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Construction of Basis Functions for Finite Element Methods in a Hilbert Space

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Abstract. The present work is devoted to construction of the optimal interpolation formula exact for trigonometric functions $\sin(\omega x)$ and $\cos(\omega x)$. Here the analytical representations of the coefficients of the optimal interpolation formula in a certain Hilbert space are obtained using the discrete analogue of the differential operator. Taking the coefficients of the optimal interpolation formula as basis functions, in the finite element methods the boundary value problems for ordinary differential equations of the second order are approximately solved. In particular, it is shown that the coefficients of the optimal interpolation formula can serve as a set of effective basis functions. Approximate solutions of the differential equations are compared using the constructed basis functions and known basis functions. In particular, we have obtained numerical results for the cases when the numbers of basis functions are 6 and 11. In both cases, we have got that the accuracy of the approximate solution to the boundary value problems for second-order ordinary differential equations found using our basis functions is higher than the accuracy of the approximate solution found using known basis functions. It is proven that the accuracy of the approximate solution increases with increasing the number of basis functions.

Key words: basis functions, ordinary differential equation, boundary value problem, finite element method, interpolation.

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Построение базисных функции в методе конечных элементов в гильбертовом пространстве

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Аннотация. Настоящая работа посвящена построению оптимальной интерполяционной формулы, точной для тригонометрических функций $\sin(\omega x)$ и $\cos(\omega x)$. Здесь аналитические представления коэффициентов оптимальной интерполяционной формулы в некотором гильбертовом пространстве получены с использованием дискретного аналога дифференциального оператора. Принимая в качестве базисных функций коэффициенты оптимальной интерполяционной формулы, в методах конечных элементов приближенно решаются краевые задачи для обыкновенных дифференциальных уравнений второго порядка. В частности, показано, что коэффициенты оптимальной интерполяционной формулы могут служить набором эффективных базисных функций. Приближенные решения дифференциальных уравнений сравниваются с использованием построенных базисных функций и известных базисных функций. В частности, мы получили численные результаты для случаев, когда количество базисных функций равно 6 и 11. В обоих случаях мы получили, что точность приближенного решения краевых задач для обыкновенных дифференциальных уравнений второго порядка, найденного с помощью наших базисных функций, выше точности приближенного решения, найденного с использованием известных базисных функций. Доказано, что точность приближенного решения возрастает с увеличением числа базисных функций.

Ключевые слова: базисные функции, обыкновенное дифференциальное уравнение, краевая задача, конечный элемент, интерполяция.

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1 Introduction

The finite element method is one of the effective methods in numerical solving many differential equations encountered in science and technology. The emergence of this method is related to the solution of problems arising in the course of space research (1950). The finite element method was first studied by M.J.Turner, R.W.Clough, N.S.Martin, and L.J.Topp (1956) (see [1]). After that, in 1963, R.J.Melosh [2] theoretically developed this method and showed that it is possible to consider the finite element method as one of the variants of the well-known Rayleigh-Ritz method. In subsequent works, the field of application of the finite element method was further expanded. In particular, it was argued in [3] and [4] that the finite element methods can be easily obtained in solving structural mechanics and hydromechanics problems using options such as the Galerkin method or the least squares method. The establishment of this fact played an important role in the theoretical foundation of the finite element method, as it allowed to use it in solving any differential equations. The scope of application of the finite element methods has expanded from tension analysis in aircraft and automobile structures to the calculation of complex systems in nuclear power plants. It should be noted that the theoretical and practical development of the finite element methods eliminated the need to solve many problems of physics by the variational method. This can be seen as an achievement of the finite element methods. For more information about the finite element methods, one can refer to [5], [6], [7] or other interesting books.

In this work, we construct basis functions using the coefficients of the optimal interpolation formula obtained in a certain Hilbert space and apply these basis functions in finite element methods to approximately solve ordinary differential equations of the second order. At the same time, we compare the accuracy of the approximate solution found using our constructed basis functions with the accuracy of the approximate solution found using known hat basis functions.

The rest of the paper is organized as follows. In the second section, we consider the problem of approximate solution of the boundary value problem for the second-order ordinary differential equation using the finite element method. In the third section, we present the optimal interpolation formula and analytical forms of the coefficients of the optimal interpolation formula in a certain Hilbert space. In the fourth section, we deal with the construction of basis functions using the coefficients of the optimal interpolation formula. In the fifth section, we apply the constructed basis functions in finite element methods and we present numerical results.

2 The finite element method for the second order linear differential equations

The concept of a boundary value problem for ordinary differential equations of the second order can be stated in general as follows (see, for instance, [9]). We consider the second order differential equation

$$Lu \equiv -\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = f(x),\ \ a \leq x \leq b \tag{1}$$

with the boundary conditions

$$\alpha_1 u(a) + \beta_1 u'(a) = \gamma_1,\ \ \alpha_2 u(b) + \beta_2 u'(b) = \gamma_2. \tag{2}$$

The differential equation (1) with the boundary conditions (2) is called a boundary value problem. Here $p \in C^1[a,b]$ and $q, f \in C[a,b]$ and $p(x) \geq k > 0, q(x) \geq 0$ for $x \in [a,b], k = const, \alpha_i, \beta_i, \gamma_i (i = 1,2)$ are given numbers.

Now we deal with the approximate solution of the boundary value problem (1)-(2) using the finite element method. We integrate (1) over the interval $[a,b]$ multiplying by an arbitrary function $v \in C^1[a,b]$ satisfying the boundary conditions (2) and equalities $v'(a) = 0$, $v'(b) = 0$. Then using the formula of integration by parts, we get

$$\int_a^b (pu'v' + quv)\, dx = \int_a^b fv dx. \tag{3}$$

It should be noted that equality (3) is in some sense equivalent to the boundary value problem (1)-(2) (see, for example, [9] page 169).

