MATHEMATICS

https://doi.org/10.26117/2079-6641-2025-50-1-62-77

Research Article

Full text in English

MSC 35B45, 35K20, 35K57, 35K59

A Diffusive Predator-Prey System with a Free Boundary

M. S. Rasulov* [1,2], ***Sh. M. Jamoldinova*** [2]

[1] V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences,
100174, Tashkent, University street, 9, Uzbekistan

[2] National Research University "TIIAME 100000, Tashkent, K. Niyoziy str., 39, Uzbekistan

Abstract. In this paper, we consider a problem with a free boundary for a diffusive predator-prey system in the one-dimensional case. Nonlinear problems with free boundary are studied using a method based on constructing a priori estimates. Therefore, first, using a method based on constructing a priori estimates, we will determine the constraints on the parameters of the problem, under which it is globally solvable. The first, fundamental estimate, gives the initial information, starting from which one can receive step by step, moving up the scale of Banach spaces. To do this, the problem is reduced to a fixed-boundary problem through a change of variables. The resulting problem has time- and space-dependent coefficients with nonlinear terms. Next, Schauder-type a priori estimates are constructed for the equation with nonlinear terms and a fixed boundary. Based on these estimates, the uniqueness of the solution to the problem is proven. Then, the global existence of a solution to the problem was proved using the Leray-Schauder fixed point theorem.

Key words: free boundary, predator-prey, reaction-diffusion, parabolic equation, aprior bounds, existence and uniqueness.

Received: 03.04.2025; Revised: 16.04.2025; Accepted: 17.04.2025; First online: 18.04.2025

For citation. Rasulov M. S., Jamoldinova Sh. M. A diffusive predator-prey system with a free boundary. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2025, **50**: 1, 62-77. EDN: GKCHHF. https://doi.org/10.26117/2079-6641-2025-50-1-62-77.

Funding. The study was conducted without the support of foundations.

Competing interests. There are no conflicts of interest regarding authorship and publication.

Contribution and Responsibility. All authors contributed to this article. Authors are solely responsible for providing the final version of the article in print. The final version of the manuscript was approved by all authors.

***Correspondence:** E-mail: rasulovms@bk.ru

МАТЕМАТИКА
https://doi.org/10.26117/2079-6641-2025-50-1-62-77
Научная статья
Полный текст на английском языке
УДК 517.956.4/.44

Диффузионная система хищник-жертва со свободной границей

М. С. Расулов[1,2], **Ш. М. Жамолдинова**[1]*

[1] Институт математики им. В. И. Романовского Академии наук Узбекистана,
100174, г. Ташкент, ул. Университетская, 9, Узбекистан
[2] Национальный исследовательский университет "ТИИИМСХ"
100000, г. Ташкент, ул. Кари Ниязов 39, Республика Узбекистан

Аннотация. В данной работе рассматривается задача со свободной границей для диффузионной системы хищник-жертва в одномерном случае. Исследование нелинейных задач со свободными границами методом, основанным на построении априорных оценок. Поэтому сначала с помощью метода, основанного на построении априорных оценок, определим ограничения на параметры задачи, при которых она глобально разрешима. Первая, основополагающая оценка, дает ту начальную информацию, отправляясь от которой можно получать шаг за шагом, двигаясь вверх по шкале банаховых пространств. Для этого задача сводится к задаче с фиксированной границей через замену переменных. Полученная задача имеет зависящие от времени и положения в пространстве коэффициенты с нелинейными слагаемыми. Далее построены априорные оценки типа Шаудера для решения уравнения с нелинейными слагаемыми и фиксированной границей. На основе полученных оценок доказана единственность решения задачи. Затем было доказано глобальное существование решения задачи с помощью теоремы Лерэ-Шаудера о неподвижной точке.

Ключевые слова: свободная граница, хищник-жертва, реакция-диффузия, параболическое уравнение, априорные границы, существование и единственность

Получение: 03.04.2025; Исправление: 16.04.2025; Принятие: 17.04.2025; Публикация онлайн: 18.04.2025

Для цитирования. Rasulov M. S., Jamoldinova Sh. M. A diffusive predator-prey system with a free boundary // *Вестник КРАУНЦ. Физ.-мат. науки.* 2025. Т. 50. № 1. C. 62-77. EDN: GKCHHF. https://doi.org/10.26117/2079-6641-2025-50-1-62-77.

Финансирование. Исследование было проведено без поддержки фондов

Конкурирующие интересы. Конфликтов интересов в отношении авторства и публикации нет.

Авторский вклад и ответственность. Авторы участвовали в написании статьи и полностью несут ответственность за предоставление окончательной версии статьи в печать.

***Корреспонденция:** E-mail: rasulovms@bk.ru

Introduction

The spreading of predator-prey species are a central topic of biology, and significant research has been devoted to better understanding the nature of such spread. Environmental problems require the use of a whole hierarchy of models capable of describing not only different levels of organization of systems, but also the interaction between these levels. However, significant advances have been made in species invasion studies through frontal spread studies (see, eg, [1–3]).

The expansion behavior of biological predator-prey populations in the environment is not just caused by simple individual random move. For example, the migration of prey are affected by the spatial distribution of food and water, in addition often affected by the indirect effects of predators which in turn affects the direction of movement of predators. So, all expansion behavior of this type in nature will have a certain direction. In mathematical models, the convection term is generally used to describe this kind of directional movement. In many studies, the term convection is linear and depends only on the density gradient of the [2, 4] components. However, in general, convection is also affected by the density of the components, which in turn leads to nonlinear convection [5–9]. Such problems are mainly focused on the study of free boundaries, i.e. studying expanding front of species. Generally, free boundaries are used to describe the invasion of biological species in mathematical models.

In [10], author first derived the free boundary problem for a predator-prey system to describe the predator aggregation in prey density areas, where the existence and uniqueness results were provided. Since then, various reaction-diffusion models have been proposed to interpret the predator-prey phenomenon [11–13]. In the articles [14–16], it is assumed that two predator-prey species propagate along the same free boundary. Similar works, but for two-species competition models, can be found in [17, 18]. Also, many mathematical models with a free boundary have been developed to study applied problems [19–24].

