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Non-Local Initial-Boundary Value Problem for a Degenerate Second Order Equation with Fractional Caputo Derivative and Riemann-Liouville Integral

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Abstract. Recently, initial-boundary problems in a rectangular domain for differential equations in partial derivatives of both even and odd order have been intensively studied. In this case, non-degenerate equations or equations that degenerate on one side of the quadrilateral are taken as the object of study. But initial boundary problems (both local and non-local) for equations with two or three lines of degeneracy remain unexplored. In this paper, in a rectangular domain, a second-order equation degenerating on two sides of the rectangular and contains the with fractional Caputo derivative and Riemann-Liouville integro operators has been considered. For this equation, an initial-boundary problem is formulated and investigated, with non-local conditions connecting the values of the desired function and its derivatives up to the third order (inclusive), taken on the sides of the rectangle. From the beginning, the uniqueness of the solution of the formulated problem was proved by the method of energy integrals. Then, the spectral problem that arises when applying the Fourier method based on the separation of variables to the considered initial-boundary problem has been investigated. The Green's function of the spectral problem was constructed, with the help of which it is equivalently reduced to an integral Fredholm equation of the second kind with a symmetric kernel, which implies the existence of a countable number of eigenvalues and eigenfunctions of the spectral problem. A theorem is proved for expanding a given function into a uniformly convergent series in terms of a system of eigenfunctions. The solution of the considered is written as the sum of a Fourier series with respect to the system of eigenfunctions of the spectral problem. The uniform convergence of this series and the series obtained from it by term-by-term differentiation is studied. An estimate for solution to problem is obtained, from which follows its continuous dependence on the given functions.

Key words: initial-boundary problem, Caputo fractional derivative, degenerate differential equation, Mittag-Leffler-type functions of two variables.

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МАТЕМАТИКА

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Научная статья

Полный текст на английском языке

УДК 517.956.4/.44

Нелокальная начально-краевая задача для вырожденного уравнения второго порядка с дробной производной Капуто и интегралом Римана-Лиувилля

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Аннотация. В последнее время интенсивно изучаются начально-краевые задачи в прямоугольной области для дифференциальных уравнений в частных производных как четного, так и нечетного порядка. При этом в качестве объекта исследования берутся невырожденные уравнения или уравнения, вырождающиеся на одной стороне четырехугольника. Однако начально-краевые задачи (как локальные, так и нелокальные) для уравнений с двумя или тремя линиями вырождения остаются неисследованными. В данной работе в прямоугольной области рассматривается уравнение второго порядка, вырождающееся на двух сторонах прямоугольника и содержащее с дробной производной Капуто и интегрооператоры Римана-Лиувилля. Для этого уравнения формулируется и исследуется начально-краевая задача с нелокальными условиями, связывающими значения искомой функции и ее производных до третьего порядка (включительно), взятые на сторонах прямоугольника. С самого начала единственность решения сформулированной задачи доказывалась методом интегралов энергии. Затем исследована спектральная задача, возникающая при применении метода Фурье, основанного на разделении переменных, к рассматриваемой начально-краевой задаче. Построена функция Грина спектральной задачи, с помощью которой она эквивалентно сводится к интегральному уравнению Фредгольма второго рода с симметричным ядром, что влечет существование счетного числа собственных значений и собственных функций спектральной задачи. Доказана теорема о разложении заданной функции в равномерно сходящийся ряд по системе собственных функций. Решение рассматриваемой задачи записывается в виде суммы ряда Фурье по системе собственных функций спектральной задачи. Исследуется равномерная сходимость этого ряда и ряда, полученного из него почленным дифференцированием. Получена оценка решения задачи, из которой следует его непрерывная зависимость от заданных функций.

Ключевые слова: начально-краевая задача, дробная производная Капуто, вырожденное дифференциальное уравнение, функции типа Миттаг-Леффлера двух переменных

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Introduction

It is well known that the theory of differential equations has a long and rich history. Until the last quarter of the twentieth century, this theory primarily focused on differential equations of integer order. With the development of fractional (differential and integral) analysis in the late twentieth century, researchers began to study differential equations involving fractional derivatives. At present, numerous scientific articles have been published that address initial, boundary, and spectral problems for differential equations (both ordinary and partial) containing differential operators of fractional order in various forms (see, for example, [1]– [5], and the references therein). The books [6] and [7] have played a significant role in the development of this area.