We use equation (3) for approximately solution of the boundary value problem (1)-(2). Let us consider the Galerkin method. Given linear independent functions $\xi_0, \xi_1, ..., \xi_n \in C^1[a,b]$ satisfying the boundary conditions (2). In that case, the approximate solution of the boundary value problem (1)-(2) is sought in the following form:

$$u_n(x) = \sum_{j=0}^{n} c_j \xi_j(x). \tag{4}$$

Since the linear independent functions $\xi_i(x), i = 0,1,...,n$ are also elements of the space $C^1[a,b]$ then putting $u_n(x)$ in place of $u(x)$ in equation (3) and taking $\xi_i(x), i = 0,1,...,n$ as $v(x)$ from equation (3) we get the following system of linear equations

$$\int_a^b (pu_n'\xi_i' + qu_n\xi_i)\, dx = \int_a^b f\xi_i dx,\ \ i = 0,1,...,n. \tag{5}$$

Taking into account (4), the system of linear equations (5) can be written in the following form:

$$\sum_{j=0}^{n} a_{ij}c_j = b_i,\ \ i = 0,1,...,n$$

or it can be written in the following matrix form

$$Ac = b,$$

where

$$A = (a_{ij})_{i,j=0}^{n},\quad c = (c_0, ..., c_n)^T,\quad b = (b_0, ..., b_n)^T$$

with

$$a_{ij} = \int_a^b \left(p\xi_i'\xi_j' + q\xi_i\xi_j\right) dx \text{ and } b_i = \int_a^b f\xi_i dx.$$

Solving the system of linear equations (5), we find the coefficients $c_j, j = \overline{0,n}$ and get the approximate solution $u_n(x)$. Since the functions $\xi_0, \xi_1, ..., \xi_n \in C^1[a,b]$ are linear independent, it follows that the symmetric bilinear form $a_{ij}(i,j = \overline{0,n})$ is positive definite. This, in turn, means that the main matrix A of the system of linear equations is positive. Therefore, the solution of the system of linear equations (5) exists and is unique. If a basis functions are conveniently chosen, then the accuracy of the approximation method improves as n increases. More detailed information about this theory of finite element methods can be found, for example, in [7] and [9].

In the next section, we consider the issue of constructing an optimal interpolation formula in a Hilbert space. In particular, we present analytical expressions of the coefficients for the optimal interpolation formula constructed in the Hilbert space $K_{2,\omega}^{(2)}$.

3 The Optimal interpolation formula in the Hilbert space

First, let's focus on the issue of construction of an optimal interpolation formula. The problem of constructing an optimal interpolation formula was first posed and studied by S.L. Sobolev in the space $W_2^{(m)}$ (see [12]). The problem of construction of optimal interpolation formulas in different Hilbert spaces was considered in the works [13]- [16].

Let the values $\varphi(x_0), \varphi(x_1), ..., \varphi(x_N)$ of the function $\varphi(x)$ at the points $x_0, x_1, ..., x_N$ of the mesh $0 \le x_0 < x_1 < ... < x_N \le 1$ be given. Here we consider the problem of approximating the function $\varphi(x)$ in a certain Hilbert space H as follows:

$$\varphi(x) \cong P_\varphi(x) \text{ for } x \in [0,1], \tag{6}$$

where

$$P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta(x)\,\varphi(x_\beta)$$

and it is the approximating function, $C_\beta(x), \beta = \overline{0,N}$ are its coefficients.

If the approximate equality (6) satisfies the conditions

$$\varphi(x_\beta) = P_\varphi(x_\beta), \beta = \overline{0,N}$$

then the function $P_\varphi(x)$ is called the interpolation function.

The difference

$$(\ell, \varphi) = \varphi(z) - P_\varphi(z)$$

at the fixed point $x = z$ ($z \in [0,1]$) is called the error of the approximating formula (6) at the point z. Here ℓ is the error functional of the interpolation formula (6), which is defined as follows

$$\ell(x,z) = \delta(x-z) - \sum_{\beta=0}^{N} C_\beta(z)\delta(x-x_\beta), \tag{7}$$

where $\delta(x)$ is the Dirac delta-function.

One of the main problems of the approximation theory is to obtain an upper estimate of the error for the interpolation formula. According to the Cauchy-Schwarz inequality

$$|(\ell,\varphi)| \le \|\ell\|_{H^*}\|\varphi\|_H$$

the error of the approximating formula (6) is estimated using the norm of the error functional ℓ in the conjugate space H^*.

In addition, the error functional (7) depends on the coefficients $C_\beta(z)$ of the approximation formula (6). If

$$\left\|\overset{\circ}{\ell}\right\|_{H^*} := \inf_{C_\beta(z)} \|\ell\|_{H^*}$$

the least value is achieve at some $C_\beta(z) = \overset{\circ}{C}_\beta(z)$ then the corresponding formula is called the optimal approximation formula. The coefficients of the optimal approximation formula are called optimal coefficients.

We suppose that functions $\varphi(x)$ belong to the following Hilbert space

$$K_{2,\omega}^{(2)} = \left\{ \varphi : [0,1] \to \mathbb{R} | \varphi' \text{ is absolutely continuous and } \varphi'' \in L_2(0,1) \right\},$$

equipped with the norm

$$\|\varphi\|_{K_{2,\omega}^{(2)}} = \left(\int_0^1 \left(\varphi''(x) + \omega^2\varphi(x)\right)^2 dx \right)^{\frac{1}{2}},$$

where $\omega \in \mathbb{R}\setminus\{0\}$ (see, for instance, [11]).

