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Bitsadze-Samarskii Type Problem for the Diffusion Equation and Degenerate Hyperbolic Equation

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Abstract. A boundary value problem of the Bitsadze-Samarskii type is studied in the article for a fractional-order diffusion equation and a degenerate hyperbolic equation with singular coefficients at lower terms in an unbounded domain. The article considers a mixed domain where the parabolic part of the domain under consideration coincides with the upper half-plane and the hyperbolic part is bounded by two characteristics of the equation under consideration and a segment of the abscissa axis. The uniqueness of the solution to the problem under consideration is proven by the method of energy integrals. The existence of a solution to the problem under consideration is reduced to the concept of solvability of a fractional-order differential equation. An explicit form of the solution to the modified Cauchy problem is given in the hyperbolic part of the mixed domain under consideration. Using this solution, due to the boundary condition of the problem, the main functional relationship between the traces of the unknown function brought to the interval of the degeneracy line of the equation is obtained. Further, using the representation of the solution of the diffusion equation of fractional order, the second main functional relationship between the traces of the sought-for function on the interval of the abscissa axis from the parabolic part of the considered mixed domain is obtained. Through the conjugation condition of the problem under study, an equation with fractional derivatives is obtained from two functional relationships by eliminating one unknown function; its solution is written out in explicit form. In the study of the boundary value problem, generalized fractional integro-differentiation operators with the Gauss hypergeometric function are employed. The properties of the Wright and Mittag-Leffler type functions are extensively utilized in the study.

Key words: boundary value problem, diffusion equation, degenerate hyperbolic equation, Gauss hypergeometric function, Wright function, uniqueness of the solution to the problem, existence of a solution to the problem.


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МАТЕМАТИКА

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Задача типа Бицадзе-Самарского для уравнения диффузии и вырождающегося гиперболического уравнения

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Аннотация. В статье изучается краевая задача типа Бицадзе-Самарского для дробного уравнения диффузии и вырождающегося гиперболического уравнения с сингулярными коэффициентами при младших членах в неограниченной области. В статье рассматривается смешанная область, в которой параболическая часть рассматриваемой области совпадает с верхней полуплоскостью, а гиперболическая часть ограничена двумя характеристиками рассматриваемого уравнения и отрезком оси абсцисс. Единственность решения рассматриваемой задачи доказывается методом интегралов энергии. Существование решения рассматриваемой задачи сводится к понятию разрешимости дробного дифференциального уравнения. Приводится явный вид решения модифицированной задачи Коши в гиперболической части рассматриваемой смешанной области. С помощью этого решения в силу граничного условия задачи получена основная функциональная связь между следами неизвестной функции, приведенными на интервал линии вырождения уравнения. Далее, используя представление решения уравнения диффузии дробного порядка, получено второе основное функциональное соотношение между следами искомой функции на отрезке оси абсцисс из параболической части рассматриваемой смешанной области. Через условие сопряжения исследуемой задачи из двух функциональных соотношений путем исключения одной неизвестной функции получено уравнение с дробными производными, решение которого выписано в явном виде. При исследовании краевой задачи используются обобщенные операторы дробного интегро-дифференцирования с гипергеометрической функцией Гаусса. При исследовании широко используются свойства функций типа Райта и Миттаг-Леффлера.

Ключевые слова: краевая задача, уравнение диффузии, вырожденное гиперболическое уравнение, гипергеометрическая функция Гаусса, функция Райта, единственность решения задачи, существование решения задачи

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Introduction and problem statement

Fractal theory explains the structure of disordered media, like porous materials, and the processes that take place within them. A fractional-order differential equation is used to describe the movement of a substance in a uniform fluid flow [1]. The topic of fractional-order diffusion is covered in [2]. Fractional-order differential equations come up in various areas, such as classical mechanics (inverse problems), heat conduction (heat flow dynamics), diffusion (electrochemical analysis of electrode surfaces), and in the study of stochastic transport processes. Problems involving fluid filtration in highly porous (fractal) media often require studying boundary value problems for fractional-order partial differential equations. Boundary value problems for the fractional-order diffusion equation were explored in [3–6]. A certain family of generalized derivatives of the Riemann–Liouville operators $D_{a+}^{\alpha,\beta}$ of orders α and β was studied in [7]. Applications of this operator are given in [8]. The unique solvability of the problem for a partial fractional derivative equation of the Riemann–Liouville type with a boundary condition containing a generalized fractional integro-differentiation operator is investigated in [9–11]. Reference [12] studies an analog of the Bitsadze–Samarskii type problem for a mixed-type partial fractional derivative equation in an unbounded domain.

