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Two Types of Waves in a Two-Layer Stratified Fluid

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Abstract. Within the framework of the linear theory of small potential oscillations the article studies the structure and characteristics of internal and surface waves in a two-layer stably stratified fluid. Dispersion relations have been obtained and analyzed. The Boussinesq approximation has been studied. We have determined the existence of two types of waves: a fast wave and a slow wave. The fast wave differs little from a surface wave in a homogeneous fluid. The main results of studying the case of $n = 1$ are as follows: obtaining expressions for fluid particles velocity components and hydrodynamic pressure in the upper layer of the considered two-layer fluid, as well as obtaining relations, which connect the oscillation amplitudes of the free surface and the interface of the considered two-layer fluid. The main results of studying the case of $n = 2$ are as follows: obtaining expressions for fluid particles velocity components and hydrodynamic pressure in the lower layer of the considered two-layer fluid, as well as obtaining dispersion relations in the considered two-layer fluid. In the case of implementing the Boussinesq approximation, the fast wave differs little from a surface wave in a homogeneous fluid, andthe velocity of slow waves is proportional to the square root of the squared ratio $\frac{\Delta\rho}{\rho}$, i.e. it is very small.

Key words: two-layer liquid, Boussinesq approximation, dispersion relation, potential oscillations

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Два типа волн в двухслойной стратифицированной жидкости

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Аннотация. В рамках линейной теории малых потенциальных колебаний изучены строение и характеристики поверхностной и внутренней волн в устойчиво стратифицированной двуслойной жидкости. Получены и проанализированы дисперсионные соотношения. Рассмотрено приближение Буссинеска. Установлено существование двух типов волн: быстрой и медленной. Быстрая волна мало отличается от поверхностной волны в однородной жидкости. Установлено, что в двуслойной жидкости существуют два типа волн: быстрая и медленная. Основными результатами изучения случая $n = 1$ верхнего слоя являются: выражения составляющих скорости частиц жидкости и гидродинамического давления в верхнем слое рассматриваемой двуслойной жидкости, а также соотношения, которое связывает амплитуды колебаний свободной поверхности и границы раздела рассматриваемой двуслойной жидкости. Основными результатами изучения случая $n = 2$ нижнего слоя являются: выражения составляющих скорости частиц жидкости и гидродинамического давления в нижнем слое рассматриваемой двуслойной жидкости, а также дисперсионные соотношения в рассматриваемой двуслойной жидкости.

Ключевые слова: двухслойная жидкость, приближение Буссинеска, дисперсионное соотношение, потенциальные колебания

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Финансирование. Работа выполнялась без поддержки фондов.

Конкурирующие интересы. Конфликтов интересов в отношении авторства и публикации нет.

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Introduction

Within the framework of the linear theory of small potential oscillations, the article studies the structure and characteristics of internal and surface waves in a two-layer stably stratified fluid. The fluid is considered to be ideal and incompressible. Dispersion relations have been obtained and analyzed. The Boussinesq approximation has been studied. We have determined the existence of two types of waves: a fast wave and a slow wave. The fast wave differs little from a surface wave in a homogeneous fluid.

Our main results of studying the waves in a two-layer fluid are as follows.

It has been determined that in a two-layer fluid there are two types of plane harmonic waves: fast ones and slow ones.

If the Boussinesq approximation is used, fast waves are not so different from waves in a homogeneous fluid. The velocity of slow waves is proportional to the square root of the squared ratio $\frac{\Delta p}{p}$, i.e. it is very small.

Both types of waves arise from external influences imposed upon a two-layer fluid.

Article [1] - [3] studies such kinds of influences in detail.

For the reader's convenience, and, more importantly, to compare the results of solving the problems of waves in a stratified fluid, the solution to the classical problem of the structure and characteristics of a plane harmonic wave on the surface of a homogeneous fluid is given below.

Our method of solving the problem of a surface wave in a homogeneous fluid differs from the traditional method, which is outlined, in particular, in monograph [4] - [7], thus, presenting some features of interest.