The optimal interpolation formula of the form (6) in the Hilbert space $K_{2,\omega}^{(2)}$ was constructed in the work [27]

The following rusult was obtained.

Theorem 1. *In the Hilbert space $K_{2,\omega}^{(2)}$ the coefficients of the optimal interpolation formula*

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta(x)\varphi(h\beta)$$

are represented as follows

$$C_0(x) = p\left(\frac{A_1}{\lambda_1} \sum_{\gamma=0}^{N} \lambda_1^\gamma G_2(x-h\gamma) + CG_2(x) + G_2(x-h) \right)$$
$$+ p\left(-\sin(\omega h)\, d_1^- + \cos(\omega h)\, d_2^- - \frac{h}{4\omega^2}\cos(\omega h + \omega x) \right) + pA_1\left(M_1 + \lambda_1^N N_1\right), \quad (8)$$

$$C_\beta(x) = p\left(\frac{A_1}{\lambda_1} \sum_{\gamma=0}^{N} \lambda_1^{|\beta-\gamma|} G_2(x-h\gamma) + G_2(x-h(\beta-1)) + CG_2(x-h\beta) \right.$$
$$\left. + G_2(x-h(\beta+1)) \right) + pA_1\left(\lambda_1^\beta M_1 + \lambda_1^{N-\beta} N_1\right), \quad \beta = 1,2,...,N-1, \quad (9)$$

$$C_N(x) = p\left(\frac{A_1}{\lambda_1}\sum_{\gamma=0}^{N}\lambda_1^{N-\gamma}G_2(x-h\gamma) + CG_2(x-1) + G_2(x-1+h)\right)$$
$$+ p\left(\sin(\omega h+\omega)\,d_1^{+} + \cos(\omega h+\omega)\,d_2^{+} - \frac{1+h}{4\omega^2}\cos(\omega h+\omega-\omega x)\right)$$
$$+ pA_1\left(\lambda_1^N M_1 + N_1\right), \quad (10)$$

where

$$d_1^{-} = \frac{k_2t_1 - k_1t_2}{a_1k_2 - a_2k_1},\ d_1^{+} = \frac{a_1t_2 - a_2t_1}{a_1k_2 - a_2k_1},\ d_2^{-} = G_2(x),\ d_2^{+} = F - \tan(\omega)d_1^{+},$$

here

$$F = \frac{G_2(x-1)}{\cos(\omega)} + \frac{\cos(\omega-\omega x)}{4\omega^2\cos(\omega)},\quad G_2(x) = \frac{\mathrm{sign}(x)}{4\omega^3}\Big(\sin(\omega x) - \omega x\cos(\omega x)\Big),$$

$$p = \frac{2\omega^3}{\sin(\omega h) - \omega h\cos(\omega h)},\quad A_1 = \frac{(2\omega h)^2\sin^4(\omega h)\lambda_1^2}{(\sin(\omega h) - \omega h\cos(\omega h))^2(\lambda_1^2-1)},$$

$$\lambda_1 = \frac{2\omega h - \sin(2\omega h) - 2\sin(\omega h)\sqrt{(\omega h)^2 - \sin^2(\omega h)}}{2(\omega h\cos(\omega h) - \sin(\omega h))},\ C = \frac{2\omega h\cos(2\omega h) - \sin(2\omega h)}{\sin(\omega h) - \omega h\cos(\omega h)},$$

$$a_1 = -C\sin(\omega h) - \sin(2\omega h) - \frac{A_1\sin(\omega h)}{\lambda_1(1+\lambda_1^2 - 2\lambda_1\cos(\omega h))},\ a_2 = -\frac{A_1\lambda_1^{N+1}\sin(\omega h)}{1+\lambda_1^2 - 2\lambda_1\cos(\omega h)},$$

$$k_1 = \frac{A_1\lambda_1^{N+1}\sin(\omega h)}{\cos(\omega)(1+\lambda_1^2 - 2\lambda_1\cos(\omega h))},$$
$$k_2 = \frac{C\sin(\omega h) + \sin(2\omega h)}{\cos(\omega)} + \frac{A_1\sin(\omega h)}{\lambda_1\cos(\omega)(1+\lambda_1^2 - 2\lambda_1\cos(\omega h))},$$

$$t_1 = \frac{h}{4\omega^2}\Bigg(C\cos(\omega x+\omega h) + 2\cos(\omega x + 2\omega h)$$
$$+ \frac{A_1}{\lambda_1}\frac{\cos(\omega x+\omega h) - 2\lambda_1\cos(\omega x) + \lambda_1^2\cos(\omega x - \omega h)}{(1+\lambda_1^2 - 2\lambda_1\cos(\omega h))^2}\Bigg)$$
$$- G_2(x)\left(1 + C\cos(\omega h) + \cos(2\omega h) + \frac{A_1}{\lambda_1}\frac{\cos(\omega h) - \lambda_1}{1+\lambda_1^2 - 2\lambda_1\cos(\omega h)}\right)$$
$$- A_1\sum_{\gamma=0}^{N}\lambda_1^{\gamma}G_2(x-h\gamma) - A_1\lambda_1^{N+1}F\frac{\cos(\omega h+\omega) - \lambda_1\cos(\omega)}{1+\lambda_1^2 - 2\lambda_1\cos(\omega h)} + \frac{h}{4\omega^2}A_1\lambda_1^{N+1}Q,$$