Let us consider a partial differential equation of the second order

$$\begin{cases} u_{xx} - D_{0+,y}^{\gamma} u = 0, & y > 0, 0 < \gamma < 1, \\ -(-y)^m u_{xx} + u_{yy} + \frac{\alpha_0}{(-y)^{1-\frac{m}{2}}} u_x + \frac{\beta_0}{y} u_y = 0, & y < 0, \end{cases} \quad (1)$$

where $D_{0+,y}^{\gamma}$ is the partial fractional Riemann-Liouville derivative of order γ ($0 < \gamma < 1$) of function $u(x, y)$ for the second variable [13] in domain $D = D^+ \cup D^- \cup I$, where D^+ is the half-plane $y > 0$, D^- is a finite domain of the half-plane $y < 0$, bounded by characteristics OC and BC of equation (1) emanating from point $O(0, 0)$ and $B(1, 0)$ and segment OB of the straight line $y = 0$, $I = \{(x, y) : 0 < x < 1, y = 0\}$. In equation (1) m, α_0, β_0 are some real numbers satisfying conditions $m > 0, |\alpha_0| < \frac{m+2}{2}, -\frac{m}{2} < \beta_0 < 1$.

Let us introduce the following notation: $\Theta(x) = \left(\frac{x}{2}; -\left(\frac{m+2}{4}x\right)^{\frac{m+2}{2}}\right)$ is the intersection point of the characteristic of equation (1) emanating from point $(x, 0)$ ($x \in I$), with characteristic OC, $I_{0+}^{\sigma,\delta,\eta}$ is the operator of generalized fractional integro-differentiation with the Gauss hypergeometric function $F(a, b, c; z)$ introduced by M.A. Saigo [14] and having the following form for real σ, δ, η and $x > 0$

$$(I_{0+}^{\sigma,\delta,\eta} f)(x) = \begin{cases} \frac{x^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^x (x-t)^{\sigma-1} F\left(\sigma+\delta, -\eta, \sigma; 1-\frac{t}{x}\right) f(t) dt, & (\sigma > 0), \\ \frac{d^n}{dx^n} (I_{0+}^{\sigma+n,\delta-n,\eta-n} f)(x), & (\sigma \leq 0, n = [-\alpha] + 1). \end{cases} \quad (2)$$

In particular [14],

$$(I_{0+}^{0,0,\eta} f)(x) = f(x), \quad (I_{0+}^{\sigma,-\sigma,\eta} f)(x) = (I_{0+}^{\sigma} f)(x),$$

$$(I_{0+}^{-\sigma, \sigma, \eta} f)(x) = (D_{0+}^{\sigma} f)(x), \tag{3}$$

where $(I_{0+}^{\sigma} f)(x)$ and $(D_{0+}^{\sigma} f)(x)$ are the Riemann–Liouville fractional integration and differentiation operators of order $\sigma > 0$.

Problem 1. Find in domain D solution $u = u(x, y)$ to equation (1) satisfying the following conditions:

$$y^{1-\gamma} u|_{y=0} = 0, \quad (-\infty < x \leq 0, 1 \leq x < \infty), \tag{4}$$

$$A_1 \left(I_{0+}^{a, b, \beta-1-a} u[\Theta_0(t)] \right)(x) + A_2 \left(I_{0+}^{a+1, b-1+\alpha, \beta-1-a} \lim_{y \rightarrow 0-} (-y)^{\beta_0} u_y(t, y) \right)(x) + \\ + A_3 \left(I_{0+}^{\gamma+a+\alpha, b-\gamma, \beta-1-a} \lim_{y \rightarrow 0-} (-y)^{\beta_0} u_y(t, y) \right)(x) = g(x), \tag{5}$$

and the conjugation conditions

$$\lim_{y \rightarrow 0+} y^{1-\gamma} u(x, y) = \lim_{y \rightarrow 0-} u(x, y), \quad \forall x \in \bar{I}, \tag{6}$$

$$\lim_{y \rightarrow 0+} y^{1-\gamma} (y^{1-\gamma} u(x, y))_y = \lim_{y \rightarrow 0-} (-y)^{\beta_0} u_y(x, y), \quad \forall x \in I. \tag{7}$$