A similar method was used to solve the problems of the structure and characteristics of surface and internal waves in a two-layer fluid.

Solving the problem of the structure and characteristics of a plane harmonic surface wave on the surface of a homogeneous fluid

Let us describe the notation we use before formulating the mathematical statement of the problem of the structure and characteristics of a plane harmonic surface wave on the surface of a homogeneous fluid.

Symbols t, x and y denote time, horizontal and vertical coordinates, considering that the y-axis is directed vertically upwards and the level of the undisturbed upper surface of the fluid coincides with the horizontal axis, that is, with the line $y = 0$.

Symbols ρ, ν, u and p stand for density, horizontal and vertical particle velocity components and hydrodynamic pressure. Given that the fluid is homogeneous, then $\rho = const$. Symbol η denotes the wave profile (shape of the free surface).

Let us now formulate a mathematical statement of the problem of the structure and characteristics of a plane harmonic surface wave on the surface of a homogeneous fluid.

Statement of the problem of the structure and characteristics of a plane harmonic surface wave on the surface of a homogeneous fluid. Obtain the solution to a system of linear equations of an ideal incompressible homogeneous fluid under an assumption that these movements are potential (i.e. vortex-free) [4], [8] - [15].

$$\rho u_t + p_x = 0, \quad \rho v_t + p_y + g\rho = 0, \quad u_x + v_y = 0, \quad u_y - v_x = 0. \tag{1}$$

which satisfies the kinematic and dynamic conditions on the free surface, as well as the impermeability condition at the bottom.

The first two equations of system (1) describe the fluid dynamics. The third equation is the consequence of the continuity equation and the fluid incompressibility condition. Finally, the fourth equation of system (1) expresses the condition of the potentiality of fluid's movement.

The condition of impermeability at the bottom is natural and its mathematical form is expressed with the following simple equation:

$$v = 0 \quad \text{for} \quad y = -h. \tag{2}$$

The conditions on the free surface are more complicated.

The kinematic condition expresses the following requirement: a fluid particle that was on the free surface before the start of the motion should remain there in the process of the motion.

For nonlinear waves the kinematic condition is of the form:

$$\eta_t - v = 0 \quad \text{for} \quad y = \eta(x). \tag{3}$$

By linearizing it, we get the following mathematical formula of the kinematic condition on the free surface:

$$\eta_t - v = 0 \quad \text{for} \quad y = 0. \tag{4}$$

The dynamic condition means that the atmospheric pressure can be neglected.

The mathematical form of the dynamic condition for nonlinear waves is as follows:

$$p = 0 \quad \text{for} \quad y = \eta(x). \tag{5}$$

Linearizing condition (5), we get the following simplified dynamic condition:

$$p(0) + p_y'(0)\eta(x) = 0. \tag{6}$$

Here, for brevity, the values of quantities p and p_y' for $y = 0$ are denoted by the symbols $p(0)$ and $p_y'(0)$. Condition (6) has uncertainty. It will be eliminated later, when an explicit expression for pressure is obtained.

Let us move on to the solution procedure of the stated problem. Let us begin with studying the hydrostatic state.

In the case of hydrostatics, velocity of fluid particles is equal to zero, $u = 0$, $v = 0$, so system of equations (1) gets greatly simplified and turns into a single hydrostatic equation:

$$p_y' + g\rho = 0. \tag{7}$$

When the fluid is at rest, the dynamic condition is of the form:

$$p = 0 \quad \text{for} \quad y = 0. \tag{8}$$

By integrating equation (7) taking into account conditions (8), we obtain

$$p = -g\rho y. \tag{9}$$

Expression (5) means that with increasing depth, the pressure increases linearly

Now we can start solving the problem of the structure and characteristics of a plane harmonic surface wave on the surface of a homogeneous fluid. Let us start solving the problem by specifying the initial expressions for the characteristics of a harmonic wave. The pressure expression takes into account that it is the sum of the hydrostatic and dynamic components.

