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Global and Blow-Up Solutions for a Nonlinear Diffusion System with a Source and Nonlinear Boundary Conditions

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Abstract. In this paper, we study the global solvability and unsolvability of a nonlinear diffusion system with nonlinear boundary conditions in the case of slow diffusion. The conditions for the global existence of the solution in time and the unsolvability of the solution of the diffusion problem in a homogeneous medium are found on the basis of comparison principle and self-similar analysis. We obtain the critical exponent of the Fujita type and the critical global existence exponent, which plays an important role in the study of the qualitative properties of nonlinear models of reaction-diffusion, heat transfer, filtration and other physical, chemical, biological processes. In the global solvability case the principal terms of the asymptotic of solutions are obtained. It is well known that iterative methods require the presence of a suitable initial approximation, resulting in a rapid convergence to the exact solution and preserving qualitative properties of nonlinear processes under study, it is a major challenge for the numerical solution of nonlinear problems. This difficulty, depending on the value of the numerical parameters of the equation is overcome by a successful choice of initial approximations, which are mainly in the calculations suggested taking asymptotic formula.

Key words: blow-up, nonlinear boundary condition, critical exponents, nonlinear diffusion system, asymptotic

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МАТЕМАТИКА

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Научная статья

Полный текст на английском языке

УДК 517.957

Глобальные решения и решения с обострением для нелинейной диффузионной системы с источником и нелинейными граничными условиями

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Аннотация. В данной работе изучается глобальная разрешимость и неразрешимость одной нелинейной системы диффузии с нелинейными граничными условиями в случае медленной диффузии. Найдены условия глобального существования решения по времени и неразрешимости решения нелинейной задачи диффузии в однородной среде на основе автомодельного анализа и метода сравнения решений. Получены критическая экспонента типа Фуджита, и критическая экспонента глобального существования решения по времени, играющих важную роль при исследованиях качественных свойств нелинейных моделей реакции – диффузии, теплопроводности, фильтрации и других физических, химических, биологических процессов. В случае глобальной разрешимости получен главный член асимптотики автомодельных решений. Известно, что итерационные методы требуют наличия подходящего начального приближения, приводящее быстрой сходимости к точному решению и сохраняющие качественные свойства изучаемых нелинейных процессов, это является основной трудностью для численного решения нелинейных задач. Эта трудность в зависимости от значения числовых параметров нелинейной системы диффузии с нелинейными граничными условиями преодолевается путем удачного выбора начальных приближений, в качестве которых при вычислениях предложено брать полученные асимптотические формулы.

Ключевые слова: обострение, нелинейное краевое условие, критические показатели, нелинейная диффузионная система, асимптотика

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Introduction

In this article, it is dealt with the doubly degenerate parabolic equations with the source

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u_i^k}{\partial x}\right|^{m-1}\frac{\partial u_i^k}{\partial x}\right) + u_i^{p_i}, \quad x \in R_+, \ t > 0, \qquad i = 1,2 \tag{1}$$

coupled through nonlinear boundary flux

$$-\left|\frac{\partial u_i^k}{\partial x}\right|^{m-1}\frac{\partial u_i^k}{\partial x}\Bigg|_{x=0} = u_{3-i}^{q_i}(0,t), \quad t > 0, \ i = 1,2 \tag{2}$$

where $m > 1, k \geq 1$ and $q_i, p_i > 0$ are numerical parameters. The following initial data should be taken into account

$$u_i|_{t=0} = u_{i0}(x), \ i = 1,2 \tag{3}$$

moreover, they are expected to be continuous, non-negative and compact in R_+.

Parabolic equations with nonlinearity (1) are found in population dynamics, heat transfer, chemical reactions and so forth. The functions $u_1(t,x)$, $u_2(t,x)$ serve as two populations' densities in terms of biology while a migration progresses or the thickness of two types of chemical reagents within a chemical process and two different types of materials' temperatures during propagation. Most of all, equations (1) can be used to characterize unsteady flows in a liquid medium with a power-law dependence of shear stress on displacement velocity under polytropic conditions.