$$t_2 = \frac{h}{4\omega^2}A_1\lambda_1^{N+1}\frac{\cos(\omega x+\omega h)-2\lambda_1\cos(\omega x)+\lambda_1^2\cos(\omega x-\omega h)}{(1+\lambda_1^2-2\lambda_1\cos(\omega h))^2}$$
$$-A_1\lambda_1^{N+1}G_2(x)\frac{\cos(\omega h)-\lambda_1}{1+\lambda_1^2-2\lambda_1\cos(\omega h)}-G_2(x-1)-A_1\lambda_1^N\sum_{\gamma=0}^{N}\lambda_1^{-\gamma}G_2(x-h\gamma)$$
$$-F\left(C\cos(\omega h+\omega)+\cos(2\omega h+\omega)+\frac{A_1}{\lambda_1}\frac{\cos(\omega h+\omega)-\lambda_1\cos(\omega)}{1+\lambda_1^2-2\lambda_1\cos(\omega h)}\right)$$
$$+\frac{h}{4\omega^2}\left(C(1+N)\cos(\omega h+\omega-\omega x)+(2+N)\cos(2\omega h+\omega-\omega x)+\frac{A_1}{\lambda_1}Q\right),$$

$$Q=\frac{\cos(\omega-\omega x+\omega h)-2\lambda_1\cos(\omega-\omega x)+\lambda_1^2\cos(\omega-\omega x-\omega h)}{(1+\lambda_1^2-2\lambda_1\cos(\omega h))^2}$$
$$+\frac{\cos(\omega-\omega x+\omega h)-\lambda_1\cos(\omega-\omega x)}{h(1+\lambda_1^2-2\lambda_1\cos(\omega h))},$$

$$M_1=-\frac{\sin(\omega h)}{1+\lambda_1^2-2\lambda_1\cos(\omega h)}d_1^-+\frac{\cos(\omega h)-\lambda_1}{1+\lambda_1^2-2\lambda_1\cos(\omega h)}d_2^-$$
$$-\frac{h}{4\omega^2}\frac{\cos(\omega x+\omega h)-2\lambda_1\cos(\omega x)+\lambda_1^2\cos(\omega x-\omega h)}{(1+\lambda_1^2-2\lambda_1\cos(\omega h))^2},$$

$$N_1 = \frac{\sin(\omega h+\omega)-\lambda_1\sin(\omega)}{1+\lambda_1^2-2\lambda_1\cos(\omega h)}d_1^+ + \frac{\cos(\omega h+\omega)-\lambda_1\cos(\omega)}{1+\lambda_1^2-2\lambda_1\cos(\omega h)}d_2^+ - \frac{h}{4\omega^2}Q.$$

In the next section we give three sets of basis functions.

4 Basis functions

In this section, we present a set of known hat basis functions, and we also construct basis functions using the coefficients of the optimal interpolation formula presented in Theorem 1. At the same time, we describe the properties of these basis functions, draw their graphs, and provide the necessary information to apply these basis functions to finite element methods.

It is known that the interval $[0,1]$ can be translated by linear transformation into any interval $[a,b]$. To simplify calculations, we consider the interval $[0,1]$ to be the interval $[a,b]$.

4.1 The hat basis functions

It is known that in linear spaces there is always a system of linear independent elements. This system of linear independent elements is considered as the basis of the space. The elements that make up the basis, depending on the linear space, are called basis functions or basis vectors.

Clearly, the hat basis functions corresponding to the partition $0 = z_0 < z_1 < ... < z_n = 1$, $z_i = i\tau, \tau = \frac{1}{n}, i = 0, 1, ..., n$ of the interval $[0, 1]$ have the following form (see, for example, [10], pp. 714-715):

$$\lambda_0(x) = \begin{cases} \frac{x-z_1}{z_0-z_1}, \ z_0 \le x \le z_1, \\ 0, \ z_1 < x \le 1, \end{cases} \tag{11}$$

$$\lambda_i(x) = \begin{cases} 0, \ x < z_{i-1}, \\ \frac{x-z_{i-1}}{z_i-z_{i-1}}, \ z_{i-1} \le x \le z_i, \\ \frac{x-z_{i+1}}{z_i-z_{i+1}}, \ z_i \le x \le z_{i+1}, \\ 0, \ z_{i+1} \le x, \end{cases} \quad (i = 1, 2, ..., n-1), \tag{12}$$

$$\lambda_n(x) = \begin{cases} 0, \ z_0 \le x < z_{n-1}, \\ \frac{x-z_{n-1}}{z_n-z_{n-1}}, \ z_{n-1} \le x \le z_n. \end{cases} \tag{13}$$

The graphs of the hat basis functions $\lambda_i(x)(i = 0, 1, ..., n)$ are shown in figure 1.

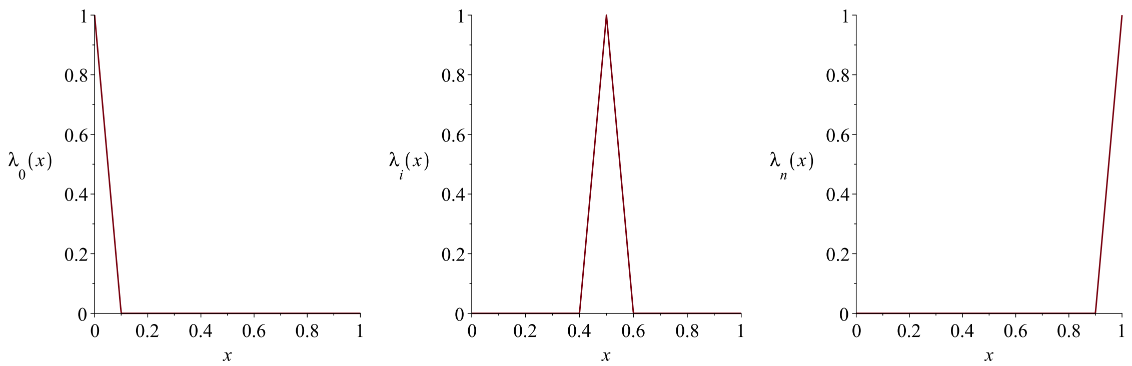

Fig. 1. The graphs of the hat functions $\lambda_0(x)$, $\lambda_i(x), i = 1, 2, ..., n-1$ and $\lambda_n(x)$ (from the left to the right).