Here $\alpha = \frac{m+2(\beta_0+\alpha_0)}{2(m+2)}$, $\beta = \frac{m+2(\beta_0-\alpha_0)}{2(m+2)}$, $0 < \alpha, \beta < \frac{1}{2}$, A_1, A_2, A_3 are real constants such that $A_1 > 0, A_2 \leq 0, A_3 \leq 0$ or $A_1 < 0, A_2 \geq 0, A_3 \geq 0$, a, b, γ are real numbers, $g(x)$ is a given function such that $g(x) \in C^1(\bar{I}) \cap C^2(I)$. We will seek solution $u(x, y)$ to the problem the class of twice differentiable functions in domain D such that $u(x, y)$ tends to zero as $(x^2 + y^2) \rightarrow \infty$,

$$y^{1-\gamma} u(x, y) \in C(\bar{D}^+), \quad u(x, y) \in C(\bar{D}^-),$$

$$y^{1-\gamma} (y^{1-\gamma} u(x, y))_y \in C(D^+ \cup I),$$

$$u_{xx} \in C(D^+ \cup D^-), \quad u_{yy} \in C(D^-).$$

Note that nonlocal boundary value problems for equation (1) in unbounded and bounded domains were studied in [15], [16], [17], and for equation (1) for $\alpha_0 = 0, \beta_0 = 0$ problems were considered in [9, 11]. In works [18], [19] nonlocal problems with shift on conjugation of two hyperbolic equations of the second order, consisting of a wave equation in one part of the domain and a degenerate hyperbolic equation of the first kind in the other part, are studied.

Uniqueness of the solution to the problem

Theorem 1. Let condition $g(x) \equiv 0$ be satisfied. Then the problem cannot have more than one solution.

Proof. Let there be a solution to the problem. We introduce the following notation

$$\lim_{y \rightarrow 0+} y^{1-\gamma} u(x, y) = \tau_1(x), \quad \lim_{y \rightarrow 0-} u(x, y) = \tau_2(x), \tag{8}$$

$$\lim_{y \rightarrow 0^+} y^{1-\gamma} (y^{1-\gamma} u(x, y))_y = \nu_1(x), \quad \lim_{y \rightarrow 0^-} (-y)^{\beta_0} u_y(x, y) = \nu_2(x). \tag{9}$$

It is known [5] that the solution to equation (1) in domain D^+ satisfying condition (4) and condition

$$\lim_{y \rightarrow 0^+} y^{1-\gamma} u(x, y) = \tau_1(x), \quad \forall x \in \bar{I}$$

is given by the following formula

$$u(x, y) = \int_0^1 G(x, y, t) \tau_1(t) dt, \tag{10}$$

where

$$G(x, y, t) = \frac{\Gamma(\gamma)}{2} y^{\frac{\gamma}{2}-1} e_{1, \frac{\gamma}{2}}^{1, \frac{\gamma}{2}} \left(-|x - t| y^{-\frac{\gamma}{2}} \right),$$

$$e_{\mu, \delta}^{\zeta, \xi} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\zeta n + \mu) \Gamma(\delta - \xi n)}, \quad \zeta > \xi, \quad \zeta > 0, \quad z \in \mathbb{C}$$

is the Wright-type function [6].

By (10), the functional relationship between $\tau_1(x)$ and $\nu_1(x)$ brought the parabolic part D^+ to the line $y = 0$, following form [20]

$$\nu_1(x) = \frac{1}{\Gamma(1 + \gamma)} \tau_1''(x). \tag{11}$$

Let us find the functional relationship between $\tau_2(x)$ and $\nu_2(x)$ brought to the line $y = 0$ from the hyperbolic part D^- of domain D .

The solution to the modified Cauchy problem (8)-(9) in domain D^- has the following form [15], [21]

$$u(x, y) = \gamma_1 \int_0^1 \tau_2 \left(x + \frac{2}{m+2} (2t-1)(-y)^{\frac{m+2}{2}} \right) t^{\beta-1} (1-t)^{\alpha-1} dt + \\ + \gamma_2 (-y)^{1-\beta_0} \int_0^1 \nu_2 \left(x + \frac{2}{m+2} (2t-1)(-y)^{\frac{m+2}{2}} \right) t^{-\alpha} (1-t)^{-\beta} dt, \tag{12}$$

where $\gamma_1 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$, $\gamma_2 = -\frac{2\Gamma(1-\alpha-\beta)}{(m+2)\Gamma(1-\alpha)\Gamma(1-\beta)}$.

