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Boundary Value Problems for the Three-Dimensional Helmholtz Equation in the Unbounded Octant, Square and Half Space

Z. O. Arzikulov*

Fergana Polytechnic Institute, 150107, Fergana, Ferganskaya str., 86, Uzbekistan

Abstract. At present, the results of the study of boundary value problems for the two-dimensional Helmholtz equation with one and two singular coefficients are known. In the presence of two positive singular coefficients in the two-dimensional Helmholtz equation, explicit solutions of the Dirichlet, Neumann and Dirichlet-Neumann problems in a quarter plane are expressed through a confluent hypergeometric function of two variables. The established properties of the confluent hypergeometric function of two variables allow us to prove the theorem of uniqueness and existence of a solution to the problems posed. In this paper, we study the Dirichlet, Neumann, and Dirichlet-Neumann problems for the three-dimensional Helmholtz equation at zero values of singular coefficients in an octant, a quarter of space, and a half-space. Uniqueness and existence theorems are proved under certain restrictions on the data. The uniqueness of solutions of which is proved using the extremum principle for elliptic equations. Using the known fundamental (singular) solution of the Helmholtz equation, solutions to the problems under study are written out in explicit forms.

Key words: confluent hypergeometric function of three variables; system of partial differential equations; asymptotic formula; three-dimensional Helmholtz equation with three singular coefficients; Dirichlet problem in the first infinite octant.


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*Correspondence:  E-mail: zafarbekarzikulov1984@gmail.com


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Краевые задачи для трехмерного уравнения Гельмгольца в неограниченном октанте, квадрате и полупространстве

О. З. Арзикулов*

Ферганский политехнический институт, 150107, ул. Ферганская, 86, г. Фергана, Узбекистан

Аннотация. В настоящее время известны результаты исследования краевых задач для двумерного уравнения Гельмгольца с одним и двумя сингулярными коэффициентами. При наличии двух положительных сингулярных коэффициентов в двумерном уравнении Гельмгольца явные решения задач Дирихле, Неймана и Дирихле-Неймана в четверти плоскости выражаются через вырожденную гипергеометрическую функцию двух переменных. Установленные свойства вырожденной гипергеометрической функции двух переменных позволяют доказать теорему единственности и существования решения поставленных задач. В данной работе изучаются задачи Дирихле, Неймана и Дирихле-Неймана для трехмерного уравнения Гельмгольца при нулевых значениях сингулярных коэффициентов в октанте, четверти пространства и полупространстве. Доказываются теоремы единственности и существования при определенных ограничениях на данные. Единственность решений которых доказывается с помощью принципа экстремума для эллиптических уравнений. Используя известное фундаментальное (сингулярное) решение уравнения Гельмгольца, решения исследуемых задач выписываются в явном виде.

Ключевые слова: вырожденная гипергеометрическая функция трех переменных; система уравнений в частных производных; асимптотическая формула; трехмерное уравнение Гельмгольца с тремя сингулярными коэффициентами; задача Дирихле в первом бесконечном октанте


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*Корреспонденция:  E-mail: zafarbekarzikulov1984@gmail.com

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Introduction

It's known, that the Helmholtz equation has a variety of applications in physics and other sciences, including the wave equation, the diffusion equation, and the Schrödinger equation for a free particle.

The Helmholtz equation often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. The Helmholtz equation, which represents a time-independent form of the wave equation, results from applying the technique of separation of variables to reduce the complexity of the analysis [15].

The two-dimensional analogue of the vibrating string is the vibrating membrane, with the edges clamped to be motionless. The Helmholtz equation was solved for many basic shapes in the 19th century: the rectangular membrane by Siméon Denis Poisson in 1829, the equilateral triangle by Gabriel Lamé in 1852, and the circular membrane by Alfred Clebsch in 1862. The elliptical drumhead was studied by Émile Mathieu, leading to Mathieu's differential equation.

Two- and more-dimensional Helmholtz equations

$$\sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \mu u = 0$$

and their related boundary-value problems have been investigated in a large number of papers [1–3, 12–14].

On the other hand, the equation has important applications. In 1952 Kapilevich [18] has solved Dirichlet and Neumann problems for multidimensional Helmholtz equation with singular coefficient

$$\sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \frac{2\alpha}{x_1} \frac{\partial u}{\partial x_1} + \mu u = 0, \quad 0 < 2\alpha < 1$$

in the half-space. In 1978 Marichev [19] has investigated two-dimensional Helmholtz equation with two singular coefficients

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\alpha}{x} \frac{\partial u}{\partial x} + \frac{2\beta}{y} \frac{\partial u}{\partial y} + \mu u = 0, \quad 0 < 2\alpha, 2\beta < 1. \quad (1)$$

There are many works [6–9, 11] devoted to the Helmholtz equation (1). For instance, in the work [10] the Dirichlet problem for the singular Helmholtz equation (1) for $\mu = -\lambda^2$ is solved explicitly.

