$$-\prod_{i=1}^{\alpha_1}\left(a+\alpha_1-i+\alpha_1 x\frac{\partial}{\partial x}+\beta_1 y\frac{\partial}{\partial y}+\gamma_1 z\frac{\partial}{\partial z}\right)$$
$$\left.\cdot\prod_{i=1}^{\alpha_2}\left(b+\alpha_2-i+\alpha_2 x\frac{\partial}{\partial x}+\beta_2 y\frac{\partial}{\partial y}+\gamma_2 z\frac{\partial}{\partial z}\right)\right]\bar{F}_C^{(3)}=0, \tag{26}$$

$$\left[\prod_{i=1}^{\beta_3}\left(c_2+\beta_3-i+\beta_3 y\frac{\partial}{\partial y}\right)\prod_{i=1}^{\beta_4}\left(c_5+\beta_4-i+\beta_4 y\frac{\partial}{\partial y}\right)y^{-1}\right.$$
$$-\prod_{i=1}^{\beta_1}\left(a+\beta_1-i+\alpha_1 x\frac{\partial}{\partial x}+\beta_1 y\frac{\partial}{\partial y}+\gamma_1 z\frac{\partial}{\partial z}\right)$$
$$\left.\cdot\prod_{i=1}^{\beta_2}\left(b_2+\beta_2-i+\alpha_2 x\frac{\partial}{\partial x}+\beta_2 y\frac{\partial}{\partial y}+\gamma_2 z\frac{\partial}{\partial z}\right)\right]\bar{F}_C^{(3)}=0 \tag{27}$$

and

$$\left[\prod_{i=1}^{\gamma_3}\left(c_3+\gamma_3-i+\gamma_3 z\frac{\partial}{\partial z}\right)\prod_{i=1}^{\gamma_4}\left(c_6+\gamma_4-i+\gamma_4 z\frac{\partial}{\partial z}\right)z^{-1}\right.$$
$$-\prod_{i=1}^{\gamma_1}\left(a+\gamma_1-i+\alpha_1 x\frac{\partial}{\partial x}+\beta_1 y\frac{\partial}{\partial y}+\gamma_1 z\frac{\partial}{\partial z}\right)$$
$$\left.\cdot\prod_{i=1}^{\gamma_2}\left(b_3+\gamma_2-i+\alpha_2 x\frac{\partial}{\partial x}+\beta_2 y\frac{\partial}{\partial y}+\gamma_2 z\frac{\partial}{\partial z}\right)\right]\bar{F}_C^{(3)}=0. \tag{28}$$

Analogous systems of partial differential equations are satisfied by the other three-variable Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$ *and* $\bar{F}_D^{(3)}$.

Proof. For the validity of the first partial differential equation (26), we substitute the defining triple series for function $\bar{F}_C^{(3)}$ into its right-hand side, so

$$\left[\prod_{i=1}^{\alpha_3}\left(c_1+\alpha_3-i+\alpha_3 x\frac{\partial}{\partial x}\right)\prod_{i=1}^{\alpha_4}\left(c_4+\alpha_4-i+\alpha_4 x\frac{\partial}{\partial x}\right)x^{-1}\right]\bar{F}_C^{(3)}$$
$$=\sum_{m=1}^{\infty}\sum_{n,p=0}^{\infty}\frac{(a)_{\alpha_1 m+\beta_1 n+\gamma_1 p}\,(b)_{\alpha_2 m+\beta_2 n+\gamma_2 p}}{\Gamma\left(c_1+\alpha_3\left(m-1\right)\right)\Gamma\left(c_2+\beta_3 n\right)\Gamma\left(c_3+\gamma_3 p\right)}$$

$$\cdot\frac{x^{m-1}}{\Gamma\left(c_4+\alpha_4\left(m-1\right)\right)}\frac{y^n}{\Gamma\left(c_5+\beta_4 n\right)}\frac{z^p}{\Gamma\left(c_6+\gamma_4 p\right)} \tag{29}$$

and

$$\left[\prod_{i=1}^{\alpha_1}\left(a+\alpha_1-i+\alpha_1 x\frac{\partial}{\partial x}+\beta_1 y\frac{\partial}{\partial y}+\gamma_1 z\frac{\partial}{\partial z}\right)\times \prod_{i=1}^{\alpha_2}\left(b_1+\alpha_2-i+\alpha_2 x\frac{\partial}{\partial x}+\beta_2 y\frac{\partial}{\partial y}+\gamma_2 z\frac{\partial}{\partial z}\right)\right]\bar{F}_C^{(3)}$$