In this article, we study a prey-predator system with a free boundary:

$$u_t - d_1 u_{xx} + m_1 v u_x = u\left(a_1 - b_1 u + c_1 v\right), \quad (t, x) \in D, \tag{1}$$

$$v_t - d_2 v_{xx} + m_2 u v_x = v\left(a_2 - b_2 u - c_2 v\right), \quad (t, x) \in Q, \tag{2}$$

$$u\left(0, x\right) = u_0\left(x\right), \quad 0 \le x \le s\left(0\right) \equiv s_0, \tag{3}$$

$$v\left(0, x\right) = v_0\left(x\right), \quad 0 \le x \le l, \tag{4}$$

$$u_x\left(t, 0\right) = 0, \quad 0 \le t \le T, \tag{5}$$

$$u\left(t, s\left(t\right)\right) = 0, \quad 0 \le t \le T, \quad u(t, x) \equiv 0, \quad s(t) < x \le l, \tag{6}$$

$$s'\left(t\right) = -\mu u_x\left(t, s\left(t\right)\right), \quad 0 \le t \le T, \tag{7}$$

$$v_x\left(t, 0\right) = 0, \quad 0 \le t \le T, \tag{8}$$

$$v_x\left(t, l\right) = 0, \quad 0 \le t \le T, \tag{9}$$

where $D = \{(t, x) : 0 < t \le T, 0 < x < s(t)\}$, $Q = \{(t, x) : 0 < t \le T, 0 < x < l\}$; $s\left(t\right)$ is a free boundary, which represent propagation fronts. $u\left(t, x\right)$ and $v\left(t, x\right)$ represent, respectively, densities of predator and prey species that are interacting and migrating in the region D and Q. The prey species migrates in the interval $(0,l)$ and the predator disperses through random diffusion only in a part of the interval $(0,l)$, namely $0 < x < s(t)$, then there is no predator in the remaining part. d_i denotes its respective diffusion rate and the real number a_i describes its net birth rate, $i = 1,2$. b_1 and c_2 are the coefficients of intra-specific competitions, and b_2 and c_1 are the coefficients of inter-specific competitions. And the parameter μ respectively represents the speed of propagation into new region for $u(t,x)$.

Assume that the data of the problem satisfy the following conditions:
(i) d_i, m_i, a_i, b_i, c_i, μ are positive constants, $i = 1,2$;
(ii) $u_0(x), v_0(x) \in C^2[0,s_0]$, $0 < u_0(x) \leq \frac{a_1}{b_1}$ in $(0,s_0)$ $\quad 0 < v_0(x) \leq \frac{a_2}{c_2}$ in $(0,l)$,
$u_0(0) = u_0(s_0) = 0$, $u_0'(s_0) < 0$, $v_0'(0) = v_0'(l) = 0$.

Problem (1)–(9) was studied in [10] in the case $m_1 = m_2 = 0$.

The rest of paper is organized as follows. In section 2 we establish two-sided bounds for $u(t,x)$, $v(t,x)$ and $s'(t)$, and then a Hölder norm bounds for $u(t,x)$, $v(t,x)$. And in section 3 we prove uniqueness and existence results.

A priori estimates

In this section we establish several a priori bounds which will be used later. First of all, let us establish the boundedness of the functions $u(t,x)$, $v(t,x)$, $s'(t)$. Further, a priori estimates for the higher derivatives are established.

Theorem 1. *Let $u(t,x)$, $v(t,x)$, $s(t)$ be a solution of (1)–(9). Then there hold estimates:*

$$0 < u(t,x) \leq M_1, \quad (t,x) \in \overline{D}, \tag{10}$$

$$0 < v(t,x) \leq M_2, \quad (t,x) \in \overline{Q}, \tag{11}$$

$$0 < s'(t) \leq M_3, \qquad 0 \leq t \leq T. \tag{12}$$

Proof. First we prove the positivity of the function $v(t,x)$. Take some arbitrary point $P \in Q$ such that $v(P) = 0$. At this point, the right-hand side of (1) should be zero. And also at this point the function $v(t,x)$ reaches its minimum value. Hence, according to the usual maximum principle $v(P) = v(P_0)$ for all $P_0 \in Q_0 = \{(t_0,x) : 0 < t_0 < T, 0 < x < l\}$ and we arrive at a contradiction. The resulting contradiction proves that $v(t,x) > 0$ in Q (the proof of the positiveness of $u(t,x)$ is similar in D).

To get the upper bounds, we proceed as follows. Consider the following initial value problem:

$$\begin{cases} \frac{d\overline{v}}{dt} = \overline{v}\,(a_2 - c_2\overline{v}), \quad t > 0, \\ \overline{v}(0) = \|v_0\|_\infty. \end{cases}$$

The comparison principle applied to the function $v - \overline{v}$, shows that

$$v(t,x) \leq \sup_{t\geq 0} \overline{v}(t) \leq \max\left\{\|v_0(x)\|, \frac{a_2}{b_2}\right\} \equiv M_2,$$

where
$$\overline{v}(t)=\frac{a_2}{b_2}e^{a_2t}\left(e^{a_2t}-1+\frac{a_2}{c_2\|v_0\|_\infty}\right)^{-1}.$$

Similarly, we construct an auxiliary problem for an ordinary differential equation:
$$\begin{cases}\frac{d\overline{u}}{dt}=\overline{u}\left(a_1+c_1M_2-b_1\overline{u}\right), \quad t>0,\\ \overline{u}(0)=\|u_0\|_\infty.\end{cases}$$

Its solution is given by the following explicit formula:
$$\overline{u}(t)=\frac{a_1+c_1M_2}{b_1}e^{(a_1+c_1M_2)t}\left(e^{(a_1+c_1M_2)t}-1+\frac{a_1+c_1M_2}{b_1\|u_0\|_\infty}\right)^{-1}.$$

Comparing $u(t,x)$ with $\overline{u}(t)$ yields that
$$u(t,x)\le\sup_{t\ge0}\overline{u}(t)\le\max\left\{\|u_0(x)\|,\frac{a_1b_2+c_1a_2}{b_1b_2}\right\}\equiv M_1.$$

We now turn to prove (12). Since the condition (6) and the positivity of the function $u(t,x)$ in the domain D, we find $u_x(t,s(t))<0$. Hence, from (7) we get $s'(t)>0$.