Initial expressions for the characteristics of a harmonic wave.

$$\eta = a\exp(i\theta), u = u_0(y)\exp(i\theta), v = v_0(y)\exp(i\theta),$$

$$p = -g\rho y + p_0(y)\exp(i\theta), \theta = \omega t - kx. \tag{10}$$

In expressions (10) a is the amplitude of the wave profile, θ is the wave phase, k is the wave number, ω is the cyclic frequency.

It should be noted that the cyclic frequency and the wave number of a harmonic wave are connected by the equation:

$$\omega = ck, \tag{11}$$

where c - wave velocity.

System of equations and boundary conditions for functions $u_0(y), v_0(y), p_0(y)$ and wave profile amplitude. After substituting the last three expressions from expressions (6) into system of equations (1) and doing trivial simplifications, we obtain a system of four ordinary differential equations

$$\omega\rho u_0(y) - kp_0(y) = 0, i\omega\rho v_o(y) + p_0'(y) = 0,$$

$$-iku_0(y) + {v_0}'(y) = 0, u_0'(y) + ikv_0(y) = 0. \tag{12}$$

A similar substitution of expressions (12) into the linearized forms of the kinematic and dynamic boundary conditions (4) and (6) and into the impermeability condition at the bottom (2), neglecting the small quadratic term in the dynamic condition, and doing trivial simplifications, leads to equations

$$v_0(0) = i\omega a, v_0(-h) = 0, \tag{13}$$

$$p_0(0) - g\rho a = 0. \tag{14}$$

Eliminating the function $u_0(y)$ from the last two equations of system (12), we obtain an ordinary linear differential equation for function $v_0(y)$, which has the form:

$$v_0''(y) - k^2 v_0(y) = 0. \tag{15}$$

The solution of differential equation (15), which satisfies boundary conditions (13), is of the form

$$v_0(y) = i\omega a \frac{\mathrm{sh}(k(y+h))}{\mathrm{sh}(kh)}. \tag{16}$$

The consequence of expression (16) and the third equation of system (12) is the following expression for function $u_0(y)$:

$$u_0(y) = \omega a \frac{\mathrm{ch}(h(y+h))}{\mathrm{sh}(kh)}. \tag{17}$$

Using the first equation of system (12) and expression (17) of function $u_0(y)$, taking into account relation (), we determine the expression of function $p_0(y)$. It has the form:

$$p_0(y) = \rho c^2 k a \frac{\mathrm{ch}(k(y+h))}{\mathrm{sh}(kh)}. \tag{18}$$

To finish the solution procedure of the stated problem we need to study condition (14).

Assuming $y = 0$ in expression (18) and using condition (14) we obtain the dispersion relation for waves on the surface of a homogeneous fluid.

$$c = \sqrt{g \frac{\mathrm{th}(kh)}{k}}. \tag{19}$$

For a fluid of infinite depth, the dispersion relation has the form $c = \sqrt{\frac{g}{k}}$.

For long waves in shallow water, the dispersion relation is as follows: $c = \sqrt{gh}$.

Evidently, long waves in shallow water have no dispersion: the velocity of long waves in shallow water doesn't depend on the wave number.

The problem of the structure and characteristics of a plane harmonic surface wave on the surface of a homogeneous fluid has been completely solved. Let us move to the solution procedure of the problem of the surface and internal waves in a two-layer fluid.

Solving the problem of the structure and characteristics of surface and internal waves in a two-layer fluid

The total depth of a two-layer fluid is equal to h. Hence, the line $y = -h$ is the bottom.

The thickness of the upper layer of a two-layer fluid is equal to d. Then, the average level of the interface is the line $y = -d$.

Density ρ, horizontal and vertical velocity components of a fluid particle u and v, and hydrodynamic pressure p are subscripted with indices 1 and 2, where index 1 refers to the upper layer, and index 2 refers to the lower layer. Since both layers are homogeneous fluids, then $\rho_1 = \mathrm{const}$, $\rho_2 = \mathrm{const}$.