Recently, there has been growing interest in the study of boundary value problems for second-order mixed-type equations involving fractional-order differential operators (see, for example, [8]– [14]).

In the aforementioned and other related works, only non-degenerate equations have been considered. However, both local and non-local boundary value problems for degenerate partial differential equations involving fractional derivatives of the unknown function remain largely unstudied. The investigation of boundary value problems for such equations is of great importance not only from a theoretical standpoint but also from a practical one, as these equations frequently arise in the mathematical modeling of various problems in gas and hydrodynamics, the theory of small surface bendings, mathematical biology, and other scientific fields.

Problem statement

Consider the following degenerate partial integro-differential equation in the domain $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t < T\}$:

$$ {}^{C}D^{\alpha}_{0t}u(t,x) + aI^{\beta}_{0t}u(t,x) + bu(t,x) = \left[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)\right]_x, \tag{1} $$

where T, α, β, α_1, β_1, a, b are given real numbers, $T > 0$, $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \alpha_1 < 1$, $0 < \beta_1 < 1$ $u(t,x)$ - is an unknown function,

$$ {}^{C}D^{\alpha}_{0t}u(t,x) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-z)^{-\alpha}u_z(z,x)\,dz $$

- fractional derivative in the sense of Caputo [15] of the function $u(t,x)$ with respect to the argument t,

$$ I^{\beta}_{0t}u(t,x) = \frac{1}{\Gamma(\beta)}\int_0^t (t-z)^{\beta-1}u(z,x)\,dz $$

-fractional integral in the Riemann-Liouville [15] sense of the function $u(t,x)$ with respect to the argument t.

Let us study the following initial-boundary value problem for equation (1).

Problem A_{pq}. Find a function $u(t,x)$ with the following properties:
1) $u(t,x)$, $x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x) \in C(\bar{\Omega})$, $[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)]_x$, ${}^{C}D^{\alpha}_{0t}u(t,x) \in C(\Omega)$;
2) in the domain Ω satisfies equation (1);
3) The following boundary conditions hold on $\partial\Omega$:

$$pu(t,0) = qu(t,1),\ q\,[x^{\alpha_1}u_x(t,x)]|_{x=0} = p\left[(1-x)^{\beta_1}u_x(t,x)\right]\big|_{x=1}, t \in [0,T]; \quad (2)$$

$$u(0,x) = \varphi(x),\ \ x \in [0,1], \quad (3)$$

where $\varphi(x)$ is a given continuous function, $\varphi(0) = 0$, $\varphi(1) = 0$, p, q are given real numbers, $p^2 + q^2 \neq 0$.

Uniqueness of the solution problem A_{pq}

Before proceeding to prove the uniqueness of the solution to the problem A_{pq}, we present the following auxiliary lemmas.

Lemma 1. *If $V(0,x) = 0$ and $\alpha \in (0,1)$, then the following inequality holds [19]:*

$$I(x) = \int_0^t V(t,x)\,{}^{C}D^{\alpha}_{0t}V(t,x)dt \geq 0.$$

Lemma 2. *If $\beta \in (0,1)$, then the following inequality holds [19]:*

$$I(x) = \int_0^t V(t,x)I^{\beta}_{0t}V(t,x)dt \geq 0.$$

Theorem 1. *If $a \geq 0$, $b \geq 0$, $p \neq q$ then problem A_{pq} cannot have more than one solution.*

Proof. Let us assume that there are two solutions $u_1(t,x)$ and $u_2(t,x)$ of the problem A_{pq}. Their difference will be denoted by $u(t,x)$. Then the function $u(t,x)$ is a solution of equation (1) under homogeneous boundary conditions.