From formula (12) and relation (2) we obtain

$$u[\theta(x)] = \gamma_1 \Gamma(\alpha) (I_{0+}^{\alpha, 0, \beta-1} \tau_2)(x) + \gamma_2 \left(\frac{m+2}{4} \right)^{1-\alpha-\beta} \Gamma(1-\beta) (I_{0+}^{1-\beta, \alpha+\beta-1, \beta-1} \nu_2)(x). \tag{13}$$

Substituting (13) into the boundary condition (5), in view of (8) and (9) and applying the following relation [14]

$$(I_{0+}^{\alpha, \beta, \eta} I_{0+}^{\gamma, \delta, \alpha+\eta} f)(x) = (I_{0+}^{\alpha+\gamma, \beta+\delta, \eta} f)(x), \quad (\gamma > 0), \tag{14}$$

we obtain

$$\begin{aligned}
 & A_1 \gamma_1 \Gamma(\alpha) \left(I_{0+}^{\alpha+a, b, \beta-1-a} \tau_2 \right) (x) + \\
 & + A_1 \gamma_2 \left(\frac{m+2}{4} \right)^{1-\alpha-\beta} \Gamma(1-\beta) \left(I_{0+}^{a+1-\beta, b+\alpha+\beta-1, \beta-1-a} \nu_2 \right) (x) + \\
 & + A_2 \left(I_{0+}^{a+1, b+\alpha-1, \beta-1-a} \nu_2 \right) (x) + A_3 \left(I_{0+}^{\gamma+a-\alpha, b-\gamma, \beta-1-a} \nu_2 \right) (x) = g(x). \quad (15)
 \end{aligned}$$

where $k_1 = A_1 \gamma_1 \Gamma(\alpha)$, $k_2 = A_1 \gamma_2 \left(\frac{m+2}{4} \right)^{1-\alpha-\beta} \Gamma(1-\beta)$. We apply operator $I_{0+}^{-\alpha-a, -b, \alpha+\beta-1}$ to both sides of equality (15). Direct calculations using formulas (14) and (3) show that

$$\tau_2(x) = -k_1 \left(I_{0+}^{1-\alpha-\beta} \nu_2 \right) (x) - k_2 \left(I_{0+}^{1-\alpha} \nu_2 \right) (x) - k_3 \left(I_{0+}^{\gamma} \nu_2 \right) (x) + g_1(x), \quad (16)$$

where

$$\begin{aligned}
 k_1 &= \frac{\gamma_2 \left(\frac{m+2}{4} \right)^{1-\alpha-\beta} \Gamma(1-\beta)}{\gamma_1 \Gamma(\alpha)}, \quad k_2 = \frac{A_2}{A_1 \gamma_1 \Gamma(\alpha)}, \\
 k_3 &= \frac{A_3}{A_1 \gamma_1 \Gamma(\alpha)}, \quad g_1(x) = \frac{1}{A_1 \gamma_1 \Gamma(\alpha)} \left(I_{0+}^{-a-\alpha, -b, \alpha+\beta-1} g \right) (x).
 \end{aligned}$$

Let us estimate the integral

$$K = \int_0^1 \tau_2(x) \nu_2(x) dx.$$

By virtue of the conjugation conditions (6), (7) and relation (11), we have

$$K = \frac{1}{\Gamma(1+\gamma)} \int_0^1 \tau_1(x) \tau_1''(x) dx.$$

Integrating by parts and assuming that $\tau_1(0) = \tau_1(1) = 0$, we obtain

$$K = -\frac{1}{\Gamma(1+\gamma)} \int_0^1 [\tau_1'(x)]^2 dx \leq 0. \quad (17)$$