Stratification stability condition is expressed with the inequality $\rho_1 < \rho_2$.

The free surface profile is denoted by symbol η. The interface profile is denoted by symbol η_1.

Let us now formulate a mathematical statement of the problem of the structure and characteristics of plane harmonic surface and internal waves in a two-layer fluid.

Statement of the problem of the structure and characteristics of plane harmonic surface and internal waves in a two-layer fluid. Obtain the solution to two systems of linear equations of an ideal incompressible homogeneous fluid under an assumption that these movements are potential

$$\rho_n u_{n,t} + p_{n,x} = 0, \rho_n v_{n,t} + p_{n,y} + g\rho_n = 0, u_{n,x} + v_{n,y} = 0, u_{n,y} - v_{n,x} = 0, n = 1,2, \quad (20)$$

which satisfies the kinematic and dynamic conditions on the free surface and the interface surface, as well as the impermeability condition at the bottom.

The impermeability condition at the bottom is the same, it has the following form:

$$v_2 = 0 \quad \text{for} \quad y = -h. \quad (21)$$

Let us move on to studying the kinematic and dynamic boundary conditions on the free surface and on the interface.

The kinematic condition on the free surface expresses the former requirement: a fluid particle that was on the free surface before the start of the motion should remain there in the process of the motion.

The mathematical formula of this requirement after linearization takes the form

$$\eta_t - v_1 = 0 \quad \text{for} \quad y = 0. \quad (22)$$

The kinematic condition on the interface has similar content. It is expressed by the requirement, such that a fluid particle which was on the interface before the start of the motion should remain there in the process of the motion.

This requirement leads to two mathematical formulas that take the following forms after linearization

$$\eta_{1,t} - v_1 = 0 \quad \text{for} \quad y = -d. \quad (23)$$

$$\eta_{1,t} - v_2 = 0 \quad \text{for} \quad y = -d. \quad (24)$$

The dynamic condition on the free surface expresses the requirement that the atmospheric pressure can be neglected.

By linearizing the dynamic condition on the free surface, we get the following relation:

$$p_1(0) + p'_{1,y}(0)\eta(x) = 0. \quad (25)$$

Let us denote the values of p_1 and $p'_{1,y}$ for $y = 0$ by symbols $p_1(0)$ and $p'_{1,y}(0)$ for brevity.

The dynamic condition on the interface arises from the requirement of pressure continuity.

Linearization of the dynamic condition at the interface of the two-layer fluid leads to the relation

$$p_1(-d) + p'_{1,y}(-d)\eta_1(x) = p_2(-d) + p'_{2,y}(-d)\eta_1(x). \quad (26)$$

Here, for brevity, the values of p_1, $p'_{1,y}$, p_2 and $p'_{2,y}$ for $y=-d$ are denoted by symbols $p_1(-d)$, $p'_{1,y}(-d)$, $p_2(-d)$ and $p'_{2,y}(-d)$.

Conditions (25) and (26) have uncertainty, the cause of which are unknown values of $p_1(-d)$, $p'_{1,y}(-d)$, $p_2(-d)$ and $p'_{2,y}(-d)$. It will be eliminated later, when explicit expressions for pressure in the upper and lower layers are obtained.

Let us move on to the solution procedure of the stated problem. Let us begin with studying the hydrostatic state.

In the case of hydrostatics, the velocity of fluid particles is equal to zero, $u_n=0$, $v_n=0$, $n=1,2$, hence, the system of equations (20) simplifies greatly and transforms into hydrostatics equations:

$$p'_{n,y}+g\rho_n=0, n=1,2. \tag{27}$$

When the fluid is at rest, the dynamic condition on the free surface is of the form:

$$p_1=0 \quad \text{for} \quad y=-d. \tag{28}$$

In the same case, the dynamic condition on the interface has the form:

$$p_1=p_2 \quad \text{for} \quad y=-d. \tag{29}$$

By integrating equation (27) taking into account conditions (28) and (29), we obtain the following distribution across the layers of hydrostatic pressure:

$$p_1=-g\rho_1 y, -d<y<0, \tag{30}$$

$$p_2=-g\rho_2 y+g(\rho_1-\rho_2)d, -h<y<d. \tag{31}$$

Now we can start solving the problem of the structure and characteristics of plane harmonic surface and internal waves in a two-layer fluid.