We multiply equation (1) by the function $u(t,x)$ and integrate over the domain Ω:

$$\int_0^1 dx \int_0^T u(t,x)\ {}^{C}D^{\alpha}_{0t}u(t,x)dt + a\int_0^1 dx \int_0^T u(t,x)I^{\beta}_{0t}u(t,x)dt + b\int_0^1 dx \int_0^T u^2(t,x)dt =$$

$$= \int_0^T dt \int_0^1 u(t,x)\left[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)\right]_x dx.$$

Applying the rule of integration by parts to the inner integral of the last term and taking into account conditions (2), we have

$$\int_0^T dt \int_0^1 x^{\alpha_1}(1-x)^{\beta_1}u_x^2(t,x)dx =$$

$$-\int_0^1 dx \int_0^T u(t,x)\,{}^{C}D_{0t}^{\alpha}u(t,x)dt - a\int_0^1 dx \int_0^T u(t,x)I_{0t}^{\beta}u(t,x)dt - b\int_0^1 dx \int_0^T u^2(t,x)dt,$$

from which, by virtue of Lemma 1, 2 and $a \geq 0$, $b \geq 0$, the equality follows:

$$\int_0^T dt \int_0^1 x^{\alpha_1}(1-x)^{\beta_1} u_x^2(t,x)dx = 0.$$

Therefore, $x^{\alpha_1}(1-x)^{\beta_1} u_x^2(t,x) = 0$, i.e. $u_x(t,x) = 0$ for $(t,x) \in \Omega$. Then $u(t,x) = \tau(t)$, where $\tau(t)$ - is an arbitrary function. Satisfying this function with the conditions $pu(t,0) = qu(t,0)$, and taking into account $p \neq q$, we obtain $u(t,x) \equiv 0$, $(x,t) \in \Omega$. Since $u(x,t) \in C(\bar{\Omega})$, then $u(t,x) \equiv 0$, $(t,x) \in \Omega$. Then, $u_1(t,x) = u_2(t,x)$, $(t,x) \in \bar{\Omega}$.

The proof of Theorem 1 is complete. □

Study of the spectral problem

With the formal application of the Fourier method to the stated problem A_{pq}, the following spectral problem arises: find those values of the parameter λ for which non-trivial solutions of the equation

$$Mv \equiv -[x^{\alpha_1}(1-x)^{\beta_1}v'(x)]' = \lambda v(x), \quad 0 < x < 1, \tag{4}$$

exist, satisfying the following conditions:

$$\begin{cases} v(x), \quad x^{\alpha_1}(1-x)^{\beta_1}v'(x) \in C[0,1]; \\ pv(0) = qv(1), q\left[x^{\alpha_1}v'(x)\right]\big|_{x=0} = p\left[(1-x)^{\beta_1}v'(x)\right]\Big|_{x=1}. \end{cases} \tag{5}$$

It is easy to show that the spectral problem $\{(4),(5)\}$ has a solution $v(x) \not\equiv 0$ only for $p \neq q$, $\lambda > 0$, and it is equivalent to the following integral equation with symmetric kernel:

$$v(x) = \lambda \int_0^1 G(x,s)\,v(s)\,ds, \tag{6}$$

where the function $G(x,s)$ is defined by formula

$$G(x,s) = \begin{cases} \dfrac{p}{p-q}\displaystyle\int_0^x \frac{dz}{z^{\alpha_1}(1-z)^{\beta_1}} + \frac{q}{p-q}\int_0^s \frac{dz}{z^{\alpha_1}(1-z)^{\beta_1}} + \frac{q^2\sigma_1}{(p-q)^2}, & x \leq s; \\ \dfrac{p}{p-q}\displaystyle\int_0^s \frac{dz}{z^{\alpha_1}(1-z)^{\beta_1}} + \frac{q}{p-q}\int_0^x \frac{dz}{z^{\alpha_1}(1-z)^{\beta_1}} + \frac{q^2\sigma_1}{(p-q)^2}, & x \geq s, \end{cases}$$

where $\sigma_1 = [\Gamma(1-\alpha_1)\Gamma(1-\beta_1)]/\Gamma(2-\alpha_1-\beta_1)$.

Then, according to the theory of integral equations, the integral equation (6) therefore, the problem (4),(5) has a countable number of eigenvalues $0 < \lambda_1 \leq$

$\lambda_2 \leq \lambda_3 \leq ... \leq \lambda_k, ...$ condensing at infinity, and the corresponding eigenfunctions $v_1(x)$, $v_2(x)$, $v_3(x), ..., v_k(x), ...$ form orthonormal system in the space $L_2(0,1)$.