Generally speaking, our further goal is to pose and investigate boundary value problems for Helmholtz equation with three singular coefficients

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{2\alpha}{x} \frac{\partial u}{\partial x} + \frac{2\beta}{y} \frac{\partial u}{\partial y} + \frac{2\gamma}{z} \frac{\partial u}{\partial z} + \mu u = 0, \quad 0 \leq 2\alpha, 2\beta, 2\gamma < 1 \quad (2)$$

in some infinite domains.

For beginning, in the present paper, we study the Dirichlet, Neumann and Dirichlet-Neumann boundary value problems for equation (2) at $\alpha = \beta = \gamma = 0$ and $\mu = -\lambda^2$ in the unbounded domains – in an octant, square of the space and half-space.

The Dirichlet problem D_3^3 for the Helmholtz equation in the first octant

Let us consider the following Helmholtz equation

$$u_{xx} + u_{yy} + u_{zz} - \lambda^2 u = 0, \quad (3)$$

in the infinite domain $\Omega_3 \equiv \{(x, y, z) : x > 0, y > 0, z > 0\}$

The Dirichlet problem D_3^3 . Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the class of functions $C(\overline{\Omega_3}) \cup C^2(\Omega_3)$, satisfying the conditions

$$u(x, y, 0) = \tau_1(x, y), \quad 0 \leq x, y < \infty, \quad (4)$$

$$u(x, 0, z) = \tau_2(x, z), \quad 0 \leq x, z < \infty, \quad (5)$$

$$u(0, y, z) = \tau_3(y, z), \quad 0 \leq y, z < \infty, \quad (6)$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \quad (7)$$

where $\tau_k(t, s)$ are given functions such that

$$\tau_1(x, y) = \frac{\tilde{\tau}_1(x, y)}{(1 + x^2 + y^2)^{(1+\varepsilon_1)/2}}, \quad \tilde{\tau}_1(x, y) \in C(0 \leq x, y < \infty), \quad \varepsilon_1 > 0, \quad (8)$$

$$\tau_2(x, z) = \frac{\tilde{\tau}_2(x, z)}{(1 + x^2 + z^2)^{(1+\varepsilon_2)/2}}, \quad \tilde{\tau}_2(x, z) \in C(0 \leq x, z < \infty), \quad \varepsilon_2 > 0, \quad (9)$$

$$\tau_3(y, z) = \frac{\tilde{\tau}_3(y, z)}{(1 + y^2 + z^2)^{(1+\varepsilon_3)/2}}, \quad \tilde{\tau}_3(y, z) \in C(0 \leq y, z < \infty), \quad \varepsilon_3 > 0. \quad (10)$$

In addition, the functions $\tau_1(x, y)$, $\tau_2(x, z)$ and $\tau_3(y, z)$ satisfy the concordance conditions in origin $\tau_1(0, 0) = \tau_2(0, 0) = \tau_3(0, 0)$, and in the lateral ribs of the domain Ω_3 :

$$\tau_1(x, 0) = \tau_2(x, 0), \quad \tau_1(0, y) = \tau_3(y, 0), \quad \tau_2(0, z) = \tau_3(0, z) \quad 0 \leq x, y, z < \infty. \quad (11)$$

Theorem. [4] *The Dirichlet problem for equation (3) in an unbounded domain Ω_3 can have at most one solution.*

Existence of the solution to the Dirichlet problem D_3^3

We will seek a solution to the Dirichlet problem (3) – (11) in the form

$$u(x, y, z) = \sum_{i=1}^3 u_i(x, y, z),$$

where $u_1(x, y, z)$, $u_2(x, y, z)$, $u_3(x, y, z)$ are a solutions to the equation (3), satisfying the boundary conditions

$$u_1(x, y, 0) = \tau_1(x, y), \quad u_1(x, 0, z) = 0, \quad u_1(0, y, z) = 0, \quad (12)$$

$$u_2(x, y, 0) = 0, \quad u_2(x, 0, z) = \tau_2(x, z), \quad u_2(0, y, z) = 0, \tag{13}$$

$$u_3(x, y, 0) = 0, \quad u_3(x, 0, z) = 0, \quad u_3(0, y, z) = \tau_3(y, z), \tag{14}$$

respectively.