Solving the problem of the structure and characteristics of plane harmonic surface and internal waves in a two-layer fluid. Let us start solving the problem of the structure and characteristics of a plane harmonic surface wave in a two-layer fluid by specifying the initial expressions for the characteristics of plane harmonic surface and internal waves in a two-layer fluid. The pressure expression takes into account that it is the sum of the hydrostatic and dynamic components.

The initial expressions for the characteristics of plane harmonic surface and internal waves in a two-layer fluid:

$$\begin{gathered}\eta=a\exp(i\theta), \eta_1=a_1\exp(i\theta), u_n=u_{n0}(y)\exp(i\theta),\\ v_n=v_{n0}(y)\exp(i\theta), n=1,2, p_1=-g\rho_1 y+p_{10}(y)\exp(i\theta), -d<y<0,\\ p_2=-g\rho_2 y+g(\rho_1-\rho_2)d+p_{20}(y)\exp(i\theta), -h<y<-d.\end{gathered} \tag{32}$$

In expressions (32) quantity a is the amplitude of the wave profile on the free surface, quantity a_1 denotes the amplitude of the wave profile on the interface, quantities θ, k and ω retain their former meanings of the wave phase, wave number and cyclic frequency.

Let us remind that the cyclic frequency and wave number of a harmonic wave are connected by relation (11) (see introduction).

System of equations and boundary conditions for functions $u_{n0}(y)$, $v_{n0}(y)$, $p_{n0}(y)$, $n = 1,2$, and wave profile amplitudes on the free surface and on the interface. After substituting the last four expressions from expressions (32) (expressions of the velocity components of fluid particles u_n and v_n and pressure expression $p_n, n = 1,2$) into the system of equations (20) and doing trivial simplifications we obtain a pair of systems, each of which contains four ordinary differential equations. They are as follows:

$$\omega\rho_n u_{n0}(y) - kp_{n0}(y) = 0, i\omega\rho_n v_{n0}(y) + p'_{n0}(y) = 0,$$

$$-iku_{n0}(y) + v'_{n0}(y) = 0, u'_{n0}(y) + ikv_{n0}(y) = 0, n = 1,2. \quad (33)$$

A similar substitution of expressions (32) into the linearized forms of the kinematic and dynamic boundary conditions on the free surface and on the interface (22)-(26) and into the impermeability condition at the bottom (21), neglecting the small quadratic term in the dynamic condition, and doing trivial simplifications, leads to six boundary conditions for functions $v_{n0}(y)$ and $p_{n0}(y)$:

$$v_{10}(0) = i\omega a, \quad (34)$$

$$v_{10}(-d) = i\omega a_1, \quad (35)$$

$$v_{20}(-d) = i\omega a_1, \quad (36)$$

$$v_{20}(-h) = 0, \quad (37)$$

$$p_{10}(0) - g\rho_1 a = 0, \quad (38)$$

$$p_{10}(-d) - p_{20}(-d) + g(\rho_1 - \rho_2)a_1 = 0. \quad (39)$$

Let us get the solution to the system of ordinary differential equations (33) which satisfy boundary conditions (34)-(39), consecutively for the cases of $n = 1$ and $n = 2$.

In the case of $n = 1$ the system of differential equations (33) takes the following form:

$$\omega\rho_1 u_{10}(y) - kp_{10}(y) = 0, i\omega\rho_1 v_{10}(y) + p'_{10}(y) = 0,$$

$$-iku_{10}(y) + v'_{10}(y) = 0, u'_{10}(y) + ikv_{10}(y) = 0. \quad (40)$$

The goal is to obtain a solution to the system of equations (40), which satisfies boundary conditions (34), (35) and (40).