The following lemmas can be proven on their own. Therefore, we present them without proof.

Lemma 3. *Let the function* $h(x)$ *satisfy the following conditions:*

$$h(x) \in C[0,1], \tag{7}$$

$$x^{\alpha_1}(1-x)^{\beta_1}h'(x) \in C[0,1], \tag{8}$$

$$ph(0) = qh(1), \tag{9}$$

$$q\left[x^{\alpha_1}h'(x)\right]|_{x=0} = p\left[(1-x)^{\beta_1}h'(x)\right]\big|_{x=1}, \tag{10}$$

$Mh(x) \in L_2(0,1)$. *Then, on the segment* $[0,1]$ *it can be expanded into an absolutely and uniformly convergent series by the system of eigen functions* $\{v_k(x)\}_{k=1}^{\infty}$ *of the problem* (4), (5)*:*

$$h(x) = \sum_{k=1}^{+\infty} h_k v_k(x),$$

where $h_k = \int_0^1 h(x)v_k(x)dx$.

Lemma 4. *The following series converge uniformly on the segment* $[0,1]$*:*

$$\sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}, \quad \sum_{k=1}^{+\infty} \frac{\left[x^{\alpha}(1-x)^{\beta}v'_k(x)\right]^2}{\lambda_k^2}.$$

Lemma 5. *If function* $h(x)$ *the conditions* (7), (9), $x^{\alpha_1/2}(1-x)^{\beta_1/2}h'(x) \in L_2(0,1)$ *hold true, then the inequality*

$$\sum_{k=1}^{+\infty} \lambda_k h_k^2 \leq \int_0^1 x^{\alpha_1}(1-x)^{\beta_1}[h'(x)]^2 dx,$$

is valid, where $h_k = \int_0^1 h(x)v_k(x)dx$.

Lemma 6. *If function* $h(x)$ *the conditions* (7)-(10), *and* $Mh(x)$ *function conditions* (7), (9) $x^{\alpha_1/2}(1-x)^{\beta_1/2}[Mh(x)]' \in L_2(0,1)$ *hold true, then the inequality*

$$\sum_{k=1}^{+\infty} \lambda_k^3 h_k^2 \leq \int_0^1 x^{\alpha_1}(1-x)^{\beta_1}\left\{[Mh(x)]'\right\}^2 dx,$$

is valid, where $h_k = \int_0^1 h(x)v_k(x)dx$.

Uniqueness and stability of a solution to problem A_{pq}

The solution to the problem A_{pq} is formally sought in the form

$$u(x,t)=\sum_{k=1}^{+\infty}u_k(t)\,v_k(x), \tag{11}$$

where $v_k(x)$, $k\in N$ are the eigen functions of the problem (4), (5) and $u_k(t)$, $k\in N$ are the unknown functions to be determined.

Substituting (11) into (1) and (3), with respect to $u_k(t)$, $k\in N$ we obtain the following problem:

$${}^{C}D_{0t}^{\alpha}u_k(t)+aI_{0t}^{\beta}u_k(t)+(\lambda_k+b)\,u_k(t)=0,\quad 0<t<T, \tag{12}$$

$$u_k(0)=\varphi_k, \tag{13}$$

where $\varphi_k=\int\limits_0^1\varphi(x)v_k(x)dx$, $k\in N$.

To solve this problem, we apply the operator I_{0t}^{α} to equation (12) and, using the properties $I_{0t}^{\alpha}\,{}^{C}D_{0t}^{\alpha}u_k(t)=u_k(t)-u_k(0)$ and $I_{0t}^{\alpha}\,I_{0t}^{\beta}u_k(t)=I_{0t}^{\alpha+\beta}u_k(t)$ and condition (13), we obtain the Volterra integral equation of the second kind:

$$u_k(t)+aI_{0t}^{\alpha+\beta}u_k(t)+(\lambda_k+b)\,I_{0t}^{\alpha}u_k(t)=\varphi_k. \tag{14}$$