$$=\sum_{m,n,p=0}^{\infty}\frac{(a)_{\alpha_1(m+1)+\beta_1 n+\gamma_1 p}\,(b)_{\alpha_2(m+1)+\beta_2 n+\gamma_2 p}}{\Gamma\left(c_1+\alpha_3 m\right)\Gamma\left(c_2+\beta_3 n\right)\Gamma\left(c_3+\gamma_3 p\right)}\frac{x^m}{\Gamma\left(c_4+\alpha_4 m\right)}\frac{y^n}{\Gamma\left(c_5+\beta_4 n\right)}\frac{z^p}{\Gamma\left(c_6+\gamma_4 p\right)} \tag{30}$$

Now, upon substituting (29) and (30) into the equation (26), if we replace the summation index m by $m+1$, we complete the demonstration of the first assertion (26) of Theorem 1 after some simplification and interpretation.

The proofs of the validity of the second assertion (27) and the third assertion (28) are similar, so we omit their proofs. □

Euler-Type Integral Representations

The Euler-type integral representations for the three-variable Mittag-Leffler-type function $\bar{F}_C^{(3)}$ are presented as Theorem 2 below. One can analogously derive the corresponding Euler-type integral representations for the other three-variable Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$ and $\bar{F}_D^{(3)}$.

Theorem 2. *If* $a, b, c_i, x, y, z \in \mathrm{C}$ $(i=\{1,\cdots,6\})$, $\alpha_k, \beta_k, \gamma_k \in \mathrm{R}$ *and* $\min\{\alpha_k, \beta_k, \gamma_k\}>0$ $(\ k=\{1,\cdots,4\})$, *then the following integral representations holds true:*

$$\bar{F}_C^{(3)}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix}\middle|x,y,z\right)=\frac{\Gamma\left(\mu\right)}{\Gamma\left(a\right)\Gamma\left(\mu-a\right)}$$

$$\cdot\int_0^1\xi^{a-1}\left(1-\xi\right)^{\mu-a-1}\bar{F}_C^{(3)}\left(\begin{matrix} \mu,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix}\middle|x\xi^{\alpha_1},y\xi^{\beta_1},z\xi^{\gamma_1}\right)d\xi \tag{31}$$

$$\left(\Re\left(\mu\right)>\Re\left(a\right)>0\right),$$

$$\bar{F}_C^{(3)}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix}\middle|x,y,z\right)=\frac{\Gamma\left(\mu\right)}{\Gamma\left(b\right)\Gamma\left(\mu-b\right)}$$

$$\cdot\int_0^1\xi^{b-1}\left(1-\xi\right)^{\mu-b-1}\bar{F}_C^{(3)}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;\mu,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix}\middle|x\xi^{\alpha_2},y\xi^{\beta_2},z\xi^{\gamma_2}\right)d\xi \tag{32}$$

$$\left(\Re\left(\mu\right)>\Re\left(b\right)>0\right),$$

$$\int_0^1\int_0^1\int_0^1\xi^{c_1-1}\eta^{c_2-1}\tau^{c_3-1}\left(1-\xi\right)^{\mu_1-1}\left(1-\eta\right)^{\mu_2-1}\left(1-\tau\right)^{\mu_3-1}$$

$$\cdot \bar{F}_C^{(3)} \left(\begin{array}{c} a, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ c_1, \alpha_3; c_2, \beta_3; c_3, \gamma_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{array} \middle| x\xi^{\alpha_3}, y\eta^{\beta_3}, z\tau^{\gamma_3} \right) d\xi d\eta d\tau$$

$$= \Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\bar{F}_C^{(3)} \left(\begin{array}{c} a, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ c_1, \alpha_3 + \mu_1; c_2, \beta_3 + \mu_2; c_3, \gamma_3 + \mu_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{array} \middle| x, y, z \right) \quad (33)$$

$$(\Re(\mu_1) > 0, \ \Re(\mu_2) > 0, \ \Re(\mu_2) > 0).$$

Proof. For proving the Euler-type integral representations (31) to (33), which are asserted by Theorem 2, we express $\bar{F}_C^{(3)}$ as a triple series, justifiably invert the order of the series and integrals involved, and then evaluate the resulting integrals by means of the well-known integral representing the classical Beta function $B(\alpha, \beta)$:

$$B(\alpha, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt & (\min\{\Re(\alpha), \Re(\beta)\} > 0), \\ \\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in C \backslash Z_0^-). \end{cases} \quad (34)$$