Lemma 1. *If a function $\tau_1(x, y)$ can be represented by the formula (8), then the function*

$$u_1(x, y, z) = \sum_{j=1}^4 (-1)^{j+1} u_{1j}(x, y, z) \tag{15}$$

is a regular solution of the equation (3) in the domain Ω_3 , satisfying the boundary conditions (7) and (12), where

$$u_{1j}(x, y, z) = \frac{z}{2\pi} \int_0^\infty \int_0^\infty \frac{\tau_1(t, s)}{\rho_{1j}^3} (1 + \lambda \rho_{1j}) e^{-\lambda \rho_{1j}} dt ds, \quad j = 1, 2, 3, 4; \tag{16}$$

$$\rho_{11} = \sqrt{(t-x)^2 + (s-y)^2 + z^2}, \quad \rho_{14} = \sqrt{(t-x)^2 + (s+y)^2 + z^2}, \tag{17}$$

$$\rho_{12} = \sqrt{(t+x)^2 + (s-y)^2 + z^2}, \quad \rho_{13} = \sqrt{(t+x)^2 + (s+y)^2 + z^2}. \tag{18}$$

Proof. Now we will show that function $u_{11}(x, y, z)$ satisfies condition (4) of the Dirichlet problem.

By setting $t = x + z\mu$, $s = y + z\nu$, we transform formula (16) for $j = 1$ to the form

$$u_{11}(x, y, z) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty \frac{\tau_1(x + z\mu, y + z\nu)}{(1 + \mu^2 + \nu^2)^{3/2}} [1 + \lambda z^2 (1 + \mu^2 + \nu^2)] e^{-\lambda z^2 (1 + \mu^2 + \nu^2)} d\mu d\nu \tag{19}$$

On the right-hand side of (19) we pass to the limit as $z \rightarrow 0$ and get

$$\lim_{z \rightarrow 0} u_{11}(x, y, z) = \tau_1(x, y) \frac{1}{2\pi} \int_0^\infty \int_0^\infty \frac{d\mu d\nu}{(1 + \mu^2 + \nu^2)^{3/2}}.$$

Using consistently the well-known identity [5, Eq. 4.623]¹

$$\int_0^\infty \int_0^\infty \varphi(a^2x^2 + b^2y^2) dx dy = \frac{\pi}{4ab} \int_0^\infty \varphi(x) dx, \tag{20}$$

we obtain

$$\lim_{z \rightarrow 0} u_{11}(x, y, z) = \tau_1(x, y).$$

Likewise, everyone can easily get

$$\lim_{z \rightarrow 0} u_{1j}(x, y, z) = 0, \quad j = 2, 3, 4.$$

¹There is a typo in this formula, the correct formula is (20).

So, the following equality is true for the function $u_1(x, y, z)$ defined in (15):

$$\lim_{z \rightarrow 0} u_1(x, y, z) = \tau_1(x, y).$$

Taking into account the obvious equalities

$$\rho_{11}|_{x=0} = \rho_{12}|_{x=0}, \quad \rho_{11}|_{y=0} = \rho_{14}|_{y=0}, \quad \rho_{12}|_{y=0} = \rho_{13}|_{y=0}, \quad \rho_{13}|_{x=0} = \rho_{14}|_{x=0},$$

we establish that for there exist a following limits

$$\lim_{x \rightarrow 0} u_1(x, y, z) = 0, \quad \lim_{y \rightarrow 0} u_1(x, y, z) = 0.$$

It is easy to see that as the point (x, y, z) tends to infinity, i.e. $R \rightarrow \infty$, function (16) tends to zero. Then, by virtue of (8), we get

$$|u_{11}(x, y, z)| \leq C_{11}z \int_0^\infty \int_0^\infty \frac{dt ds}{(1+t^2+s^2)^{(1+\varepsilon_1)/2} \rho_{11}^2} \quad (21)$$

At the right-hand side of inequality (21), change the variables as follows: $t = R\mu$, $s = R\nu$. We obtain the inequalities

$$|u_{11}| \leq \frac{C_{11}}{R^{\varepsilon_1}} \cdot \frac{z}{R} K(x, y, z; R)$$

where

$$K(x, y, z; R) = \int_0^\infty \int_0^\infty \frac{d\mu d\nu}{\left(\frac{1}{R^2} + \mu^2 + \nu^2\right)^{(1+\varepsilon_1)/2} \left[1 + \mu^2 + \nu^2 - \frac{2}{R}(\mu x + \nu y)\right]}.$$

Let us show that this proper double integral is bounded. Indeed, using the formula (20) and passing to the limit as $R \rightarrow \infty$, we calculate

$$\lim_{R \rightarrow \infty} K(x, y, z; R) = \frac{1}{4} \Gamma\left(\frac{1+\varepsilon_1}{2}\right) \Gamma\left(\frac{2+\varepsilon_1}{2}\right), \quad (22)$$

where $\Gamma(z)$ is a famous gamma function.