To solve equation (14), we apply the method of successive approximations:

$$u_{k,m}(t)=u_{k,0}(t)-aI_{0t}^{\alpha+\beta}u_{k,m}(t)-(\lambda_k+b)\,I_{0t}^{\alpha}u_{k,m}(t),\quad m=1,2,3,...,u_{k,0}(t)=\varphi_k. \tag{15}$$

Using formula $I_{0t}^{\alpha}\,I_{0t}^{\beta}u_k(t)=I_{0t}^{\alpha+\beta}u_k(t)$, we calculate $u_{k,m}(t)$:

$$u_{k,m}(t)=\sum_{n=0}^{m}\sum_{j=0}^{n}C_n^j(-a)^{n-j}(-\lambda_k-b)^jI_{0t}^{\gamma(n-j)+\alpha j}u_{k,0}(t), \tag{16}$$

where $\gamma=\alpha+\beta$, $C_n^j=\dfrac{n!}{j!(n-j)!}$.

According to the theory of integral equations [18], if there exists $\lim\limits_{m\to+\infty}u_{k,m}(t)$ uniformly in t, then its limit function is a solution of the integral equation (14).

Passing to the limit at $m\to+\infty$ in (16) and substituting the expression $u_{k,0}(t)$, we obtain a solution to equation (14) in the form

$$u_k(t)=\varphi_k\sum_{n=0}^{+\infty}\sum_{j=0}^{n}C_n^j(-a)^{n-j}(-\lambda_k-b)^jI_{0t}^{\gamma(n-j)+\alpha j}(1), \tag{17}$$

Using the formula $I_{0t}^{\alpha_2}(1)=\dfrac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}$, we can write (17) in the form

$$u_k(t)=\varphi_k\sum_{n=0}^{+\infty}\sum_{j=0}^{n}C_n^j(-a)^{n-j}(-\lambda_k-b)^j\frac{t^{\gamma(n-j)+\alpha j}}{\Gamma(\gamma(n-j)+\alpha j+1)}. \tag{18}$$

We will prove that the series (18) is convergent a little later.

Using the formula [20] $\sum\limits_{k=0}^{+\infty}\sum\limits_{n=0}^{k} B_{k,n} = \sum\limits_{n=0}^{+\infty}\sum\limits_{k=n}^{+\infty} B_{k,n}$, we can write (18) in the form

$$u_k(t) = \varphi_k \sum_{j=0}^{+\infty}\sum_{n=j}^{+\infty} C_n^j (-a)^{n-j}(-\lambda_k - b)^j \frac{t^{\gamma(n-j)+\alpha j}}{\Gamma(\gamma(n-j)+\alpha j+1)}.$$

If we introduce the notation $n = j + m$ here, we can write the last equation in the form

$$u_k(t) = \varphi_k \sum_{j=0}^{+\infty}\sum_{m=0}^{+\infty} C_{m+j}^j (-a)^m (-\lambda_k - b)^j \frac{t^{\gamma m+\alpha j}}{\Gamma(\gamma m+\alpha j+1)}.$$

This equation can be written using the Mittag-Leffer function of two variables as follows:

$$u_k(t) = \varphi_k E_2\left(\begin{matrix}1,1,1;1,0;\\ 1,\gamma,\alpha;1,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)t^\alpha\end{matrix}\right), \quad t\in[0,T], \tag{19}$$

where E_2 -Mittag-Leffer function of two variables [21]

$$E_2\left(\begin{matrix}\gamma_1,\alpha_1,\beta_1;\gamma_2,\alpha_2;\\ \delta_1,\alpha_3,\beta_2;\delta_2,\alpha_4;\delta_3,\beta_3;\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right) =$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma_1)_{\alpha_1 m+\beta_1 n}(\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1+\alpha_3 m+\beta_2 n)} \frac{x^m}{\Gamma(\delta_2+\alpha_4 m)} \frac{y^n}{\Gamma(\delta_3+\beta_3 m)}, \tag{20}$$

$x, y, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3 \in R$,

$$(\gamma_1)_{\alpha_1 m+\beta_1 n} = \frac{\Gamma(\gamma_1+\alpha_1 m+\beta_1 n)}{\Gamma(\gamma_1)}.$$

If $\alpha_3+\alpha_4-\alpha_1-\alpha_2>0$ and $\beta_2+\beta_3-\beta_1>0$, then the series (20) is absolutely and uniformly convergent for arbitrary $\forall(x,y)$ [22].