The details are being left as an exercise for the interested reader. □

Laplace Transform

Named after the French scholar and polymath, Pierre-Simon Laplace (1749–1827), the Laplace transform is defined by

$$\mathcal{L}\{f(t) : s\} = \int_0^\infty e^{-st} f(t)\, dt \qquad (\Re(s) > 0), \quad (35)$$

provided that the integral exists. The need for *simultaneous* operational calculus (based upon multidimensional Laplace transformation) presents itself naturally when problems dependent on several variables are to be treated operationally (see, for example [44]). The multidimensional Laplace transform defined by

$$\mathcal{L}_n\{f(t_1, \cdots, t_n) : s_1, \cdots, s_n\} = \int_0^\infty \cdots \int_0^\infty \exp(-s_1 t_1 - \cdots - s_n t_n) f(t_1, \cdots, t_n)\, dt_1 \cdots dt_n$$

$$(\Re(s_j) > 0 \quad (j = \{1, \cdots n\})),$$

So that, obviously, $\mathcal{L} = \mathcal{L}_1$.

Theorem 3. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ denote the operators of the one-dimensional, the two-dimensional and the three-dimensional Laplace transforms, respectively. Then the following Laplace transformations are valid:*

$$\mathcal{L}_3 \left\{ t_1^{c_1-1} t_2^{c_2-1} t_3^{c_3-1} \bar{F}_C^{(3)} \left(\begin{array}{c} a, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ c_1, \alpha_3; c_2, \beta_3; c_3, \gamma_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{array} \middle| x t_1^{\alpha_3}, y t_2^{\beta_3}, z t_3^{\gamma_3} \right) : s_1, s_2, s_3 \right\}$$

$$= \frac{1}{s_1^{c_1}} \frac{1}{s_2^{c_2}} \frac{1}{s_3^{c_3}} \bar{F}_C^{(3)} \left(\begin{array}{c} a, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ 1, 0; 1, 0; 1, 0; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{array} \middle| \frac{x}{s_1^{\alpha_3}}, \frac{y}{s_2^{\beta_3}}, \frac{z}{s_3^{\gamma_3}} \right) \quad (36)$$

$$(\min\{\Re(\alpha_3), \Re(\beta_3), \Re(\gamma_3)\} > 0),$$

$$\mathcal{L}_2\left\{t_1{}^{c_1-1}t_2{}^{c_2-1}\bar{F}_C^{(3)}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4; \end{matrix}\left|xt_1{}^{\alpha_3},yt_2^{\beta_3},z\right.\right):s_1,s_2\right\} \tag{37}$$

$$=\frac{1}{s_1{}^{c_1}}\frac{1}{s_2^{c_2}}\bar{F}_C^{(3)}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ 1,0;1,0;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4; \end{matrix}\left|\frac{x}{s_1^{\alpha_3}},\frac{y}{s_2^{c_2}},z\right.\right)$$

$$(\min\{\Re(\alpha_3),\Re(\beta_3)\}>0)$$

and

$$\mathcal{L}_1\left\{t_1{}^{c_1-1}\bar{F}_C^{(3)}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4; \end{matrix}\left|xt_1^{\alpha_3},y,z\right.\right):s_1\right\} \tag{38}$$

$$=\frac{1}{s_1{}^{c_1}}\bar{F}_C^{(3)}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ 1,0;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4; \end{matrix}\left|\frac{x}{s_1^{\alpha_3}},y,z\right.\right)$$

$$(\Re(\alpha_3)>0),$$

provided that each member of the equations (41), (42) and (43) exists. Analogous one-dimensional, two-dimensional and three-dimensional Laplace transformations hold true also for the other three-variable Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$ *and* $\bar{F}_D^{(3)}$.