Using (22), we obtain

$$|u_{11}(x, y, z)| \leq \frac{C}{R^{\varepsilon_1}}, \quad \varepsilon_1 > 0, \quad (23)$$

where C is a positive constant.

Now, it intermediately follows from estimate (23) that the function (16) vanishes at infinity. \square

Similar statements are also true for functions $u_2(x, y, z)$ and $u_3(x, y, z)$ that satisfy the conditions (13) and (14), respectively, and vanish at infinity, where

$$u_i(x, y, z) = \sum_{j=1}^4 (-1)^{j+1} u_{ij}(x, y, z), \quad i = 2, 3,$$

$$u_{2j}(x, y, z) = \frac{y}{2\pi} \int_0^\infty \int_0^\infty \frac{\tau_2(t, s)}{\rho_{2j}^3} (1 + \lambda \rho_{2j}) e^{-\lambda \rho_{2j}} dt ds, \tag{24}$$

$$\rho_{21} = \sqrt{(t-x)^2 + y^2 + (s-z)^2}, \quad \rho_{24} = \sqrt{(t-x)^2 + y^2 + (s+z)^2}, \tag{25}$$

$$\rho_{22} = \sqrt{(t+x)^2 + y^2 + (s-z)^2}, \quad \rho_{23} = \sqrt{(t+x)^2 + y^2 + (s+z)^2}, \tag{26}$$

$$u_{3j}(x, y, z) = \frac{x}{2\pi} \int_0^\infty \int_0^\infty \frac{\tau_3(t, s)}{\rho_{3j}^3} (1 + \lambda \rho_{3j}) e^{-\lambda \rho_{3j}} dt ds, \tag{27}$$

$$\rho_{31} = \sqrt{x^2 + (t-y)^2 + (s-z)^2}, \quad \rho_{34} = \sqrt{x^2 + (t-y)^2 + (s+z)^2}, \tag{28}$$

$$\rho_{32} = \sqrt{x^2 + (t+y)^2 + (s-z)^2}, \quad \rho_{33} = \sqrt{x^2 + (t+y)^2 + (s+z)^2}. \tag{29}$$

Thus, we have proved the following

Theorem 2. *If a given functions τ_1, τ_2 and τ_3 satisfy the conditions (8) – (11), then a function $u(x, y, z) = \sum_{i=1}^3 \sum_{j=1}^4 (-1)^{j+1} u_{ij}(x, y, z)$, where the functions u_{ij} are defined in (16), (24) and (27), is a regular solution of equation (3) in the domain Ω_3 , satisfying the conditions (4) to (7).*

Other problems for the Helmholtz equation in the first infinite octant

Let us consider the Helmholtz equation (3) in the infinite domain Ω_3 .

The Neumann problem N_3^3 . Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the domain Ω_3 , satisfying the conditions

$$\lim_{z \rightarrow 0} \frac{\partial u(x, y, z)}{\partial z} = \nu_1(x, y), \quad 0 < x, y < \infty, \tag{30}$$

$$\lim_{y \rightarrow 0} \frac{\partial u(x, y, z)}{\partial y} = \nu_2(x, z), \quad 0 < x, z < \infty, \tag{31}$$

$$\lim_{x \rightarrow 0} \frac{\partial u(x, y, z)}{\partial x} = \nu_3(y, z), \quad 0 < y, z < \infty, \tag{32}$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \tag{33}$$

where $\nu_k(t, s)$ are given functions such that

$$\nu_1(x, y) = \frac{\tilde{\nu}_1(x, y)}{(1 + x^2 + y^2)^{\varepsilon_1/2}}, \quad \tilde{\nu}_1(x, y) \in C(0 < x, y < \infty), \quad \varepsilon_1 > 0, \tag{34}$$

$$\nu_2(x, z) = \frac{\tilde{\nu}_2(x, z)}{(1 + x^2 + z^2)^{\varepsilon_2/2}}, \quad \tilde{\nu}_2(x, z) \in C(0 < x, z < \infty), \quad \varepsilon_2 > 0, \tag{35}$$

$$\nu_3(y, z) = \frac{\tilde{\nu}_3(y, z)}{(1 + y^2 + z^2)^{\varepsilon_3/2}}, \quad \tilde{\nu}_3(y, z) \in C(0 < y, z < \infty), \quad \varepsilon_3 > 0. \tag{36}$$