Next we prove an upper bound of $s'(t)$. To do so, we define
$$\Omega=\left\{(t,x):0<t<T, s(t)-\frac{1}{N}<x<s(t)\right\},$$
where N is a positive constant to be determined later. And we construct an auxiliary function on Ω as follows
$$w(t,x)=M_1\left[2N(s(t)-x)-N^2(s(t)-x)^2\right].$$

We will choose N so that $w(t,x)\ge u(t,x)$ holds over Ω. Direct calculations show that, for $(t,x)\in\Omega$,
$$w_t=2M_1Ns'(t)(1-N(s(t)-x)),$$
$$-w_{xx}=2M_1N^2,$$
$$-w_x=2M_1N\left(1-N(s(t)-x)\right).$$

It follows from $s'(t)>0$, (10) and (11) that
$$\begin{aligned}w_t-d_1w_{xx}+m_1vw_x&\ge\left(2d_1N^2-2m_1M_2N\right)M_1\ge(a_1+c_1M_2)M_1\\&>u(a_1-b_1u+c_1v)=u_t-d_1u_{xx}+m_1vu_x\end{aligned}$$
on Ω, provided that
$$N\ge\frac{(a_1+c_1M_2)+\sqrt{m_1^2+2d_1(a_1+c_1M_2)}}{2d_1}.$$

Also note that
$$w\left(t,s(t)-\frac{1}{N}\right)=M_1\ge u\left(t,s(t)-\frac{1}{N}\right),$$

$$w(t, s(t)) = 0 = u(t, s(t)).$$

Thus, if we can choose N such that $u_0(x) \le w(0, x)$ for $x \in \left[s_0 - \frac{1}{N}, s_0\right]$, then we can apply the maximum principle to $w - u$ over Ω to deduce that $u(t, x) \le w(t, x)$ for $(t, x) \in \Omega$. It would then follow that

$$u_x(t, s(t)) \ge w_x(t, s(t)) = -2NM_1,$$

$$s'(t) = -\mu u_x(t, s(t)) \le M_3 \equiv 2NM_1\mu.$$

To complete the proof, we need only find some N independent of T such that

$$u_0(x) \le w(0, x) \quad \text{for} \quad x \in \left[s_0 - \frac{1}{N}, s_0\right],$$

then we can apply the maximum principle to $w - u$.

By the definition of w, we have

$$w(0, x) = M_1 \left[2N\left(s_0 - x\right) - N^2\left(s_0 - x\right)^2\right]$$

if $x \in \left[s_0 - \frac{1}{N}, s_0 - \frac{1}{2N}\right]$, then $w(0, x) \ge \frac{3}{4}M_1$; if $x \in \left[s_0 - \frac{1}{2N}, s_0\right]$, then

$$w_x(0, x) = -2M_1N\left[1 - N\left(s_0 - x\right)\right] \le -M_1N.$$

Therefore upon choosing

$$N = \max\left\{\frac{(a_1 + c_1M_2) + \sqrt{m_1^2 + 2d_1(a_1 + c_1M_2)}}{2d_1}, \frac{4\left\|u_0\right\|_{C^1([0,s_0])}}{3M_1}\right\}$$

we will have

$$w_x(0, x) \le u_0'(x) \quad \text{for} \quad x \in \left[s_0 - \frac{1}{2N}, s_0\right]$$

Since $w\left(0, s_0\right) = u_0\left(s_0\right) = 0$, the above inequality implies

$$w(0, x) \ge u_0(x) \text{ for } x \in \left[s_0 - \frac{1}{2N}, s_0\right]$$

Moreover, for $x \in \left[s_0 - \frac{1}{N}, s_0 - \frac{1}{2N}\right]$, we have

$$w(0, x) \ge \frac{3}{4}M_1, \quad u_0(x) \le \frac{\left\|u_0\right\|_{C^1([0,s_0])}}{N} \le \frac{3}{4}M_1.$$

Therefore $u_0(x) \le w(0, x)$ for $x \in \left[s_0 - \frac{1}{N}, s_0\right]$.

This completes the proof. □

Initial estimates have been received. Now, using the results of the work [25], we establish a priori estimates for u_x, v_x and higher derivatives.

Further, for each equation of the system, we separately formulate the corresponding problem:

$$u_t - d_1u_{xx} + m_1vu_x = u\left(a_1 - b_1u + c_1v\right), \quad (t, x) \in D, \tag{13}$$

$$u\left(0, x\right) = u_0\left(x\right), \quad 0 \le x \le s_0, \tag{14}$$

$$u_x\,(t,0)=0,\quad u\,(t,s(t))=0,\quad 0\le t\le T, \tag{15}$$

$$s'\,(t)=-\mu u_x\,(t,s\,(t))\,,\quad 0\le t\le T, \tag{16}$$

and

$$v_t-d_2v_{xx}+m_2uv_x=v\,(a_2-b_2u-c_2v)\,,\quad (t,x)\in Q, \tag{17}$$

$$v\,(0,x)=v_0(x),\quad 0\le x\le l, \tag{18}$$

$$v_x\,(t,0)=0,\quad 0\le t\le T, \tag{19}$$

$$v_x\,(t,l)=0,\quad 0\le t\le T. \tag{20}$$

First, we obtain a priori estimates for $v(t,x)$. The equation (17) satisfies the conditions of Theorem 3 in [25]:

$$\frac{|B_2|}{A_2}\le K_2(v_x^2+1),\quad K_2>0. \tag{21}$$

where $A_2=d_2$, $B_2=v\,(a_2-b_2u-c_2v)-m_2uv_x$.

Therefore, it takes place:

Theorem 2. *If the conditions* (21) *are fulfilled, then the following estimate is valid*

$$|v_x(t,x)|\le M_4(M_1,M_2,d_2,K_2,\delta),\quad (t,x)\in Q^{\delta}, \tag{22}$$

where $Q^{\delta}=\{(t,x):0<\delta\le t\le T,\delta\le x\le 1-\delta\}$.