Proof. The above results can easily be proved on using the definitions of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\bar{F}_C^{(3)}$ in conjunction with the following familiar formula:

$$\mathcal{L}\left\{t^{\lambda-1}:s\right\}=\frac{\Gamma(\lambda)}{s^{\lambda}} \qquad (\Re(\lambda)>0;\ \Re(s)>0).$$

□ **Remark 3.** The Eulerian integral defining the classical Laplace transform in (37) as well as its following s-multiplied version studied by the American transmission theorist, John Renshaw Carson (1886–1940):

$$\mathcal{LC}\{f(\tau):s\}=s\int_0^{\infty}e^{-s\tau}f(\tau)\,d\tau=F_{\mathcal{LC}}(s), \tag{39}$$

which has one distinct advantage over the Laplace transform (35) in the fact that the Laplace-Carson transform of a constant in (39) is the same constant. Regrettably, many obviously trivial and inconsequential variations have been and continue to be made in the parameter (or index) s or in the integration variable t or τ (or in both s and t or τ), ridiculously giving a "new" name to each of such parametric and argument variations of the classical Laplace transform in (35) or its s-multiplied version in (39) by forcing-in some obviously redundant (or superfluous) parameters. Yet another somehow missed-out instance of such trivialities can be exemplified by Yang's attempt to produce what he called a "new" integral transform by replacing the parameter (or index) s in (35) by $\frac{1}{s}$ (see, [45] and [46]). Such demonstratively trivial and obviously inconsequential parametric and argument variations as those that we have recalled above continue to flood the literature merely to unnecessarily repeat or translate the already-published developments using the Laplace transform itself rather successfully.

Connections with the Riemann-Liouville Operators of Fractional Calculus

The Riemann–Liouville fractional integral operator ${}^{RL}\mathfrak{I}^{\alpha}_{\tau+}$ of order α $(\alpha \in \mathbb{C};\ \Re(\alpha) > 0)$ is defined as follows (see, for example, [5], [18] and [40]):

$$ {}^{RL}\mathfrak{I}^{\alpha}_{\tau+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{x} (x-t)^{\alpha-1} f(t)\, dt \quad (x > \tau;\ \Re(\alpha) > 0). \tag{40} $$

Correspondingly, the Riemann–Liouville fractional derivative operator ${}^{RL}D^{\alpha}_{\tau+}$ of order α $(\alpha \in \mathbb{C};\ n-1 < \Re(\alpha) < n;\ n \in \mathbb{N})$ is defined by

$$ {}^{RL}D^{\alpha}_{\tau+} f(x) = \left(\frac{d}{dx}\right)^{n} \left\{ {}^{RL}\mathfrak{I}^{n-\alpha}_{\tau+} f(x) \right\} \qquad (\Re(\alpha) \geq 0; \quad n = [\Re(\alpha)] + 1), \tag{41} $$

where f is locally integrable $\Re(\alpha)$ denotes the real part of the complex number $\alpha \in \mathbb{C}$ and $[\Re(\alpha)]$ means the greatest integer in $\Re(\alpha)$.

Theorem 4 below lists the applications of the Riemann-Liouville fractional integral and fractional derivative operators involving the three-variable Mittag-Leffler-type function $\bar{F}^{(3)}_{C}$. Analogous results for the other three-variable Mittag-Leffler-type functions $\bar{F}^{(3)}_{A}$, $\bar{F}^{(3)}_{B}$ and $\bar{F}^{(3)}_{D}$ can be derived similarly.

Theorem 4. *Let* $a, b, c_i, w_i \in \mathbb{C}$ $(i = \{1,2,3\})$, $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$ *and* $\min\{\alpha_k, \beta_k, \gamma_k\} > 0$ $(k = \{1, \cdots, 4\})$. *Then*

$$ {}^{RL}\mathfrak{I}^{\alpha}_{\tau+} \left\{ (x-\tau)^{a-1} \bar{F}^{(3)}_{C} \left(\begin{matrix} a, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ c_1, \alpha_3; c_2, \beta_3; c_3, \gamma_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{matrix} \middle| \sigma_1, \sigma_2, \sigma_3 \right) \right\} $$

$$ = \frac{\Gamma(a-\alpha)}{\Gamma(a)} (x-\tau)^{a-1} \bar{F}^{(3)}_{C} \left(\begin{matrix} a-\alpha, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ c_1, \alpha_3; c_2, \beta_3; c_3, \gamma_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{matrix} \middle| \sigma_1, \sigma_2, \sigma_3 \right) \tag{42} $$

$$ (\Re(a) > \Re(\alpha) > 0), $$

where

$$ \sigma_1 = w_1 (x-\tau)^{\alpha_1}, \quad \sigma_2 = w_2 (x-\tau)^{\beta_1} \quad \text{and} \quad \sigma_3 = w_3 (x-\tau)^{\gamma_1}. \tag{43} $$