Here are the first-order derivatives of the hat basis functions $\lambda_i(x)(i = 1, ..., n-1)$ above is determined by the equation

$$\lambda_i'(x) = \begin{cases} 0, \ x < z_{i-1}, \\ \frac{1}{z_i-z_{i-1}}, \ z_{i-1} \le x \le z_i, \\ \frac{1}{z_i-z_{i+1}}, \ z_i \le x \le z_{i+1}, \\ 0, \ z_{i+1} \le x. \end{cases} \tag{14}$$

It can be seen that the hat basis functions $\lambda_i(x)(i = 0, 1, ..., n)$ are continuous in the interval $[0, 1]$, and its first-order derivatives $\lambda_i'(x)(i = 0, 1, ..., n)$ have a first-order discontinuity in the interval $[0, 1]$.

When we approximately solve the above boundary value problem $(1)-(2)$ using the hat basis functions $\lambda_i(x)(i = 0, 1, ..., n)$, we take approximate solution as follows

$$u_n(x) = \sum_{i=0}^{n} c_i \lambda_i(x). \tag{15}$$

4.2 Construction of basis functions using optimal coefficients

At this stage, based on equations (8)−(10) we present an analytical representation of the coefficients for $N = 1$. Then we construct a set of basis functions using the analytical representation of the coefficients.

For $N = 1(x_0 = 0, x_1 = 1)$, from equations (8)-(10) we have the following:

$$C_0(x) = \frac{\sin(\omega x - \omega x_1)}{\sin(\omega x_0 - \omega x_1)}, \; C_1(x) = \frac{\sin(\omega x - \omega x_0)}{\sin(\omega x_1 - \omega x_0)}, \; x \in [0, 1]. \tag{16}$$

The graphs of the coefficients $C_0(x)$ and $C_1(x)$ are presented in figure 2.

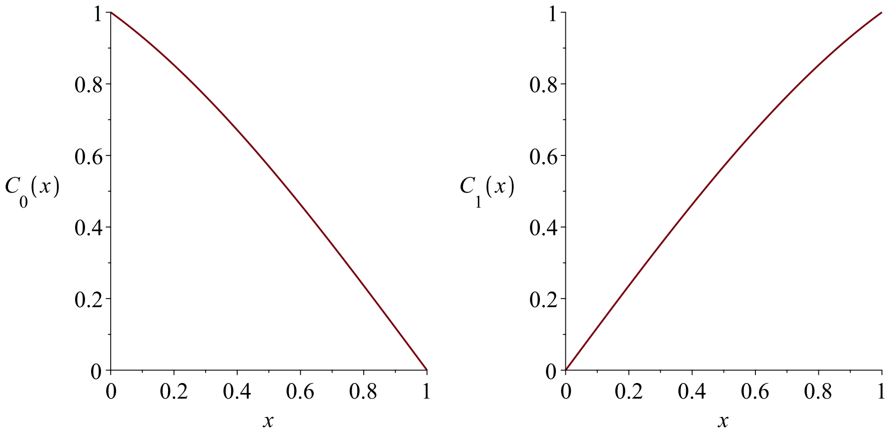

Fig. 2. The graphs of coefficients $C_0(x)$ and $C_1(x)$ defined by (16)(from the left to the right).

Now, using the coefficients $C_0(x)$ and $C_1(x)$ as the shape functions on the interval $[0, 1]$, we construct a set of basis functions $\mu_i(x)$ $(i = 0, 1, ..., n)$. Here we consider the interval $[0, 1]$ to be divided by $0 = z_0 < z_1 < ... < z_n = 1$, where $z_i = i\tau$, $\tau = \frac{1}{n}$, $i = 0, 1, ..., n$.

The first function $\mu_0(x)$ has the following form

$$\mu_0(x) = \begin{cases} \frac{\sin(\omega x - \omega z_1)}{\sin(\omega z_0 - \omega z_1)}, & z_0 \le x \le z_1, \\ 0, & z_1 \le x \le 1, \end{cases} \tag{17}$$

Then using the shape functions (11) for the intervals (z_{i-1}, z_i) and (z_i, z_{i+1}) we describe the functions $\mu_i(x), i = 1, 2, ..., n-1$ as follows

$$\mu_i(x) = \begin{cases} 0, & z_0 \le x \le z_{i-1}, \\ \frac{\sin(\omega x - \omega z_{i-1})}{\sin(\omega z_i - \omega z_{i-1})}, & z_{i-1} \le x \le z_i, \\ \frac{\sin(\omega x - \omega z_{i+1})}{\sin(\omega z_i - \omega z_{i+1})}, & z_i \le x \le z_{i+1}, \\ 0, & z_{i+1} \le x \le 1. \end{cases} \tag{18}$$