It is not difficult to show that the following formulas are valid for the function E_2:

$$I_{0t}^\beta\left(E_2\left(\begin{matrix}1,1,1;1,0;\\ 1,\gamma,\alpha;1,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)t^\alpha\end{matrix}\right)\right) = t^\beta E_2\left(\begin{matrix}1,1,1;1,0;\\ 1+\beta,\gamma,\alpha;1,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)t^\alpha\end{matrix}\right),$$

$${}^C D_{0t}^\alpha\left(E_2\left(\begin{matrix}1,1,1;1,0;\\ 1,\gamma,\alpha;1,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)t^\alpha\end{matrix}\right)\right) = -(\lambda_k+b)E_{\alpha,1}(-(\lambda_k+b)t^\alpha) +$$

$$-at^{\gamma-\alpha}E_2\left(\begin{matrix}2,1,1;1,0;\\ 1+\gamma-\alpha,\gamma,\alpha;2,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)t^\alpha\end{matrix}\right),$$

$$E_2\left(\begin{matrix}1,1,1;1,0;\\ 1,\gamma,\alpha;1,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)t^\alpha\end{matrix}\right) =$$

$$= E_{\alpha,1}(-(\lambda_k+b)t^\alpha) - at^\gamma E_2\left(\begin{matrix}2,1,1;1,0;\\ \gamma+1,\gamma,\alpha;2,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)t^\alpha\end{matrix}\right),$$

$$E_2\left(\begin{matrix}1,1,1;1,0;\\ 1+\gamma-\alpha,\gamma,\alpha;1,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)\,t^\alpha\end{matrix}\right)-E_2\left(\begin{matrix}2,1,1;1,0;\\ 1+\gamma-\alpha,\gamma,\alpha;2,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)\,t^\alpha\end{matrix}\right)=$$

$$=(\lambda_k+b)\,t^\alpha E_2\left(\begin{matrix}2,1,1;1,0;\\ 1+\gamma,\gamma,\alpha;2,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)\,t^\alpha\end{matrix}\right),$$

where $E_{\alpha_2,\beta_2}(z)$ - Mittag-Leffler function [15]:

$$E_{\alpha_2,\beta_2}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha_2 k+\beta_2)},\ \alpha_2>0.$$

Using these formulas, it can be proven that (19) is a solution to problem (12), (13).

It can be shown directly that (19) can be written in the form

$$u_k(t)=\varphi_k\int_0^1\int_0^{+\infty}\frac{e^{-\eta}}{\sqrt{\xi(1-\xi)}}e^{1,\,1/2}_{1,\,-\alpha}\left[-at^\gamma(1-\xi)^\gamma\eta\right]e^{1,\,1/2}_{1,\,-\gamma}\left[-(\lambda_k+b)\,t^\alpha\xi^\gamma\eta\right]d\eta\,d\xi,\quad t\in[0,T],\tag{21}$$

where $e^{\mu_3,\delta_3}_{\alpha_3,\beta_3}(z)$ - Wright-type function [5]:

$$e^{\mu_3,\delta_3}_{\alpha_3,\beta_3}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha_3 k+\mu_3)\Gamma(\delta_3-\beta_3 k)},\ \alpha_3>0,\ \alpha_3>\beta_3.\tag{22}$$

If $z<0$, $\mu_3\geq 0$, $\delta_3\geq\beta_3$, the following inequality holds for function (22) [see page 87 in [5]]:

$$e^{\mu_3,\delta_3}_{\alpha_3,\beta_3}(z)<\frac{1}{\Gamma(\mu_3)\Gamma(\delta_3)}\tag{23}$$

If we say $a\geq 0$, $b\geq 0$ and note that $\lambda_k>0$, then using inequality (23), we obtain the following inequality for (21):

$$|u_k(t)|\leq|\varphi_k|,\ t\in[0,T].\tag{24}$$

Substituting (19) into (11), we find a formal solution to the problem A_{pq} :

$$u(t,x)=\sum_{k=1}^{+\infty}\varphi_k E_2\left(\begin{matrix}1,1,1;1,0;\\ 1,\gamma,\alpha;1,1;1,1;\end{matrix}\middle|\begin{matrix}-at^\gamma\\ -(\lambda_k+b)t^\alpha\end{matrix}\right)v_k(x),\quad \gamma=\alpha+\beta.\tag{25}$$