Now we find a lower bound for the integral K . For $g(x) = 0$, equality (16) takes the following form

$$\begin{aligned}
 \tau_2(x) &= -k_1 \left(I_{0+}^{1-\alpha-\beta} \nu_2 \right) (x) - k_2 \left(I_{0+}^{1-\alpha} \nu_2 \right) (x) - k_3 \left(I_{0+}^{\gamma} \nu_2 \right) (x) = \\
 &= -\frac{k_1}{\Gamma(1-\alpha-\beta)} \int_0^x \nu_2(t) (x-t)^{-\alpha-\beta} dt - \frac{k_2}{\Gamma(1-\alpha)} \int_0^x \nu_2(t) (x-t)^{-\alpha} dt - \\
 &\quad - \frac{k_3}{\Gamma(\gamma)} \int_0^x \nu_2(t) (x-t)^{\gamma-1} dt,
 \end{aligned}$$

and, therefore,

$$K = -\frac{k_1}{\Gamma(1-\alpha-\beta)} \int_0^1 v_2(x) dx \int_0^x (x-t)^{-\alpha-\beta} v_2(t) dt -$$

$$-\frac{k_2}{\Gamma(1-\alpha)} \int_0^1 v_2(x) dx \int_0^x (x-t)^{-\alpha} v_2(t) dt - \frac{k_3}{\Gamma(\gamma)} \int_0^1 v_2(x) dx \int_0^x (x-t)^{\gamma-1} v_2(t) dt.$$

Next, we use the well-known formula for the gamma function $\Gamma(\sigma)$ [22]

$$\int_0^\infty s^{\sigma-1} \cos(ks) ds = \frac{\Gamma(\sigma)}{k^\sigma} \cos\left(\frac{\sigma\pi}{2}\right), \quad (k > 0, 0 < \sigma < 1).$$

Assuming that $k = |x - t|$, $\sigma = \alpha + \beta$, we obtain

$$|x - t|^{-\alpha-\beta} =$$

$$= \frac{1}{\Gamma(\alpha + \beta) \cos(\frac{\pi(\alpha+\beta)}{2})} \int_0^\infty s^{\alpha+\beta-1} \cos(s|x-t|) ds, \quad (0 < \alpha + \beta < 1),$$

for $k = |x - t|$, $\sigma = \alpha$ we obtain

$$|x - t|^{-\alpha} = \frac{1}{\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \int_0^\infty s^{\alpha-1} \cos(s|x-t|) ds,$$

for $k = |x - t|$, $\sigma = 1 - \gamma$ we obtain

$$|x - t|^{\gamma-1} = \frac{1}{\Gamma(1-\gamma) \cos(\frac{\pi(1-\gamma)}{2})} \int_0^\infty s^{-\gamma} \cos(s|x-t|) ds.$$

Applying these formulas and the Dirichlet formula for the permutation of the order of integration in the repeated, we arrive at the following relation

$$K = -\frac{2k_1 \sin(\frac{\pi(\alpha+\beta)}{2})}{\pi} \int_0^\infty s^{\alpha+\beta-1} \left[\left(\int_0^1 v_2(x) \cos(sx) dx \right)^2 + \left(\int_0^1 v_2(x) \sin(sx) dx \right)^2 \right] ds -$$

$$-\frac{2k_2 \sin \frac{\alpha\pi}{2}}{\pi} \int_0^\infty s^{\alpha-1} \left[\left(\int_0^1 v_2(x) \cos(sx) dx \right)^2 + \left(\int_0^1 v_2(x) \sin(sx) dx \right)^2 \right] ds -$$

$$-\frac{2k_3 \sin \frac{(1-\gamma)\pi}{2}}{\pi} \int_0^\infty s^{-\gamma} \left[\left(\int_0^1 v_2(x) \cos(sx) dx \right)^2 + \left(\int_0^1 v_2(x) \sin(sx) dx \right)^2 \right] ds \geq 0. \quad (18)$$

From (17) and (18), it follows that $K = 0$, and, consequently, according to (17)

$$\int_0^1 [\tau_1(x)]^2 dx = 0.$$

Hence, by virtue of equalities $\tau_1(0) = \tau_1(1) = 0$, we obtain $\tau_1(x) = 0$ for all $x \in \bar{I}$.

Thus, according to formula (10), makes it possible to assert that $u(x, y) \equiv 0$ in domain \bar{D}^+ .