Finally, we express $\mu_n(x)$ by the following equality

$$\mu_n(x) = \begin{cases} 0, & z_0 \le x \le z_{n-1}, \\ \frac{\sin(\omega x - \omega z_{n-1})}{\sin(\omega z_n - \omega z_{n-1})}, & z_{n-1} \le x \le z_n. \end{cases} \tag{19}$$

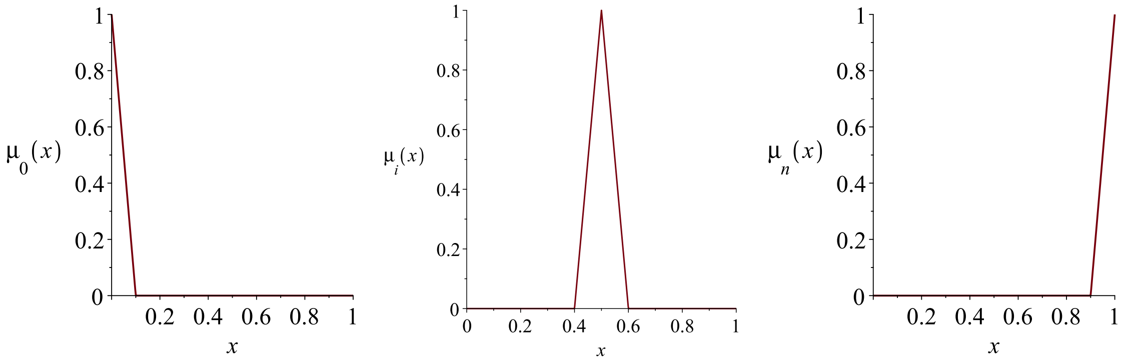

Fig. 3. The graphs of the functions $\mu_0(x)$, $\mu_i(x), i = 1, 2, ..., n-1$ and $\mu_n(x)$ (from the left to the right).

The graphs of the basis functions $\mu_i(x) (i = 0, 1, ..., n)$ are shown in figure 3.

It is easy to check that the functions $\mu_i(x), i = 0, 1, ..., n$ are independent on the interval $[0, 1]$.

The first-order derivatives of the basis functions $\mu_i(x), i = 1, ..., n-1$ are determined by the equation

$$\mu_i'(x) = \begin{cases} 0, \ z_0 \le x \le z_{i-1}, \\ \frac{\omega \cos(\omega x - \omega z_{i-1})}{\sin(\omega z_i - \omega z_{i-1})}, \ z_{i-1} \le x \le z_i, \\ \frac{\omega \cos(\omega x - \omega z_{i+1})}{\sin(\omega z_i - \omega z_{i+1})}, \ z_i \le x \le z_{i+1}, \\ 0, \ z_{i+1} \le x \le 1. \end{cases} \tag{20}$$

It can be seen that the basis functions $\mu_i(x) (i = 0, 1, ..., n)$ are continuous in the interval $[0, 1]$, and its first-order derivatives $\mu_i'(x) (i = 0, 1, ..., n)$ have a first-order discontinuity in the interval $[0, 1]$.

When we approximately solve the above boundary value problem (1)-(2) using the basis functions $\mu_i(x) (i = 0, 1, ..., n)$, the approximate solution we get as follows

$$\vartheta_n(x) = \sum_{i=0}^{n} d_i \mu_i(x). \tag{21}$$

4.3 The optimal coefficients as basis functions

At this stage, using Theorem 1, we get the coefficients of the interpolation formula constructed above as basis functions. From Theorem 1 for $N = n$ we get

$$\nu_i(x) = C_i(x), \ i = 0, 1, ..., n, \tag{22}$$

where $C_0(x), C_i(x), i = \overline{1, n-1}$ and $C_n(x)$ are defined by equations (8), (9) and (10), respectively.

The graphs of the basis functions $\nu_i(x) (i = 0, 1, ..., n)$ are shown in figure 4.

When we approximately solve the above boundary value problem (1)-(2) using the basis functions $\nu_i(x) (i = 0, 1, ..., n)$, we get the approximate solution as follows

$$\omega_n(x) = \sum_{i=0}^{n} e_i \nu_i(x). \tag{23}$$

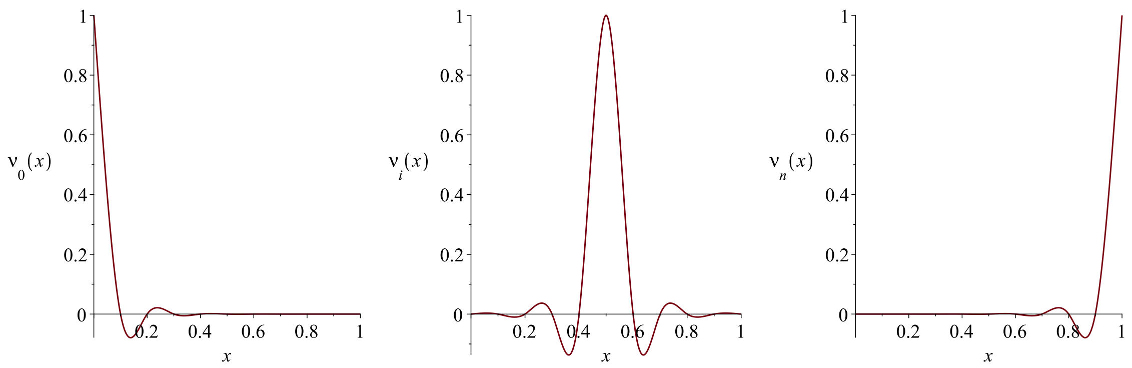

Fig. 4. The graphs of the functions $\nu_0(x)$, $\nu_i(x), i = 1, 2, ..., n-1$ and $\nu_n(x)$ given by formula (22)(from the left to the right).