Theorem 2. *If $a\geq 0$, $b\geq 0$, $p\neq q$ and function $\varphi(x)$ satisfy the conditions of Lemma 6, then of the series* (25) *determines the unique solution of the problem A_{pq}.*

Proof. To prove the theorem, it is sufficient to prove that the series (25) and the series corresponding to the functions $x_1^\alpha(1-x)^{\beta_1}u_x(x,t)$ converge uniformly in $\overline{\Omega}$, and the series corresponding to the functions $[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)]_x$, ${}^C\!D^\alpha_{0t}u(t,x)\in C(\Omega)$ converges uniformly on any compact $\Omega_1\subset\Omega$.

Consider the series corresponding to the function $[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)]_x$.

Using inequality (24), then from (25) follows the inequality

$$\left|[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)]_x\right| \le \sum_{k=1}^{+\infty}|\varphi_k|\left|[x^{\alpha_1}(1-x)^{\beta_1}v_k'(x)]'\right|.$$

Hence, by virtue of equation (4), on any compact set $\Omega_1 \subset \Omega$, we have

$$\left|[x^{\alpha_1}(1-x)^{\beta_1}u_x(t,x)]_x\right| \le \sum_{k=1}^{+\infty}|\lambda_k\varphi_k v_k(x)|. \tag{26}$$

Based on the Cauchy-Schwartz inequality, we have

$$\sum_{k=1}^{+\infty}|\lambda_k\varphi_k v_k(x)| = \sum_{k=1}^{+\infty}\left|\lambda_k^{3/2}\varphi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le \left(\sum_{k=1}^{+\infty}\lambda_k^3\varphi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right)^{1/2}.$$

Here,by Lemmas 6 and 4, the series on the right-hand side converge uniformly with respect to $x \in [0,1]$. Then the series on the left-hand side converges uniformly with respect to $x \in [0,1]$. Therefore, the series (26) converges absolutely and uniformly on the compact set $\Omega_1 \subset \Omega$. The convergence of the remaining series are proved similarly. Theorem 2 has been proved. □

Theorem 3. *If the conditions of Theorem 2 are met, then the following estimate is true for the solution of problem* A_{pq}*:*

$$\|u(x,t)\|_{C(\overline{\Omega})} \le C_1\|\varphi'(x)\|_{L_{2,r}}, \tag{27}$$

where $C_1 = \sup\limits_{[0,1]}\sqrt{G(x,x)}$, $\|f(x)\|_{L_{2,r}} = \int\limits_0^1 r(x)f^2(x)dx$, $r(x) = x^{\alpha_1}(1-x)^{\beta_1}$.

Proof. By virtue of Lemma 4, Lemma 5 and the inequality $|u_k(t)| \le |\varphi_k|$, the following inequalities hold:

$$|u(x,t)| = \left|\sum_{k=1}^{+\infty}u_k(t)v_k(x)\right| \le \left|\sum_{k=1}^{+\infty}|\varphi_k||v_k(x)|\right| = \left|\sum_{k=1}^{+\infty}\sqrt{\lambda_k}\varphi_k\right|\left|\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le$$

$$\le \left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2} \le \left[G(x,x)\int_0^1 x^{\alpha}(1-x)^{\beta}[\varphi'(x)]^2dx\right]^{1/2} \le C_1\|\varphi'(x)\|_{L_{2,r}(0,1)}.$$

From this, inequality (27) follows. Theorem 3 is proven.□

Conclusion

In this paper, a non-local initial-boundary value problem for a degenerate second-order equation with a fractional Caputo derivative and a Riemann–Liouville integral is considered in a rectangular domain. Using the method of separation of variables, a solution to the problem is obtained in the form of a series that converges absolutely and uniformly in the closure of the domain under consideration. In addition, the uniqueness of the solution and its continuous dependence on the given functions are established.

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