In addition, the following estimate is true in the domain $\{(t,x)\in\overline{Q},|u|\le M_1,|v|\le M_2,|v_x|\le M_4\}$

$$|v|_{\frac{2}{3}}^{Q^{2\delta}}\le M_5(M_1,M_2,d_2,K_2,\delta).$$

If $v(t,x)$ *is summable in* $\overline{Q}$ *with a square generalized derivatives* v_{tx}, v_{xx}, *then there hold estimates*

$$|v|_{1+\alpha}^{Q^{2\delta}}\le M_6(M_1,M_2,d_2,M_4,K_2,\delta),\quad 0<\alpha<1, \tag{23}$$

$$|v|_{2+\alpha}^{Q^{4\delta}}\le M_7(M_1,M_2,d_2,M_4,K_2,\delta). \tag{24}$$

If $v_x(t,0)=v_x(t,l)=0$, *then the estimates* (22)–(24) *are true in* $\overline{Q}$.

Next, we will prove a priori bounds for $u(t,x)$. Due to the boundary conditions (13)–(16), we cannot use the results of the work [25]. Therefore, we introduce the following transformation

$$t=t,\quad y=\frac{x}{s\,(t)}$$

to straighten the free boundary. Then the domain D is transformed into the domain $D_1=\{(t,y):0<t<T,\ 0<y<1\}$, and the bounded function $U\,(t,y)=u\,(t,x)$ is a solution to the problem

$$U_t-A_1U_{yy}=B_1(U,V,U_y),\quad (t,y)\in D_1, \tag{25}$$

$$U\,(0,y)=U_0\,(y)\equiv u_0(s_0y),\quad 0\le y\le 1, \tag{26}$$

$$U_y\,(t,0)=0,\quad U\,(t,1)=0,\quad 0\le t\le T, \tag{27}$$

where $D_1 = \{(t,y) : 0 < t \le T, \quad 0 < y < 1\}$, $A_1 = \frac{d_1}{s^2(t)}$, $B_1(U,V,U_y) = U\,(a_1 - b_1U + c_1V) - \frac{m_1U - ys'(t)}{s(t)}U_y$, $s'(t) = -\frac{\mu}{s(t)}U_y(t,s(t))$.

Without loss of generality, we may assume that $U(0,y)=0$.

Theorem 3. *Suppose that a function* $U(t,y)$ *continuous in* $\overline{D}_1$ *satisfies the conditions of problem* (25)-(27) *Assume that, for* $(t,y)\in\overline{D}_1$, $|U|\le M_1$ *and any* U_y, *continuous functions* A_1 *and* B_1 *satisfy the condition*

$$\begin{gathered} A_1 \ge A_0 > 0,\\ \frac{|B_1|}{A_1} \le K_1(U_y^2+1),\quad K_1>0. \end{gathered} \tag{28}$$

Under the conditions (28), *the following estimate is valid*

$$|U_y(t,y)| \le M_8(M_1,M_2,A_0,\frac{d_1}{s_0^2},K_1,\delta),\quad (t,y)\in D_1^{\delta}. \tag{29}$$

In addition, in the domain $\{(t,y)\in\overline{D}_1, |U|\le M_1, |V|\le M_2, |U_y|\le M_8\}$ *we have the estimate*

$$|U|_{\frac{2}{3}}^{D_1^{2\delta}} \le M_9(M_1,M_2,\frac{d_1}{s^2(t)},K_2,\delta).$$

And if it is also known that $U(t,y)$ *possesses in* $\overline{D}_1$ *summable with a square generalized derivatives* U_{ty}, U_{yy}, *then*

$$|U|_{1+\alpha}^{D_1^{2\delta}} \le M_{10}(M_1,M_2,A_0,\frac{d_1}{s_0^2},M_8,K_1,\delta),\quad 0<\alpha<1, \tag{30}$$

If $U_y(t,0)=U(t,1)=0$, *then the estimates* (29) *and* (30) *are also valid in* $\overline{D}_1$.

Proof. Since the estimates $|u\,(t,x)|\le M_1$, $|v\,(t,x)|\le M_2$, $|s'\,(t)|\le M_3$, have been established, respectively, the boundedness of the functions $U\,(t,y)$ and $V\,(t,y)$ is obtained, by virtue of Theorem 3 in the paper [25], the inner estimates hold (29), (30).

If $U(t,1)\equiv 0$, then we redefine the function $U(t,y)$ in the rectangle $R_+ = \{(t,y) : 0\le t\le T, 1-4\delta < y < 1+4\delta\}$, $0<\delta<\frac{1}{5}$ by the formula $U(t,y) = -W(t,1-4\delta(2y-1))$, $V(t,y)=V(t,y)$. The function constructed in the rectangle $D_1^{\delta}\cup R_+$ (for which we retain the notation $U(t,y)$) is continuous together with their derivative and everywhere, except for points of the line $y=1$, satisfies an equation of the form (25) for which the conditions (26) and (27) are satisfied:

$$\begin{cases} W_t = 64\delta A_1 W_{yy} + B_1(t,y,-W,W_y,V), & (t,y)\in R_+,\\ W(0,y)=0, & 1-4\delta\le y\le 1+4\delta,\\ W(t,1-4\delta)=0,\quad W(t,1+4\delta)=0, & \delta\le t\le T. \end{cases} \tag{31}$$

Now, applying Theorem 2 in [25] to the problem (31), we obtain the estimate $|W_y|\le M_5$ in the domain R_+. Hence,

$$|U_y| \le M_8(M_1,M_2,A_0,\frac{d_1}{s_0^2},K_1,\delta)\quad (t,y)\in R_+.$$

Similarly, we set $U(t,y) = W(t,4\delta(1-2y))$, $V(t,y) = V(t,y)$ in the rectangle $R_- = \{(t,y) : 0 \le t \le T, -4\delta < y < 4\delta\}$ and the system (25)–(27) takes the following form

$$\begin{cases} W_t = 64\delta A_1 W_{yy} + B_1(t,y,-W,W_y,V), & (t,y) \in R_+, \\ W(0,y) = 0, & -4\delta \le y \le 4\delta, \\ W_y(t,-4\delta) = 0, \quad W_y(t,4\delta) = 0, & \delta \le t \le T. \end{cases} \tag{32}$$

Estimates in the domain $R_- \cup D_1^\delta \cup R_+$ give a general estimate of $|U_y| \le M_8$ in $\overline{D}_1$.