For the Riemann–Liouville fractional derivative operator ${}^{RL}D^{\alpha}_{\tau+}$, *it is asserted that*

$$ {}^{RL}D^{\alpha}_{\tau+} \left\{ (x-\tau)^{a+\alpha-1} \bar{F}^{(3)}_{C} \left(\begin{matrix} a, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ c_1, \alpha_3; c_2, \beta_3; c_3, \gamma_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{matrix} \middle| \sigma_1, \sigma_2, \sigma_3 \right) \right\} $$

$$ = \frac{\Gamma(a+\alpha)}{\Gamma(a)} (x-\tau)^{a-1} \bar{F}^{(3)}_{C} \left(\begin{matrix} a+\alpha, \alpha_1, \beta_1, \gamma_1; b, \alpha_2, \beta_2, \gamma_2; \\ c_1, \alpha_3; c_2, \beta_3; c_3, \gamma_3; c_4, \alpha_4; c_5, \beta_4; c_6, \gamma_4; \end{matrix} \middle| \sigma_1, \sigma_2, \sigma_3 \right) \tag{44} $$

$$ (\Re(a) > \Re(\alpha) > 0), $$

where σ_1, σ_2 *and* σ_3 *are given by* (43).

Proof. For the Riemann-Liouville fractional integral operator defined by (40), it is easily seen that (see [18])

$$^{RL}\mathfrak{I}^{\alpha}_{\tau+}\{(x-\tau)^{\mu}\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(x-\tau)^{\mu+\alpha} \quad (\Re(\alpha)>0;\ \Re(\mu)>-1). \tag{45}$$

Similarly, from the definition (44) of the Riemann-Liouville fractional derivative operator $^{RL}D^{\alpha}_{\tau+}$, we have (see [18])

$$^{RL}D^{\alpha}_{\tau+}\{(x-\tau)^{\mu}\} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-\tau)^{\mu-\alpha} \quad (\Re(\alpha)\geq 0;\ \Re(\mu)>-1). \tag{46}$$

The assertions (42) and (44) of Theorem 4 can now be established by using the formulas (45) and (46), respectively, in conjunction with the triple-series representing the three-variable Mittag-Leffler-type function $\bar{F}^{(3)}_{C}$. □

Some corollaries and consequences of the above developments are recorded below.

Result 1. For $n \in N$, we have

$$\left(\frac{d}{dx}\right)^{n}\left\{(x-\tau)^{a+n-1}\bar{F}^{(3)}_{C}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix} |\sigma_1,\sigma_2,\sigma_3\right)\right\}$$

$$=\frac{\Gamma(a+n)}{\Gamma(a)}(x-\tau)^{a-1}\bar{F}^{(3)}_{C}\left(\begin{matrix} a+n,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix} |\sigma_1,\sigma_2,\sigma_3\right) \tag{47}$$

where σ_1, σ_2 and σ_3 are given, as before, by (43).

Result 2. For $\nu_1,\nu_2,\nu_3 \in C$ $(\min\{\Re(\nu_1),\ \Re(\nu_2),\ \Re(\nu_3)\}>0)$, we have

$$^{RL}\mathfrak{I}^{\nu_1}_{\tau+}\,{}^{RL}\mathfrak{I}^{\nu_2}_{\tau+}\,{}^{RL}\mathfrak{I}^{\nu_3}_{\tau+}\Big\{(x-\tau)^{c_1-1}(y-\tau)^{c_2-1}(z-\tau)^{c_3-1}$$

$$\cdot\bar{F}^{(3)}_{C}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix} |\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3\right)\Big\}$$

$$=(x-\tau)^{c_1+\nu_1-1}(y-\tau)^{c_2+\nu_2-1}(z-\tau)^{c_3+\nu_3-1}$$

$$\cdot\bar{F}^{(3)}_{C}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1+\nu_1,\alpha_3;c_2+\nu_2,\beta_3;c_3+\nu_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix} |\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3\right) \tag{48}$$

where

$$\bar{\sigma}_1 = w_1(x-\tau)^{\alpha_3}, \quad \bar{\sigma}_2 = w_2(y-\tau)^{\beta_3}, \quad \bar{\sigma}_3 = w_3(z-\tau)^{\gamma_3}, \tag{49}$$

it being tacitly assumed that the Riemann-Liouville fractional integral operators $^{RL}\mathfrak{I}^{\nu_1}_{\tau+}$, $^{RL}\mathfrak{I}^{\nu_2}_{\tau+}$ and $^{RL}\mathfrak{I}^{\nu_3}_{\tau+}$ apply, individually and respectively, on the first, second and third variables of the the threevariable Mittag-Leffler-type function $\bar{F}^{(3)}_{C}$.