It should be noted that various geometric curves and surfaces, as well as Bezier curves [17], [18], [19], trigonometric B-splines [18], [20], [21] can be generated using the basis functions $\mu_i(x)(i = 0, 1, ..., n)$ and $\nu_i(x)(i = 0, 1, ..., n)$.

In the next section, we consider the application of these basis functions to the solution of the boundary value problems for second-order ordinary differential equations using finite element methods.

5 Numerical Results

In this section, we approximately solve the boundary value problems for ordinary differential equations of the second order using the Galerkin method, applying the basis functions constructed above.

Example 1. (Example 3 a), 726 p., [10]) Solve the following boundary value problem using the Galerkin method:

$$-x^2u'' - 2xu' + 2u = -4x^2, \ u(0) = u(1) = 0. \tag{24}$$

It is known that this boundary value problem has an exact solution $u(x) = x^2 - x$.

Case 1. In this case, we approximately solve the boundary value problem (24) using basis functions $\lambda_i(x)(i = 0, 1, ..., n)$. Since the boundary conditions in the boundary value problem (24) are homogeneous, the approximate solution has the form

$$u_n(x) = \sum_{i=1}^{n-1} c_i \lambda_i(x).$$

The absolute value of the error for the approximate solution $u_n(x)$ is presented in figure 5.

Case 2. In this case, we approximately solve the boundary value problem (24) using basis functions $\mu_i(x)(i = 0, 1, ..., n)$. Since the boundary conditions in the boundary value problem (24) are homogeneous, the approximate solution has the form

$$\vartheta_n(x) = \sum_{i=1}^{n-1} d_i \mu_i(x).$$

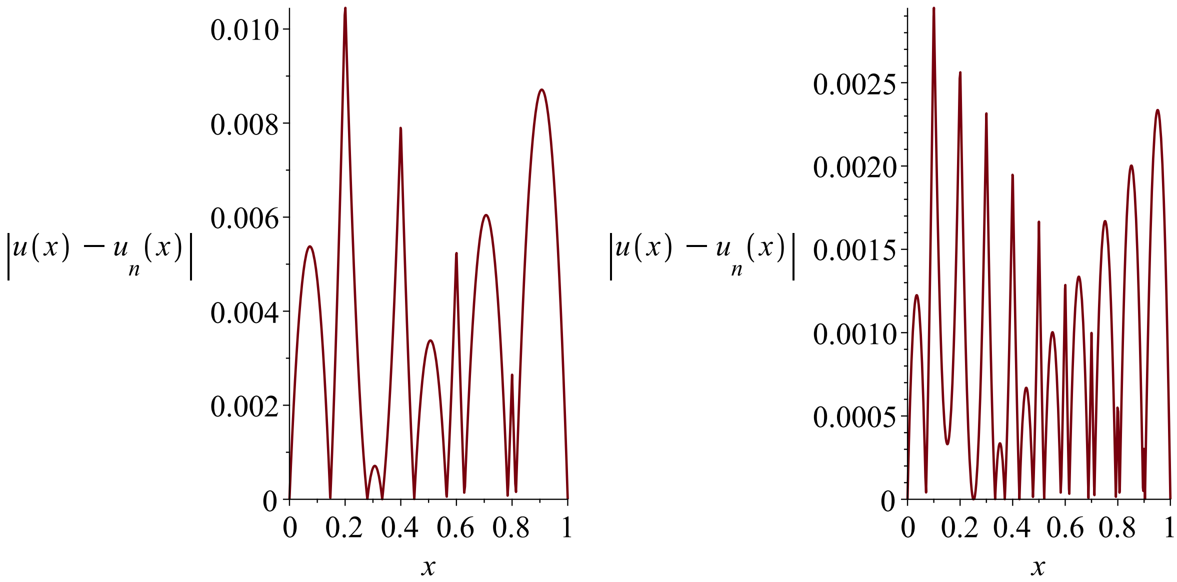

Fig. 5. The graphs of the error $|u(x) - u_n(x)|$ for $n = 5$ (on the left) and $n = 10$ (on the right), respectively (for the boundary value problem (24)).

The absolute value of the error for the approximate solution $\vartheta_n(x)$ is presented in figure 6.