By virtue of the conjugation condition (6) $\tau_2(x) \equiv \tau_1(x)$ and so $\tau_2(x) \equiv 0 \forall x \in [0, 1]$, and by virtue of (7), (9), (11), also $v_1(x) \equiv 0 \forall x \in [0, 1]$. Then $u(x, y) \equiv 0$ in the domain \bar{D}^- as a solution to the modified Cauchy problem with zero data, which proves the uniqueness of the solution to the original problem. \square

Existence of a solution to the problem

Theorem 2. *Let conditions $\alpha > \max\{-\alpha, \beta - 1\}$, $\gamma > 1 - \beta$ be satisfied. Then the solution to the problem exists.*

Proof. Differentiate both sides of relation (16) with respect to x twice:

$$\begin{aligned} \frac{d^2}{dx^2} \tau_2(x) = & -k_1 \frac{d^2}{dx^2} \left(I_{0+}^{1-\alpha-\beta} v_2 \right) (x) - k_2 \frac{d^2}{dx^2} \left(I_{0+}^{1-\alpha} v_2 \right) (x) - k_3 \frac{d^2}{dx^2} \left(I_{0+}^{\gamma} v_2 \right) (x) + \\ & + \frac{d^2}{d(x)^2} g_1(x), \end{aligned}$$

or (assuming that $\tau_1(x) = \tau_2(x) = \tau(x)$, $v_1(x) = v_2(x) = v(x)$)

$$\left(D_{0+}^{1+\alpha+\beta} v \right) (x) - \lambda \left(D_{0+}^{1+\alpha} v \right) (x) - \delta \left(D_{0+}^{2-\gamma} v \right) (x) - \mu v(x) = g_2(x), \quad (19)$$

where $\lambda = -\frac{k_2}{k_1}$, $\delta = -\frac{k_3}{k_1}$, $\mu = -\frac{\Gamma(1+\gamma)}{k_1}$, $g_2(x) = \frac{1}{k_1} g_1''(x)$. In the monograph [23], the equation with fractional derivatives is considered

$$\left(D_{0+}^{\alpha} y \right) (x) - \lambda \left(D_{0+}^{\beta} y \right) (x) - \delta \left(D_{0+}^{\gamma} y \right) (x) - \mu y(x) = f(x),$$

where $x > 0$, $\alpha > \beta > \gamma > 0$, $\lambda, \mu, \delta \in \mathbb{R}$, $l-1 < \alpha \leq l$, $l \in \mathbb{R}$, and its solution is written out in the following form

$$y(x) = \int_0^x (x-t)^{\alpha-1} G_{\gamma, \beta, \alpha; \lambda}(x-t) f(t) dt.$$

Here

$$\begin{aligned} G_{\gamma, \beta, \alpha; \lambda}(z) = & \sum_{n=0}^{\infty} \left(\sum_{i+v=n} \right) \frac{\mu^i \delta^v}{i! v!} z^{(\alpha-\beta)n + \beta i - (\beta-\gamma)v} \times \\ & \times {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ \left((\alpha-\beta)n + \beta i + \alpha + (\beta-\gamma)v, \alpha - \beta \right) \end{matrix} \middle| \lambda z^{\alpha-\beta} \right]. \\ {}_p\Psi_q(z) = & \sum_{k=0}^{\infty} c_k z^k, \quad c_k = \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{1}{k!}, \quad (k \in \mathbb{N}_0 = \{0, 1, \dots\}), \end{aligned}$$

$$z, a_i, b_j \in \mathbb{C}, \quad \alpha_i, \beta_j \in \mathbb{R}, \quad i = \overline{1, p}, \quad j = \overline{1, q}.$$

For equation (19), this solution takes the following form

$$v(x) = \int_0^x (x-t)^{\alpha+\beta} G_{2-\alpha, 1+\alpha, 1+\alpha+\beta; \lambda}(x-t) g_2(t) dt,$$

$$G_{2-\alpha, 1+\alpha, 1+\alpha+\beta; \lambda}(x-t) = \sum_{n=0}^{\infty} \left(\sum_{i+v=n} \right) \frac{\mu^i \delta^v}{i!v!} (x-t)^{\beta n + (1+\alpha)i - (\alpha+\gamma-1)v} \times$$

$$\times {}_1\Psi_1 \left[\begin{matrix} (n+1, 1) \\ (\beta n + (1+\alpha)i + (1+\alpha+\beta) + (\alpha+\gamma-1)v, \beta) \end{matrix} \middle| \lambda(x-t)^\beta \right].$$

This completes the proof of the existence of a solution to the original problem. \square

Conclusion

The article investigated a nonlocal boundary value problem for a fractional diffusion equation and a degenerate hyperbolic equation in an unbounded domain. The main results obtained are new. We can use these results to investigate various boundary value problems for differential equations with a partial fractional derivative.

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