Naturally, the solution procedure of the stated problem should start with finding an expression for function $v_{10}(y)$.

To do this, first of all, an ordinary differential equation for function $v_{10}(y)$ should be obtained by eliminating function $u_{10}(y)$ from the last two equations of the system of ordinary differential equations (40).

Following this simple procedure, we obtain an ordinary linear differential equation

$$v''_{10} - k^2 v_{10} = 0. \quad (41)$$

Solution of differential equation (41) satisfying conditions (34) and (35) is of the form:

$$\nu_{10}(y) = i\omega((ch(kd)a - a_1)\frac{sh(ky)}{sh(kd)} + ach(ky)), -d < y < 0. \quad (42)$$

Let us now move on to carrying out a procedure for determining the expressions of functions $u_{10}(y)$ and $p_{10}(y)$.

The expression of function $u_{10}(y)$ can be found from the third equation of the system of differential equations (40) by differentiating expression (42) of function $\nu_{10}(y)$.

The result is the following expression of the function $u_{10}(y)$:

$$u_{10}(y) = \omega(ash(ky) + (ch(kd)a - a_1)\frac{ch(ky)}{sh(kd)}), -d < y < 0. \quad (43)$$

Next, a procedure for determining the expression of function $p_{10}(y)$ is carried out.

By using the first equation of the system of equations (40), expression (43) of function $u_{10}(y)$ and considering relation (11), we can obtain an expression of function $p_{10}(y)$. It is as follows:

$$p_{10}(y) = \rho_1 c\omega(ash(ky) + (ch(kd)a - a_1)\frac{ch(ky)}{sh(kd)}), -d < y < 0. \quad (44)$$

Letting $y = 0$ in expression (44) of function $p_{10}(y)$, using condition (38) and relation (11) (see introduction) and doing trivial simplifications, we obtain the following relation for the oscillation amplitudes of the upper boundary and the interface of the considered two-layer fluid:

$$c^2ka_1 = (c^2kch(kd) - gsh(kd))a \quad (45)$$

It should be noted that the oscillation amplitude of the interface is proportional to the oscillation amplitude of the free surface.

For the completeness of results, we provide the expressions of fluid particle velocity components and hydrodynamic pressure in the upper layer of the considered two-layer fluid. They are the consequences of the initial expressions of these values (32) and expressions (42), (43) and (44) of the functions $u_{10}(y)$, $\nu_{10}(y)$ and $p_{10}(y)$.

Expressions of the velocity components of fluid particles and hydrodynamic pressure in the upper layer of a two-layer fluid

$$u_1 = \omega(ash(ky) + (ch(kd)a - a_1)\frac{ch(ky)}{sh(kd)})exp(i\theta),$$

$$\nu_1 = i\omega((ch(kd)a - a_1)\frac{sh(ky)}{sh(kd)} + ach(ky))exp(i\theta),$$

$$p_1 = -g\rho_1 y + \rho_1 c\omega(ash(ky) + (ch(kd)a - a_1)\frac{ch(ky)}{sh(kd)})exp(i\theta), -d < y < 0.$$

Studying the case of $n = 1$ has been completed.

The main results of studying the case of $n = 1$ are as follows: obtaining expressions for fluid particles velocity components and hydrodynamic pressure in the upper layer of the considered two-layer fluid, as well as obtaining relations, which connect the oscillation amplitudes of the free surface and the interface of the considered two-layer fluid.

In the case of $n=2$ the system of ordinary differential equations (33) takes the following form:

$$\omega\rho_2 u_{20}(y)-kp_{20}(y)=0, i\omega\rho_2 v_{20}(y)+p_{20}^{'}(y)=0,$$
$$-iku_{20}(y)+v_{20}^{'}(y)=0, u_{20}^{'}(y)+ikv_{20}(y)=0. \tag{46}$$

The goal is to get a solution to the system of equations (46), which satisfies boundary conditions (36), (37) and (39).