Result 3. For $\nu_1,\nu_2,\nu_3 \in C$ $(\min\{\Re(\nu_1),\ \Re(\nu_2),\ \Re(\nu_3)\}>0)$, we have

$$^{RL}D^{\nu_1}_{\tau+}\,{}^{RL}D^{\nu_2}_{\tau+}\,{}^{RL}D^{\nu_3}_{\tau+}\Big\{(x-\tau)^{c_1-1}(y-\tau)^{c_2-1}(z-\tau)^{c_3-1}$$

$$\cdot\bar{F}^{(3)}_{C}\left(\begin{matrix} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4;\end{matrix} |\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3\right)\Big\}$$

$$= (x-\tau)^{c_1-\nu_1-1}(y-\tau)^{c_2-\nu_2-1}(z-\tau)^{c_3-\nu_2-1}$$

$$\cdot \bar{F}_C^{(3)}\left(\begin{array}{c} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1-\nu_1,\alpha_3;c_2-\nu_2,\beta_3;c_3-\nu_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4; \end{array} |\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3\right) \quad (50)$$

where $\bar{\sigma}_1$, $\bar{\sigma}_2$ and $\bar{\sigma}_3$ are given by (49), it being tacitly assumed that the Riemann-Liouville fractional derivative operators ${}^{RL}D_{\tau+}^{\nu_1}$, ${}^{RL}D_{\tau+}^{\nu_2}$ and ${}^{RL}D_{\tau+}^{\nu_3}$ apply, individually and respectively, on the first, second and third variables of the the three-variable Mittag-Leffler-type function $\bar{F}_C^{(3)}$.

Result 4. For $\nu_1 = p$, $\nu_2 = q$ and $\nu_3 = r$ $(p,q,r \in \mathbb{N}_0)$ the above result (50) reduces to the following simple form:

$$\frac{\partial^{p+q+r}}{\partial x^p \partial y^q \partial z^r}\left\{(x-\tau)^{c_1-1}(y-\tau)^{c_2-1}(z-\tau)^{c_3-1}\right.$$

$$\left.\cdot \bar{F}_C^{(3)}\left(\begin{array}{c} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1,\alpha_3;c_2,\beta_3;c_3,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4; \end{array} |\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3\right)\right\}$$

$$= (x-\tau)^{c_1-p-1}(y-\tau)^{c_2-q-1}(z-\tau)^{c_3-r-1}$$

$$\cdot \bar{F}_C^{(3)}\left(\begin{array}{c} a,\alpha_1,\beta_1,\gamma_1;b,\alpha_2,\beta_2,\gamma_2; \\ c_1-p,\alpha_3;c_2-q,\beta_3;c_3-r,\gamma_3;c_4,\alpha_4;c_5,\beta_4;c_6,\gamma_4; \end{array} |\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3\right) \quad (51)$$

where $\bar{\sigma}_1$, $\bar{\sigma}_2$ and $\bar{\sigma}_3$ are given by (49).

Conclusion

In this article, we have first investigated various families of multi-variable hypergeometric functions including (for example) the four Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ of n variables for $n \in \mathbb{N}\backslash\{1,2\}$ and their generalized forms which are known as the Srivastava-Daoust hypergeometric functions of two and more variables. We have then introduced and studied a set of four three-variable Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$, $\bar{F}_C^{(3)}$ and $\bar{F}_D^{(3)}$,which are analogous to the the Lauricella functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ of three variables. Among the various properties and characteristics of the three-variable Mittag-Leffler-type functions $\bar{F}_A^{(3)}$, $\bar{F}_B^{(3)}$ and $\bar{F}_D^{(3)}$, which we have presented in this article, include their relationships with other extensions and generalizations of the classical Mittag-Leffler functions, their three-dimensional convergence regions, their Euler-type integral representations, their one as well as three-dimensional Laplace transforms, their connections with the Riemann-Liouville operators of fractional calculus, and the systems of partial differential equations which are associated with them. We believe that, analogous to the lines of the developments in [27] and [28] based upon the two-variable Mittag-Leffler-type functions $E_1(x,y)$ and $E_2(x,y)$, our obtained results are potentially useful in similar future investigations.

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