Further, using Theorem 3 in [25], we obtain the estimate

$$|U|_{1+\alpha} \le M_{10}(M_1, M_2, A_0, \frac{d_1}{s_0^2}, M_8, K_1) \quad \text{in} \quad \overline{D}_1.$$

The estimate for the highest derivative in $\overline{D}_1$ is established using the results for the linear parabolic equations [26, 27]. □

Theorem 4. *Let the coefficients of equation*

$$a\,(t,x)\,u_{xx} + b\,(t,x)\,u_x + c\,(t,x)\,u - u_t = f\,(t,x)\,, \quad (t,x) \in \overline{D}_1 \tag{33}$$

satisfy the Hölder conditions

$$a\,(t,x) \ge a_0 > 0, \quad |a|_\beta^{\overline{D}_1} + |b|_\beta^{\overline{D}_1} + |c|_\beta^{\overline{D}_1} = M_{11} < \infty,$$

Let the function $u\,(t,x)$ *be a solution of the parabolic equation* (33), $|u|_{2+\beta}^{\overline{D}_1} < \infty$, $M_1 = \max|u\,(t,x)|$. *Then*

$$|u|_{2+\beta}^{D_1^{4\delta}} \le M_{12}(|f|_\beta^{\overline{D}_1} + M_1)$$

where M_{12} *depends on* β, M_{11}, δ.

If $u|_\Gamma = 0$, *then there exists* $M_{12} = M_{12}\,(\beta, a_0, M_{11})$, *such that* $|u|_{2+\beta}^{D_1^{4\delta}} \le M_{12}\,|f|_\beta^{\overline{D}_1}$.

Uniqueness and existence of a solution

To prove the uniqueness of the solution, we derive an integral representation equivalent to (8)

$$\begin{aligned} \frac{d_1}{\mu} s\,(t) = \frac{d_1}{\mu} s_0 + \int_0^{s_0} u_0\,(\xi)\,d\xi - \int_0^{s(t)} u\,(t,\xi)\,d\xi + m_1 \int_0^t u\,(\eta,0)\,\nu\,(\eta,0)\,d\eta \\ + \int_0^t d\eta \int_0^{s(\eta)} u\,(a_1 - b_1 u + c_1 \nu + m_1 \nu_x)\,d\xi. \end{aligned} \tag{34}$$

Theorem 5. *Suppose that the conditions of Theorem 1 is satisfied. Then the solution of problem* (1)–(9) *is unique.*

Proof. The theorem is proved first for small $t \in [0, t_0]$, and then it is established that it is valid for any $0 < t < \infty$. Let $s_1\,(t)\,, u_1\,(t,x)\,, \nu_1\,(t,x)$ and $s_2\,(t)\,, u_2\,(t,x)\,, \nu_2\,(t,x)$

be solution of (1)–(9) and, in addition, let $y(t) = \min(s_1(t), s_2(t))$, $z(t) = \max(s_1(t), s_2(t))$. Due to (34) we get

$$
\begin{aligned}
\frac{d_1}{\mu_1}|s_1(t)-s_2(t)| \le & \int\limits_0^{y(t)} |u_1(t,\xi)-u_2(t,\xi)|\,d\xi + \int\limits_{y(t)}^{z(t)} |u_i(t,\xi)|\,d\xi \\
& + m_1 M_2 \int\limits_0^t |u_1(\eta,0)-u_2(\eta,0)|\,d\eta \\
& + m_1 M_1 \int\limits_0^t |v_1(\eta,0)-v_2(\eta,0)|\,d\eta \\
& + \int\limits_0^t d\eta \int\limits_0^{y(\eta)} |\left(a_1 - b_1(u_1+u_2) - c_1 v_2 - m_1 v_{2x}\right) U|\,d\xi \\
& + \int\limits_0^t d\eta \int\limits_0^{y(\eta)} |c_1 u_1 V - m_1 u_1 V_x|\,d\xi \\
& + \int\limits_0^t d\eta \int\limits_{y(\eta)}^{z(\eta)} |u_i\left(a_1 - b_1 u_i + c_1 v_i\right)|\,d\xi
\end{aligned}
\tag{35}
$$

where u_i, v_i (i=1,2) are solutions between $y(t)$ and $z(t)$, i.e.,

$$
u_i(t,x) = \begin{cases} u_1(t,x) & \text{if} \quad s_2(t) < s_1(t), \\ u_2(t,x) & \text{if} \quad s_2(t) > s_1(t). \end{cases}
$$

Next, we need to estimate the differences $V(t,x) = v_1(t,x) - v_2(t,x)$, $U(t,x) = u_1(t,x) - u_2(t,x)$.

From problem (17)–(20) for the function $V(t,x)$ we find

$$
\begin{cases}
V_t = d_2 V_{xx} + r_{21} V_x + r_{22}(t,x) V + r_{23}(t,x) U(t,x), & (t,x) \in Q, \\
V(0,x) = 0, \quad 0 \le x \le l, \\
V_x(t,0) = 0, \quad V_x(t,l) = 0, \quad 0 \le t \le T,
\end{cases}
\tag{36}
$$

where the coefficients $r_{21}(t,x)$, $r_{22}(t,x)$ are bounded functions, i.e.

$$
r_{21}(t,x) = -m_2 u_1, \quad r_{21}(t,x) = a_2 - b_2 u_1 - c_2(v_1+v_2), \quad r_{23}(t,x) = -m_2 v_{2x} - b_2 v_2.
$$

From problem (36) by the maximum principle we find

$$
|V(t,x)| \le M_{13} \cdot \max_D |U(t,x)|\, t,
$$

where M_{13} depends on $r_{22}(t,x)$.