Again, naturally, the solution procedure of the stated problem should start with finding the expression for function $v_{20}(y)$.

The differential equation for function $v_{20}(y)$ can be obtained in a standard way. It has the following form:

$$v_{20}^{''}-k^2 v_{20}=0. \tag{47}$$

The solution to the differential equation (47) which satisfies the condition of impermeability at the bottom (37), is as follows:

$$v_{20}(y)=Ash(k(y+h)), -h<y<-d. \tag{48}$$

The value of the coefficient A is obtained by substituting the expression (48) into boundary condition (36). This substitution produces the following equation:

$$A=i\omega\frac{1}{sh(k(h-d))}a_1.$$

This means that the solution of the differential equation (47), which satisfies boundary conditions (36) and (37), has the following form:

$$v_{20}(y)=i\omega\frac{sh(k(y+h))}{sh(k(h-d))}a_1, -h<y<-d. \tag{49}$$

Let us move on to determining the expressions of functions $u_{20}(y)$ and $p_{20}(y)$.

The expression of function $u_{20}(y)$ can be found from the third equation of the system of differential equations (46) by differentiating expression (49). The result is the following expression for function $u_{20}(y)$:

$$u_{20}(y)=\omega\frac{ch(k(y+h))}{sh(k(h-d))}a_1, -h<y<-d. \tag{50}$$

Then a procedure for finding the expression of function $p_{20}(y)$ is carried out.

By using the first equation of the system of equations (46), expression (50) of the function $u_{20}(y)$ and considering relation (11), we obtain the expression for function $p_{20}(y)$. It is of the following form:

$$p_{20}(y)=\rho_2 c^2 k\frac{ch(k(y+h))}{sh(k(h-d))}a_1, -h<y<-d. \tag{51}$$

Using the relation (45), which connects the oscillation amplitude of the upper boundary and the interface of the considered two-layer fluid, turns the expression of function $p_{20}(y)$ (51) into the following expression:

$$p_{20}(y)=\rho_2(c^2 k\,ch(kd)-g\,sh(kd))\frac{ch(k(y+h))}{sh(k(h-d))}a, -h<y<-d. \quad (52)$$

Now we can move on to the final stage of solving the problem of the structure and characteristics of plane harmonic surface and internal waves in a two-layer fluid.

The content of this stage is simplification of condition (39). Equation (39) shows that it is necessary to obtain the values of $p_{10}(-d)$ and $p_{20}(-d)$. Letting $y=-d$ in expressions (44) and (52) and using the relation (11) (see introduction), we can obtain the following values for the mentioned quantities:

$$p_{10}(-d)=-\rho_1 c^2 k(a\,sh(kd)-cth(kd)(ch(kd)a-a_1)),$$

$$p_{20}(-d)=\rho_2 cth(k(h-d))(c^2 k\,ch(kd)-g\,sh(kd))a.$$

By substituting these values into equation (39), using relation (45) and doing necessary simplifications we obtain the following equation:

$$(\rho_1 sh(kd)sh(k(h-d))+\rho_2 ch(kd)ch(k(h-d)))(c^2k)^2,$$

$$-g\rho_2 sh(kh)c^2k+g^2(\rho_2-\rho_1)sh(kd)sh(k(h-d))=0.$$

Its solution produces dispersion relations for plane harmonic waves in a two-layer fluid.

$$c=\sqrt{g\frac{\rho_2 sh(kh)+\varepsilon\sqrt{D}}{2k(\rho_1 sh(kd)sh(k(h-d))+\rho_2 ch(kd)ch(k(h-d)))}}, \varepsilon=\pm 1, \quad (53)$$

where $D = sh(kh)(\rho_2^2 sh(kh) - 4(\rho_2 - \rho_1)sh(k(h-d))(\rho_1 sh(kd)sh(k(h-d)) + \rho_2 ch(kd)ch(k(h-d)))$.