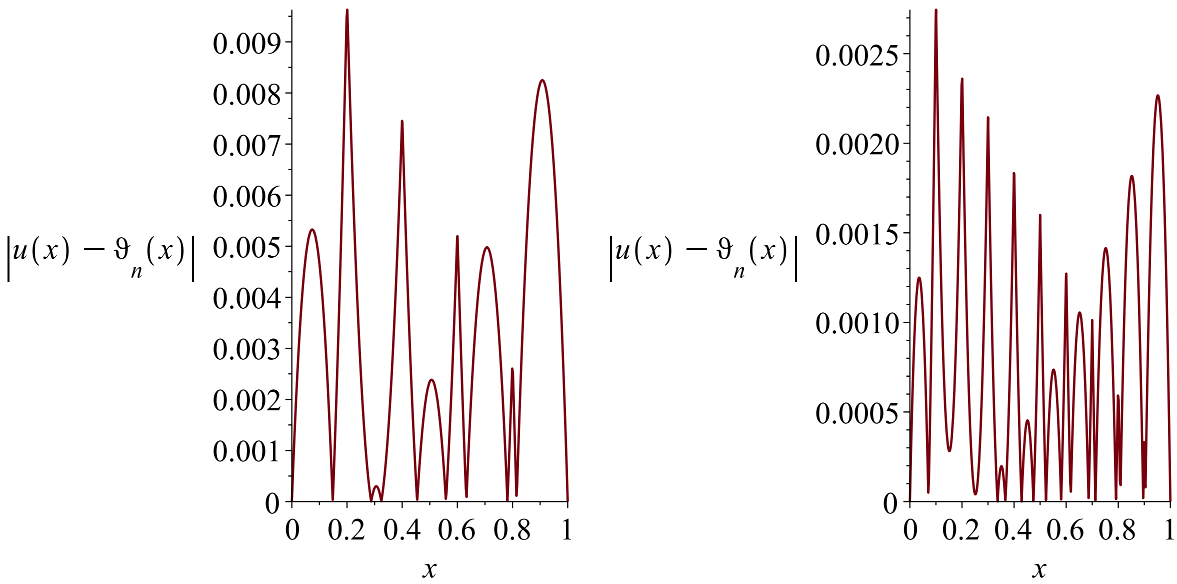

Fig. 6. The graphs of the error $|u(x) - \vartheta_n(x)|$ for $n = 5$ (on the left) and $n = 10$ (on the right), respectively (for the boundary value problem (24)).

Case 3. Finally, we approximately solve the boundary value problem (24) using basis functions $\nu_i(x)(i = 0, 1, ..., n)$. If we take into account that the boundary conditions here are also homogeneous, the approximate solution of the boundary problem has the form

$$\omega_n(x) = \sum_{i=1}^{n-1} e_i \nu_i(x).$$

The absolute value of this approximate solution error $\omega_n(x)$ is presented in figure 7.

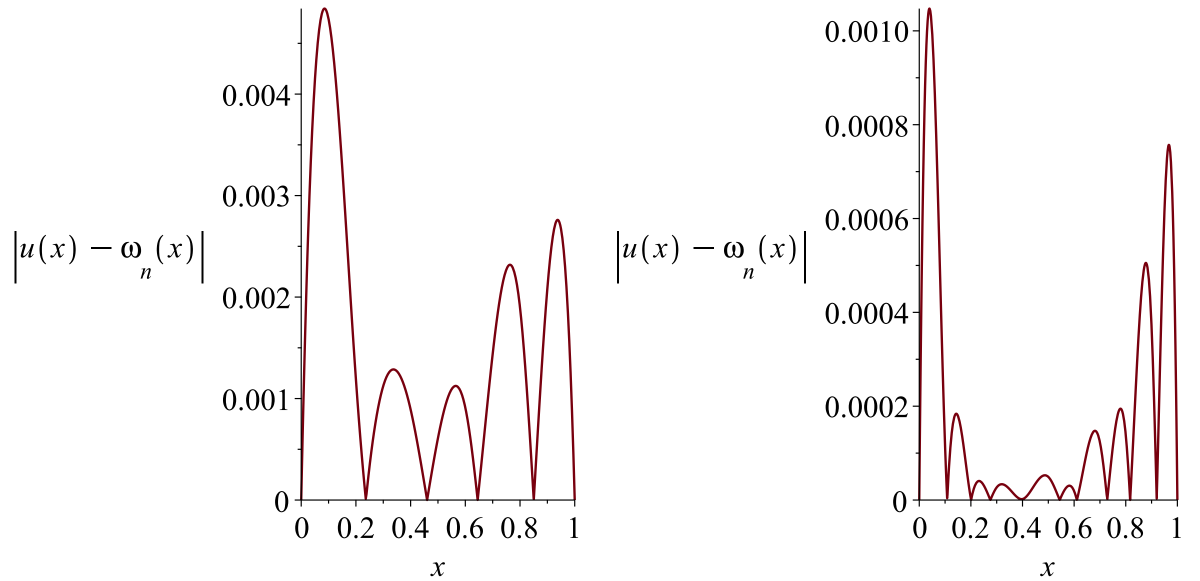

Fig. 7. The graphs of the error $|u(x) - \omega_n(x)|$ for $n = 5$ (on the left) and $n = 10$ (on the right), respectively (for the boundary value problem (24)).

The following conclusions can be done from these numerical results:

i) In the first two cases, the order of approximation of the approximate solution is the same.

ii) Effective basis functions can be formed even when $N = 1$ in equations (8)-(10).

iii) Accuracy of the approximate solution found using basis functions $\nu_i(x)(i = 0, 1, ..., n)$ is better than the accuracy of the approximate solution found using basis functions $\lambda_i(x)$ $(i = 0, 1, ..., n)$ and $\mu_i(x)$ $(i = 0, 1, ..., n)$.

It should be noted that using the above basis function one can approximately calculate definite integrals and construct B$-$ splines as well as they can be applied for construct various geometric curves.

6 Conclusion

This paper briefly reviews the history of the emergence and development of finite element methods. At the same time, the essence of the finite element method was clarified and the objects of study were mentioned. As the main result of the work, we can say that the set of basis functions is constructed from the coefficients of the optimal interpolation formula constructed in the Hilbert space $K^{(2)}_{2,\omega}$ with $N = 1$ and arbitrary $N \in \mathbb{N}$, and these basis functions are applied to boundary value problems of the finite element method for ordinary differential equations of the second order, approximately solved and numerical results obtained. In addition, it was shown that the accuracy of the approximate solution for arbitrary $N \in \mathbb{N}$ is better than the accuracy of the approximate solution for $N = 1$ and it was proven that the order of approximation of the approximate solution found using the basis functions we constructed is the same as the order of approximation of the approximate solution found using the hat basis functions.

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