Set $U(t,x) = u_1(t,x) - u_2(t,x)$. Then the problem (13)–(16) can be rewritten as follows:

$$\begin{cases} U_t = d_1 U_{xx} + r_{11}(t,x)\,U_x + r_{12}(t,x)\,U + r_{13}(t,x)\,V, & 0 \le t < t_0,\ 0 < x < y(t) \\ U(0,x) = 0, & 0 \le x \le y(0), \\ U_x(t,0) = 0, & 0 \le t \le t_0, \\ U(t,y(t)) \le N \max\limits_{0\le\eta\le t} |s_1(\eta) - s_2(\eta)|, & 0 \le t \le t_0, \end{cases} \tag{37}$$

where $r_{12}(t,x) = m_1 v_1$, $r_{12}(t,x) = a_1 - b_1(u_1+u_2) + c_2 v_2$, $r_{13}(t,x) = c_1 u_1 - m_1 u_{2x}$.

Applying the maximum principle to the problem (37), we have

$$|U(t,x)| \le N \max_{0\le\eta\le t} |s_1(\eta) - s_2(\eta)| + M_{14} \max_{\overline{Q}} |V(t,x)|\, t, \tag{38}$$

where M_{14} depends on M_{13} and $r_{13}(t,x)$.

Let us now estimate the components of (35):

$$I_1 \le \int_0^{y(t)} |u_1(t,\xi) - u_2(t,\xi)|\, d\xi, \quad I_2 \le \int_{y(t)}^{z(t)} |u_i(t,\xi)|\, d\xi \le M_1 \cdot \max_{0\le\eta\le t} |s_1(t) - s_2(t)|^2,$$

$$I_3 = m_1 M_2 \int_0^t |u_1(\eta,0) - u_2(\eta,0)|\, d\eta \le m_1 M_2 \max_{0\le\eta\le t} |s_1(\eta) - s_2(\eta)| \cdot t,$$

$$I_4 = m_1 M_1 \int_0^t |v_1(\eta,0) - v_2(\eta,0)|\, d\eta \le m_1 M_1 \max_{0\le\eta\le t} |s_1(\eta) - s_2(\eta)| \cdot t,$$

$$I_5 \le \int_0^t d\eta \int_0^{y(\eta)} |(a_1 - b_1(u_1+u_2) - c_1 v_2 - m_1 v_{2x})\, U|\, d\xi \le M_{15} \int_0^t d\eta \int_0^{y(\eta)} |u_1(t,\xi) - u_2(t,\xi)|\, d\xi,$$

$$I_6 = \int_0^t d\eta \int_0^{y(\eta)} |c_1 u_1 V - m_1 u_1 V_x|\, d\xi \le M_{16} \max_{0\le\eta\le t} |s_1(\eta) - s_2(\eta)| \cdot t,$$

$$I_7 = \int_0^t d\eta \int_{y(\eta)}^{z(\eta)} |u_i(a_1 - b_1 u_i + c_1 v_i)|\, d\xi \le M_{17} \max_{0\le\eta\le t} |s_1(t) - s_2(t)| \cdot t,$$

where $M_{15} = a_1 + m_1 M_4$, $M_{16} = (c_1 M_2 + m_1 M_4) M_1$, $M_{17} = (a_1 + c_1 M_2) M_1$.

Let be $A(t_0) = \max\limits_{0\le t\le t_0} |s_1(t) - s_2(t)| > 0$. Then $A(t_0) \le M_3 t_0$, $t_0 < 1$.

At the same time, we find

$$I_1 \le \int_0^{y(t_0)} |u_1(t,\xi) - u_2(t,\xi)|\, d\xi,\ I_2 \le M_1 A^2(t_0),\ I_3 \le m_1 M_2 A(t_0)\, t_0,\ I_4 \le m_1 M_1 A(t_0)\, t_0,$$

$$I_5 \le M_{15} \int_0^t d\eta \int_0^{y(\eta)} \max_D |U|\, d\xi,\ I_6 \le M_{16} A(t_0)\, t_0,\ I_7 \le M_{17} A(t_0)\, t_0.$$

we have

$$A(t_0) \le \int_0^{y(t)} |U(t,\xi)|\, d\xi + M_{15} \int_0^t d\eta \int_0^{y(\eta)} \max_D |U|\, d\xi + M_1 A^2(t_0) + M_{18} A(t_0)\, t_0. \quad (39)$$

where $M_{18} = m_1 M_1 + m_1 M_2 + M_{16} + M_{17}$.

Next, we divide (39) by $A(t_0)$ and we get

$$1 \le \int_0^{y(t)} \frac{\max |U(t,\xi)|}{A(t_0)} d\xi + \int_0^t d\eta \int_0^{y(\eta)} \frac{\max |U|}{A(t_0)} d\xi + (M_1 M_3 + M_{18}) t_0. \quad (40)$$

Now for the function $U(t,x) = u_1(t,x) - u_2(t,x)$ from (38) we have

$$|U(t,x)| \le N A(t_0) + M_{13} M_{14} |U(t,x)|\, t^2, \quad 0 \le t \le t_0,$$

or

$$|U(t,x)| \le M_{19} A(t_0),$$

where

$$M_{19} = \frac{N}{1 - M_{13} M_{14} t_0^2},\ \ t_0 < \frac{1}{\sqrt{M_{13} M_{14}}}.$$