Theorem 3. *If a given functions ν_1, ν_2 and ν_3 satisfy the conditions (34) – (36), then a function*

$$u(x, y, z) = -\frac{1}{2\pi} \sum_{i=1}^3 \sum_{j=1}^4 \int_0^{\infty} \int_0^{\infty} \frac{\nu_i(t, s) e^{-\lambda \rho_{ij}}}{\rho_{ij}} dt ds, \quad i = 1, 2, 3; \quad j = 1, 2, 3, 4$$

is a regular solution of equation (3) in the domain Ω_3 , satisfying the boundary conditions (30) to (33), where ρ_{ij} are defined in (17), (18), (25), (26), (28), (29).

The Dirichlet-Neuman problem $D_3^2 N_3^1$. Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the class of functions $C(\overline{\Omega}_3) \cup C^2(\Omega_3)$, satisfying the conditions

$$u(x, y, 0) = \tau_1(x, y), \quad 0 \leq x, y < \infty, \quad (37)$$

$$u(x, 0, z) = \tau_2(x, z), \quad 0 \leq x, z < \infty, \quad (38)$$

$$\lim_{x \rightarrow 0} \frac{\partial u(x, y, z)}{\partial x} = \nu_3(y, z), \quad 0 < y, z < \infty, \quad (39)$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \quad (40)$$

where $\tau_1(t, s), \tau_2(t, s), \nu_3(t, s)$ are given functions such that

$$\tau_1(x, y) = \frac{\tilde{\tau}_1(x, y)}{(1 + x^2 + y^2)^{(1+\varepsilon_1)/2}}, \quad \tilde{\tau}_1(x, y) \in C(0 \leq x, y < \infty), \quad \varepsilon_1 > 0, \quad (41)$$

$$\tau_2(x, z) = \frac{\tilde{\tau}_2(x, z)}{(1 + x^2 + z^2)^{(1+\varepsilon_2)/2}}, \quad \tilde{\tau}_2(x, z) \in C(0 \leq x, z < \infty), \quad \varepsilon_2 > 0, \quad (42)$$

$$\nu_3(y, z) = \frac{\tilde{\nu}_3(y, z)}{(1 + y^2 + z^2)^{\varepsilon_3/2}}, \quad \tilde{\nu}_3(y, z) \in C(0 < y, z < \infty), \quad \varepsilon_3 > 0. \quad (43)$$

In addition, the functions $\tau_1(x, y), \tau_2(x, z)$ satisfy the concordance conditions in origin and in the lateral ribs of the domain Ω_3 :

$$\tau_1(x, 0) = \tau_2(x, 0), \quad 0 \leq x < \infty. \quad (44)$$

Theorem 4. *If a given functions τ_1, τ_2 and ν_3 satisfy the conditions (41) – (44), then a function*

$$\begin{aligned} u(x, y, z) = & \frac{z}{2\pi} \sum_{j=1}^4 (-1)^{[\frac{j-1}{2}]} \int_0^{\infty} \int_0^{\infty} \frac{\tau_1(t, s)}{\rho_{1j}^3} (1 + \lambda \rho_{1j}) e^{-\lambda \rho_{1j}} dt ds \\ & + \frac{y}{2\pi} \sum_{j=1}^4 (-1)^{[\frac{j-1}{2}]} \int_0^{\infty} \int_0^{\infty} \frac{\tau_2(t, s)}{\rho_{2j}^3} (1 + \lambda \rho_{2j}) e^{-\lambda \rho_{2j}} dt ds \\ & - \frac{1}{2\pi} \sum_{j=1}^4 (-1)^{j+1} \int_0^{\infty} \int_0^{\infty} \frac{\nu_3(t, s)}{\rho_{3j}} e^{-\lambda \rho_{3j}} dt ds, \end{aligned}$$

is a regular solution of equation (3) in the domain Ω_3 , satisfying the boundary conditions (37) to (40), where ρ_{ij} are defined in (17),(18),(25),(26),(28),(29).

Hereinafter, $[\alpha]$ denotes an integer part of the number α .