The fact that two dispersion relations have been obtained means that there are two types of waves in a two-layer fluid.

The wave, for which $\varepsilon=1$, will be called fast, and the wave, for which $\varepsilon=-1$, will be called slow.

The justification for these names will be obtained by considering the Boussinesq approximation.

Under an external influence on a two-layer fluid, both types of plane harmonic waves arise.

For the completeness of results, we provide the expressions of fluid particle velocity components and hydrodynamic pressure in the lower layer of the considered two-layer fluid. They are the consequences of the initial expressions of these values (32) and expressions (49), (50) and (52) of functions $u_{20}(y)$, $v_{20}(y)$ and $p_{20}(y)$.

Expressions of the velocity components of fluid particles and hydrodynamic pressure in the lower layer of a two-layer fluid.

$$u_2=\omega\frac{ch(k(y+h))}{sh(k(h-d))}a_1\exp(i\theta),$$

$$v_2 = i\omega \frac{sh(k(y+h))}{sh(k(h-d))} a_1 exp(i\theta),$$

$$p_2 = -g\rho_2 y + g(\rho_1 - \rho_2)d + \rho_2(c^2 k ch(kd) - g sh(kd)) \frac{ch(k(y+h))}{sh(k(h-d))} a, -h < y < -d.$$

Studying the case of $n = 2$ has been completed.

The main results of studying the case of $n = 2$ are as follows: obtaining expressions for fluid particles velocity components and hydrodynamic pressure in the lower layer of the considered two-layer fluid, as well as obtaining two dispersion relations of the considered two-layer fluid.

To finish solving the problem of waves in a two-layer fluid, we consider the Boussinesq approximation.

In the theory of oscillations in a stratified fluid, the Boussinesq approximation is often used. It is of the following form:

$$\Delta\rho = \rho_2 - \rho_1 \ll \rho_1. \tag{54}$$

This approximation allows studying fast and slow waves in more detail. Let us slightly simplify the notation by assuming $\rho_1 = \rho$. In the case of a fast wave, $\varepsilon = 1$.

By neglecting small terms, we obtain:

$$c = \sqrt{g \frac{th(kh)}{k}}.$$

Thus, a fast wave in a two-layer fluid differs little from a surface wave in a homogeneous fluid.

This means, in particular, that for a fast wave in a two-layer fluid, the expressions for the free surface profile, the interface profile, the fluid particles velocity components and the hydrodynamic pressure in both layers of the considered two-layer fluid are just for the same values in a homogeneous fluid. In particular, the ratio of the oscillation amplitude of the interface and the free surface is equal to $\frac{sh(k(h-d))}{sh(kh)}$ in the case of a fluid of depth h, and $exp(-kd)$ in the case of an infinitely deep fluid.

The case of $\varepsilon = -1$ is of much greater interest. In this cast the dispersion relation gets greatly simplified and takes the following form:

$$c = \sqrt{g \frac{sh(k(h-d))\Delta\rho}{k\rho}}.$$

Since the ratio of $\Delta\rho/\rho$ is very small, the velocity of the second wave is very small as well. This was the reason for giving such names to fast and slow waves.

Conclusions

It has been determined that in a two-layer fluid there are two types of waves: a slow wave and a fast wave.

The main results of studying the case of $n = 1$ are as follows: obtaining expressions for fluid particles velocity components and hydrodynamic pressure in the upper layer of the considered two-layer fluid, as well as obtaining relations, which connect the oscillation amplitudes of the free surface and the interface of the considered two-layer fluid.

The main results of studying the case of $n = 2$ are as follows: obtaining expressions for fluid particles velocity components and hydrodynamic pressure in the lower layer of the considered two-layer fluid, as well as obtaining dispersion relations in the considered two-layer fluid.

In the case of implementing the Boussinesq approximation, the fast wave differs little from a surface wave in a homogeneous fluid, and the velocity of slow waves is proportional to the square root of the squared ratio $\Delta\rho/\rho$, i.e. it is very small.

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