Let us estimate the integral term

$$\int_0^{y(t)} \frac{\max |U(t,\xi)|}{A(t_0)} d\xi = \int_0^{y(0)} \frac{\max |U(t,\xi)|}{A(t_0)} d\xi + \int_{y(0)}^{y(t)} \frac{\max |U(t,\xi)|}{A(t_0)} d\xi$$

$$\le \int_0^{y(0)} \frac{\max |U(t,\xi)|}{A(t_0)} d\xi + M_{19} t_0.$$

Consider the auxiliary problem

$$\begin{cases} W_t = d_1 W_{xx} + r_{11}(t,x) W_x + r_{12}(t,x) W + r_{14} V, & 0 \le t < t_0,\ 0 < x < y(t) \\ W(0,x) = 0, & 0 \le x \le y(0), \\ W_x(t,0) = 0, \quad W(t, y(0)) = 1, & 0 \le t \le t_0, \end{cases}$$

where

$$r_{14}(t,x) = \frac{r_{13}(t,x)}{M_{19} A(t_0)}.$$

Hence, according to the maximum principle

$$|W(t,x)| \le \max_Q |r_{14} V(t,x)|\, t_0 + 1 = M_{10}.$$

Let us introduce the function

$$Z(t,x)=\frac{U(t,x)}{M_{19}A(t_0)}-W(t,x),\quad 0<x<y(0),\ 0\le t\le t_0.$$

We find

$$Z_t=d_1Z_{xx}+r_{11}(t,x)Z_x+r_{12}(t,x)Z,$$
$$Z(0,x)=0,\quad Z_x(t,0)=0,$$
$$Z(t,y(0))=\frac{U(t,y(0))}{M_{19}A(t_0)}-W(t,y(0))\le 0.$$

Hence, according to the maximum principle

$$\frac{U(t,x)}{M_{19}A(t_0)}\le W(t,x),\ 0\le t\le t_0.$$

Since $\lim\limits_{t\to 0}W(t,x)=0,\quad 0\le x\le y(0)$, then

$$\lim_{t\to 0}\int\limits_0^{y(0)}W(t,x)\,dx=0.$$

Therefore,

$$\lim_{t\to 0}\int\limits_0^{y(0)}\frac{|U(t,\xi)|\,\xi}{M_{19}A(t_0)}d\xi\to 0.$$

Since the right side of (40) tends to zero as $t_0\to 0$, then (40) does not hold for sufficiently small t_0 and we arrive at a contradiction. Hence $s_1(t)\equiv s_2(t)$ and further $u_1(t,x)\equiv u_2(t,x)$, $v_1(t,x)=v_2(t,x)$, $0\le t\le t_0$.

The uniqueness of the problem solution for any $0<t<\infty$ is established as follows.

Let $t_1=\sup\{t: s_1(\eta)=s_2(\eta), 0\le\eta\le t\}$. If $t_1=\infty$, then the issue will be resolved. Otherwise, assuming that the parameter t_1 is bounded and repeating the above calculations in the interval $t_1\le t\le t_1+\Delta t$, we again arrive at a contradiction. □

Theorem 6. *Suppose that the conditions of Theorems 2 and 3 are satisfied. Then, in* $D\cup Q$ *there exists a solution* $u(t,x)\in C^{2+\alpha}(\overline{D})$, $v(t,x)\in C^{2+\alpha}(\overline{Q})$, $s(t)\in C^{1+\alpha}(0\le t\le T)$ *of problem* (1)–(9).

Proof. To prove the solvability of the nonlinear problem, we can use different theorems from the theory of nonlinear equations taking into account the fact that the theorem on uniqueness of the classical solution is true for this problem. We use the Leray-Schauder principle [26], the established a priori estimates $|\cdot|_{1+\alpha}$ for all possible solutions of nonlinear problems, and the theorems on solvability in Hölder classes for the linear problems. Moreover, the theorems on existence for systems are the same as the theorem on existence for a single equation because each equation can be regarded as a linear equation for $u(t,x)$ and $v(t,x)$ with Hölder-continuous coefficients.

Problem (1)–(9) is considered simultaneously with a one-parameter family of problems of the same type. The linear problem specifies the transformation $w=$

$F(\omega, k), 0 \leq k \leq 1$, and the Leray-Schauder principle is applied to this transformation. This operator is nonlinear and depends on $k = 1$. The fixed points of this operator for $k = 1$ are solutions of the problem.

By $H^{1+\alpha}$, $\alpha \in (0, 1)$, we denote a Banach space of functions $u(t, x)$, $v(t, x)$ on $\overline{D}$ and $\overline{Q}$ with the norm $\|\cdot\| = |\cdot|_{1+\alpha}$, which satisfy the corresponding initial and boundary conditions of the problems (13)–(16) and (17)–(20).

For every function $\bar{u}, \bar{v} \in H^{1+\alpha}$ and any number $k \in [0, 1]$, by u^k and v^k, we denote the solutions of the linear problems (13)–(16) and (17)–(20). These solutions exist and are unique; moreover, $u^k, v^k \in C^{2+\alpha}$, $s(t) \in C^{1+\alpha}$. In this case, in D and Q, we perform the transition to the parabolic equation with Hölder-continuous coefficients in a fixed domain.

The uniform continuity and complete continuity of the operator of transformation F with respect to k, estimates for the solutions uniform in k, and the solvability of the linear problems follow from the established a priori estimates for the Hölder norms. A more detailed exposition of the technique can be found, for example, in (Chap.VII, [26]; Chap.V, [27]).

This completes the proof of the theorem. □

Conclusion

This paper is devoted to the study of the free boundary problem for diffusive predator-prey system with nonlinear advection term. The solution of this problem is of great practical importance in many fields, such as biological and ecological sciences, and others. The paper presents methods for obtaining a priori Schauder-type estimates for solving the (1)-(9) problem. Based on the estimates obtained, the uniqueness and existence of the solution is proved.

The proposed techniques and results can be further applied to other types of boundary value problems in mathematical physics.

References

1. Cantrell R. S., Cosner C. *Spatial ecology via reaction-diffusion equations.* Chichester, UK.: John Wiley and Sons Ltd., 2003. 411 pp. (In Russian)

2. Jialu T., Ping L. Global dynamics of a modified Leslie-Gower predator-prey model with Beddington-De Angelis functional response and prey-taxis // *J. Electronic Research Archive*, 2022. vol. 30, no. 3, pp. 929-942 DOI: 10.3934/era.2022048.

3. Pao C. V. *Nonlinear parabolic and elliptic equations.* New York.: Plenum Press, 1992. 777 pp.

4. Elmurodov A. N., Rasulov M. S. On a uniqueness of solution for a reaction-diffusion type system with a free boundary // *Lobachevskii Journal of Mathematics*, 2022. vol. 43, no. 8, pp. 2099-2106 DOI: 10.1134/S1995080222110087.