The Dirichlet-Neumann problem $D_3^1N_3^2$. Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the domain Ω_3 , satisfying the conditions

$$\lim_{z \rightarrow 0} \frac{\partial u(x, y, z)}{\partial z} = \nu_1(x, y), \quad 0 < x, y < \infty, \tag{45}$$

$$\lim_{y \rightarrow 0} \frac{\partial u(x, y, z)}{\partial y} = \nu_2(x, z), \quad 0 < x, z < \infty, \tag{46}$$

$$u(0, y, z) = \tau_3(y, z), \quad 0 \leq y, z < \infty, \tag{47}$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \tag{48}$$

where $\nu_1(t, s), \nu_2(t, s), \tau_3(t, s)$ are given functions such that

$$\nu_1(x, y) = \frac{\tilde{\nu}_1(x, y)}{(1 + x^2 + y^2)^{\varepsilon_1/2}}, \quad \tilde{\nu}_1(x, y) \in C(0 < x, y < \infty), \quad \varepsilon_1 > 0, \tag{49}$$

$$\nu_2(x, z) = \frac{\tilde{\nu}_2(x, z)}{(1 + x^2 + z^2)^{\varepsilon_2/2}}, \quad \tilde{\nu}_2(x, z) \in C(0 < x, z < \infty), \quad \varepsilon_2 > 0, \tag{50}$$

$$\tau_3(y, z) = \frac{\tilde{\tau}_3(y, z)}{(1 + y^2 + z^2)^{(1+\varepsilon_3)/2}}, \quad \tilde{\tau}_3(y, z) \in C(0 \leq y, z < \infty), \quad \varepsilon_3 > 0. \tag{51}$$

Theorem 5. *If a given functions ν_1, ν_2 and τ_3 satisfy the conditions (49) – (51), then a function*

$$u(x, y, z) = -\frac{1}{2\pi} \sum_{j=1}^4 (-1)^{[\frac{j}{2}]} \int_0^\infty \int_0^\infty \frac{\nu_1(t, s)}{\rho_{1j}} e^{-\lambda \rho_{1j}} dt ds - \frac{1}{2\pi} \sum_{j=1}^4 (-1)^{[\frac{j}{2}]} \int_0^\infty \int_0^\infty \frac{\nu_2(t, s)}{\rho_{2j}} e^{-\lambda \rho_{2j}} dt ds + \frac{x}{2\pi} \sum_{j=1}^4 \int_0^\infty \int_0^\infty \frac{\tau_3(t, s)}{\rho_{3j}^3} (1 + \lambda \rho_{3j}) e^{-\lambda \rho_{3j}} dt ds$$

is a regular solution of equation (3) in the domain Ω_3 , satisfying the conditions (45) to (48), where ρ_{ij} are defined in (17),(18),(25),(26),(28),(29).

Boundary value problems for the Helmholtz equation in the first infinite square

Let us consider the Helmholtz equation (3) in the infinite domain

$$\Omega_2 \equiv \{(x, y, z) : -\infty < x < \infty, y > 0, z > 0\}.$$

The Dirichlet problem D_3^2 . Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the domain Ω_2 , satisfying the conditions

$$u(x, y, 0) = \tau_1(x, y), \quad -\infty < x < \infty, 0 \leq y < \infty, \tag{52}$$

$$u(x, 0, z) = \tau_2(x, z), \quad -\infty < x < \infty, 0 \leq z < \infty \quad (53)$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \quad (54)$$

where $\tau_1(t, s)$ and $\tau_2(t, s)$ are a given functions such that

$$\tau_1(x, y) = \frac{\tilde{\tau}_1(x, y)}{(1 + x^2 + y^2)^{\varepsilon_1/2}}, \quad \tilde{\tau}_1(x, y) \in C(-\infty < x < \infty, 0 \leq y < \infty), \varepsilon_1 > 0, \quad (55)$$

$$\tau_2(x, z) = \frac{\tilde{\tau}_2(x, z)}{(1 + x^2 + z^2)^{\varepsilon_2/2}}, \quad \tilde{\tau}_2(x, z) \in C(-\infty < x < \infty, 0 \leq z < \infty), \varepsilon_2 > 0, \quad (56)$$

Theorem 6. *If a given functions τ_1 and τ_2 satisfy the conditions (55) and (56), then a function*

$$\begin{aligned} u(x, y, z) = & \\ = \frac{z}{16\pi} & \left[\int_0^\infty ds \int_{-\infty}^\infty \frac{\tau_1(t, s) (1 + \lambda\rho_{11}) e^{-\lambda\rho_{11}}}{\rho_{11}^3} dt - \int_0^\infty ds \int_{-\infty}^\infty \frac{\tau_1(t, s) (1 + \lambda\rho_{14}) e^{-\lambda\rho_{14}}}{\rho_{14}^3} dt \right] \\ + \frac{y}{16\pi} & \left[\int_0^\infty ds \int_{-\infty}^\infty \frac{\tau_2(t, s) (1 + \lambda\rho_{21}) e^{-\lambda\rho_{21}}}{\rho_{21}^3} dt - \int_0^\infty ds \int_{-\infty}^\infty \frac{\tau_2(t, s) (1 + \lambda\rho_{24}) e^{-\lambda\rho_{24}}}{\rho_{24}^3} dt \right] \end{aligned}$$

is a regular solution of equation (3) in the domain Ω_2 , satisfying the conditions (52) to (54), where ρ_{ij} are defined in (17) and (25).

The Dirichlet-Neuman problem $D_3^1 N_3^1$. Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the domain Ω_2 , satisfying the conditions

$$u(x, y, 0) = \tau_1(x, y), \quad -\infty < x < \infty, 0 \leq y < \infty, \quad (57)$$

$$\lim_{y \rightarrow 0} \frac{\partial u(x, y, z)}{\partial y} = \nu_2(x, z), \quad -\infty < x < \infty, 0 \leq z < \infty, \quad (58)$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \quad (59)$$

where $\tau_1(t, s)$, $\nu_2(t, s)$ are given functions such that

$$\tau_1(x, y) = \frac{\tilde{\tau}_1(x, y)}{(1 + x^2 + y^2)^{\varepsilon_1/2}}, \quad \tilde{\tau}_1(x, y) \in C(-\infty < x < +\infty, y < \infty), \varepsilon_1 > 0, \quad (60)$$

$$\nu_2(x, z) = \frac{\tilde{\nu}_2(x, z)}{(1 + x^2 + z^2)^{\varepsilon_2/2}}, \quad \tilde{\nu}_2(x, z) \in C(-\infty < x < +\infty, 0 < z < \infty), \varepsilon_2 > 0, \quad (61)$$

Theorem 7. *If a given functions τ_1 and ν_2 satisfy the conditions (60) and (61), then a function*

$$\begin{aligned} u(x, y, z) = & \\ = \frac{z}{16\pi} & \left[\int_0^\infty ds \int_{-\infty}^\infty \frac{\tau_1(t, s) (1 + \lambda\rho_{11}) e^{-\lambda\rho_{11}}}{\rho_{11}^3} dt + \int_0^\infty ds \int_{-\infty}^\infty \frac{\tau_1(t, s) (1 + \lambda\rho_{14}) e^{-\lambda\rho_{14}}}{\rho_{14}^3} dt \right] \end{aligned}$$

$$-\frac{1}{16\pi} \left[\int_0^\infty ds \int_{-\infty}^\infty \frac{\nu_2(t, s) e^{-\lambda\rho_{21}}}{\rho_{21}} dt - \int_0^\infty ds \int_{-\infty}^\infty \frac{\nu_2(t, s) e^{-\lambda\rho_{24}}}{\rho_{24}} dt \right]$$

is a regular solution of equation (3) in the domain Ω_2 , satisfying the conditions (57) – (59), where ρ_{ij} are defined in (17) and (25).

Let us consider the following Helmholtz equation (3) in the infinite domain $\Omega_2 \equiv \{(x, y, z) : -\infty < x < \infty, y > 0, z > 0\}$.

The Neumann problem N_3^2 . Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the domain Ω_2 , satisfying the conditions

$$\lim_{z \rightarrow 0} \frac{\partial u(x, y, z)}{\partial z} = \nu_1(x, y), \quad -\infty < x < +\infty, 0 < y < \infty, \tag{62}$$

$$\lim_{y \rightarrow 0} \frac{\partial u(x, y, z)}{\partial y} = \nu_2(x, z), \quad -\infty < x < +\infty, 0 < z < \infty, \tag{63}$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \tag{64}$$

where $\nu_k(t, s)$ are given functions such that

$$\nu_1(x, y) = \frac{\tilde{\nu}_1(x, y)}{(1 + x^2 + y^2)^{\varepsilon_1/2}}, \quad \tilde{\nu}_1(x, y) \in C(-\infty < x < +\infty, 0 < y < \infty), \varepsilon_1 > 0, \tag{65}$$

$$\nu_2(x, z) = \frac{\tilde{\nu}_2(x, z)}{(1 + x^2 + z^2)^{\varepsilon_2/2}}, \quad \tilde{\nu}_2(x, z) \in C(-\infty < x < +\infty, 0 < z < \infty), \varepsilon_2 > 0, \tag{66}$$