5. Ding N., Cai J., Xu L. A free boundary problem with nonlinear advection and Dirichlet boundary condition // *Nonlinear analysis: Real world applications*, 2023. vol. 69, pp. 103719 DOI:10.1016/j.nonrwa.2022.103719.

6. Elmurodov A. N. Two-phase problem with a free boundary for systems of parabolic equations with a nonlinear term of convection // *Vestnik KRAUNC. Fiz.-Mat. Nauki*, 2021. vol. 36, no. 3, pp. 110–122 DOI: 10.26117/2079-6641-2021-36-3-110-122.

7. Rasulov M. S. Two free boundaries problem for a parabolic equation // *Vestnik KRAUNC. Fiz.-Mat. Nauki*, 2023. vol. 42, no. 1, pp. 108–121 DOI:10.26117/2079-6641-2023-42-1-108-121.

8. Takhirov J. O., Rasulov M. S. Problem with free boundary for systems of equations of reaction-diffusion type // *Nonlinear analysis: Real world applications*, 2023. vol. 69, pp. 1968-1980 DOI:10.1007/s11253-018-1481-4.
9. Wang R., Wang L., Wang Zhi-Ch. Free boundary problem of a reaction-diffusion equation with nonlinear convection term // *Journal of Mathematical Analysis and Applications*, 2018. vol. 467, no. 2, pp. 1233-1257 DOI: 10.1016/j.jmaa.2018.07.065.
10. Lin Z. G. A free boundary problem for a predator-prey model // *Nonlinearity*, 2007. vol. 20, no. 8, pp. 1883-1892 DOI: 10.1088/0951-7715/20/8/004.
11. Elmurodov A. N., Rasulov M. S. A free boundary problem for a predator-prey system // *Lobachevskii Journal of Mathematics*, 2023. vol. 44, no. 7, pp. 2898-2909 DOI: 10.1134/S1995080223070375.
12. Liu Y, Guo Z, Smaily M. El, Wang L. Biological invasion in a predator-prey model with a free boundary // *Boundary Value Problems*, 2019. vol. 33, pp. 2898-2909 DOI: 10.1186/s13661-019-1147-7.
13. Wang R., Wang L., Wang Zhi-Ch. A free boundary problem for the predator-prey model with double free boundaries // *J.Dynam.Diff.Equations*, 2017. vol. 29, no. 3, pp. 957-979 DOI: 10.1007/s10884-015-9503-5.
14. Liu Y, Guo Z, Smaily M. El, Wang L. A Leslie-Gower predator-prey model with a free boundary // *Discrete and Continuous Dynamical Systems - S*, 2019. vol. 12, no. 7, pp. 2063-2083 DOI: 10.3934/dcdss.2019133.
15. Wang M. Spreading and vanishing in the diffusive prey-predator model with a free boundary // *Communications in Nonlinear Science and Numerical Simulation*, 2019. vol. 23, no. 1-3, pp. 311-327 DOI: 10.1016/j.cnsns.2014.11.016.
16. Wang M. On some free boundary problems of the prey-predator model // *Journal of Differential Equations*, 2014. vol. 256, no. 10, pp. 3365-3394 DOI: 110.1016/j.jde.2014.02.013.
17. Guo J. S., Wu C. H. On a free boundary problem for a two-species weak competitor system // *J. Dyn. Diff. Equat.*, 2012. vol. 24, no. 4, pp. 873-895 DOI: 10.1007/s10884-012-9267-0.
18. Wang M., Zhao J. Free Boundary Problems for a Lotka-Volterra // *J. Dyn. Diff. Equat.*, 2014. vol. 26, pp. 655–672 DOI: 10.1007/s10884-014-9363-4.
19. Meirmanov A. M., Galtsev O. V., Galtseva O. A Some free boundary problems arising in rock mechanics // *J.Math. Sci.*, 2022. vol. 260, pp. 492–523 DOI: 10.1007/s10958-022-05708-z.
20. Rasulov M. S. Free boundary problem for a system of parabolic equations of the reaction-diffusion type // *Izv. Vyssh. Uchebn. Zaved. Mat.*, 2025. no. 2, pp. 79-90 DOI:10.26907/0021-3446-2025-2-79-90.
21. Rasulov M. S. On a Stefan problem for a reaction-diffusion equation with Dirichlet condition // *Uzbek Mathematical Journal*, 2024. vol. 68, no. 4, pp. 121-127 DOI: 10.29229/uzmj.2024-4-14.
22. Takhirov J. O. A free boundary problem for a reaction-diffusion equation in biology // *Indian J. Pure Appl. Math.*, 2019. vol. 50, no. 1, pp. 95-112 DOI: 10.1007/s13226-019-0309-8.
23. Tan Q. J. Global existence of classical solutions for a class of diffusive ecological models with two free boundaries and cross-diffusion // *Nonlinear analysis: Real world applications*, 2021. vol. 60, pp. 103302 DOI: 10.1016/j.nonrwa.2021.103302.
24. Younes G. A., Khatib N. E. Volpert V. A. Existence of Solution of a Free Boundary Problem for Reaction–Diffusion Systems // *J. Math. Sci.*, 2024. vol. 283, pp. 150–166 DOI: 10.1007/s10958-024-07245-3.
25. Kruzhkov S. N. Nonlinear parabolic equations in two independent variables // *Tr. MMO*, 1967. vol. 16, pp. 355-373 (In Russian).
26. Friedman A. *Uravneniya s chastnimi proizvodnimi parabolicheskogo tipa.* Moskow.: Mir, 1968. 428 pp. (In Russian)
27. Ladyzhenskaja O. A., Solonnikov V. A., Uralceva N. N. *Lineynie i kvazilineynie uravneniya parabolicheskogo tip.* Moskow.: Nauka, 1967. 736 pp. (In Russian)

Information about the authors

Rasulov Mirojiddin Sobirjonovich – Ph. D. (Phys.& Math.), Senior Researcher, V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan, ORCID 0000-0003-0704-6012.

Jamoldinova Shohista Muzaffarjon qizi – Researcher, V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan, ORCID 0009-0007-9039-9682.