Theorem 8. If a given functions ν_1 and ν_2 satisfy the conditions (65) and (66), then a function

$$u(x, y, z) = -\frac{1}{16\pi} \left[\int_0^\infty ds \int_{-\infty}^\infty \frac{\nu_1(t, s) e^{-\lambda\rho_{11}}}{\rho_{11}} dt + \int_0^\infty ds \int_{-\infty}^\infty \frac{\nu_1(t, s) e^{-\lambda\rho_{14}}}{\rho_{14}} dt \right] - \frac{1}{16\pi} \left[\int_0^\infty ds \int_{-\infty}^\infty \frac{\nu_2(t, s) e^{-\lambda\rho_{21}}}{\rho_{21}} dt + \int_0^\infty ds \int_{-\infty}^\infty \frac{\nu_2(t, s) e^{-\lambda\rho_{24}}}{\rho_{24}} dt \right]$$

is a regular solution of equation (3) in the domain Ω_2 , satisfying the conditions (62) – (64), where ρ_{ij} are defined in (17) and (25).

Boundary value problems for the Helmholtz equation in the half-space

Let us consider the following Helmholtz equation (3) in the infinite domain $\Omega_1 \equiv \{(x, y, z) : -\infty < x, y < \infty, z > 0\}$

The Dirichlet problem D_3^1 . Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the domain Ω_1 , satisfying the conditions

$$u(x, y, 0) = \tau_1(x, y), \quad -\infty < x, y < \infty, \tag{67}$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \quad (68)$$

where $\tau_1(t, s)$ are given functions such that

$$\tau_1(x, y) = \frac{\tilde{\tau}_1(x, y)}{(1 + x^2 + y^2)^{\varepsilon_1/2}}, \quad \tilde{\tau}_1(x, y) \in C(-\infty < x, y < +\infty), \quad \varepsilon_1 > 0, \quad (69)$$

Note, that the Dirichlet problem for Helmholtz equation (3) in case of half-space is found in the books for students [16, 17].

Theorem 9. *If a given function τ_1 satisfies the condition (69), then a function*

$$u(x, y, z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \frac{\tau_1(t, s) (1 + \lambda \rho_{11}) e^{-\lambda \rho_{11}}}{\rho_{11}^3} dt \quad (70)$$

is a regular solution of equation (3) in the domain Ω_1 , satisfying the conditions (67) and (68), where ρ_{11} is defined in (17).

The Neumann problem N_3^1 . Find a regular solution $u(x, y, z)$ to the Helmholtz equation (3) in the domain Ω_1 , satisfying the conditions

$$\lim_{z \rightarrow 0} \frac{\partial u(x, y, z)}{\partial z} = \nu_1(x, y), \quad -\infty < x < +\infty, -\infty < y < +\infty, \quad (71)$$

$$\lim_{R \rightarrow \infty} u(x, y, z) = 0, \quad R = \sqrt{x^2 + y^2 + z^2}, \quad (72)$$

where $\nu_1(t, s)$ is a given function such that

$$\nu_1(x, y) = \frac{\tilde{\nu}_1(x, y)}{(1 + x^2 + y^2)^{\varepsilon_1/2}}, \quad \tilde{\nu}_1(x, y) \in C(-\infty < x < +\infty, -\infty < y < +\infty), \quad \varepsilon_1 > 0, \quad (73)$$

Note, that the Neumann problem for Helmholtz equation (3) in case of half-space is found in [16, 17].

Theorem 10. *If a given function ν_1 satisfies the condition (73), then a function*

$$u(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \frac{\nu_1(t, s) e^{-\lambda \rho_{11}}}{\rho_{11}} dt \quad (74)$$

is a regular solution of equation (3) in the domain Ω_1 , satisfying the boundary conditions (71) and (72), where ρ_{11} is defined in (17).

Remark 1. The solutions (70) and (74) of the Dirichlet and Neumann problems are found in [17].

Remark 2. Uniqueness of solutions of the problems posed is proved using the extremum principle for elliptic equations.

Conclusion

In this paper we presented solutions of 9 boundary value problems for the three-dimensional Helmholtz equation in infinite domains in explicit and convenient forms for


further research. The results of this paper can be useful in the study of boundary value problems for a three-dimensional equation of mixed type, because from the explicit solutions of the posed problems it is easy to derive functional relationships between the traces of the desired solution and its derivative along the normal, brought to the plane of change of the equation type.

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Information about the author



Arzikulov Zafarjon Odilovich ✉ – Doktorant, Fac. of Phys. & Department of Higher Mathematics, Fergana Politechnical Institute Fergana, Uzbekistan,  ORCID 0009-0004-2965-4566.