It has already established that the local existence of weak solutions for the problem (1)-(3) is determined by the usual integration method and can be easily established as well as the comparison principle (see [9] - [13] and [1] - [5]).

Global existence and blow-up conditions of nonlinear parabolic systems are intensively studied (see [1]- [3], [5, 11] and references therein). In particular, there is a great interest in critical Fujita exponents for various non-linear parabolic equations in mathematical physics (see [3, 10–12] and references therein).

In [5], investigated following problem

$$\begin{cases} u_t = \left(u^k\right)_{xx}, & x > 0, \ 0 < t < T, \\ -(u^k)_x(0,t) = u^q(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & x > 0 \end{cases} \tag{4}$$

and the heat conduction problem

$$\begin{cases} u_t = \left(|u_x|^{k-1}u_x\right)_x, & x > 0, \ 0 < t < T, \\ -|u_x|^{k-1}u_x(0,t) = u^q(0,t), & 0 < t < T, \\ u(x,0) = u_0(x), & x > 0 \end{cases} \tag{5}$$

with $k \geq 1$, $q > 0$ and u_0 has compact support. It has been proved that, for problem (4), the critical global exponent is $q_0 = \frac{1}{2}(k+1)$ where $q_c = k+1$ is the critical exponent of Fujita type in contrast with (5), the critical exponent of Fujita type $q_c = 2k$ and the critical exponent of global solvability is $q_0 = \frac{2k}{k+1}$.

In [6] Z.R.Rakhmonov and A.I.Tillaev studied the problem

$$\begin{cases} \rho(x)u_t = \left(|u_x|^{k-2}u_x\right)_x + \rho(x)u^\beta, & (x,t) \in R_+ \times (0,+\infty) \\ -|u_x|^{k-2}u_x(0,t) = u^m(0,t), & t > 0 \\ u(x,0) = u_0(x) > 0, & x \in R_+ \end{cases} \tag{6}$$

with $k > 2$, $\beta, m > 0$, $\rho(x) = x^{-n}$, $n \in R$, $u_0(x)$ - is a bounded, continuous, nonnegative and nontrivial initial data. They established that:

-if $0 < \beta \leq 1$ and $0 < m \leq \frac{(2-n)(k-1)}{k-n}$ the problem has global solution;

-if $\beta < 1$ and $m > \frac{(2-n)(k-1)}{k-n}$ the problem has blow-up solution.

In [9] Zhaoyin Xiang, Chunlai Mu and Yulan Wang studied the next problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u^{m_1}}{\partial x}\right|^{p_1-2} \dfrac{\partial u^{m_1}}{\partial x}\right) \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial v^{m_2}}{\partial x}\right|^{p_2-2} \dfrac{\partial v^{m_2}}{\partial x}\right) \end{cases}, \qquad (x,t) \in R_+ \times (0,T) \tag{7}$$

$$\begin{cases} -\left|\dfrac{\partial u^{m_1}}{\partial x}\right|^{p_1-2} \dfrac{\partial u^{m_1}}{\partial x}\Bigg|_{x=0} = v^{q_1}(0,t) \\ -\left|\dfrac{\partial v^{m_2}}{\partial x}\right|^{p_2-2} \dfrac{\partial v^{m_2}}{\partial x}\Bigg|_{x=0} = u^{q_2}(0,t) \end{cases}, \qquad t \in (0,T) \tag{8}$$

$$\begin{cases} u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases}, \qquad x \in R_+ \tag{9}$$

where $m_i > 1$, $p_i > 2$, $q_i > 0$, $i = 1,2$. They discovered that:

(i) if $q_1 q_2 \leq ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1 p_2$ then every nonnegative solution of the problem (7)-(9) is global in time;

(ii) if $q_1 q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1 p_2$ then the problem (7)-(9) solutions that explode in a finite amount of time exist.

When $q_1 q_2 > ((p_1-1)(p_2-1)(m_1+1)(m_2+1))/p_1 p_2$.

(i) if $\min\{\alpha_1+\beta_1, \alpha_2+\beta_2\} > 0$, then solution of the problem (7)-(9) is global in time;

(ii) if $\max\{\alpha_1+\beta_1, \alpha_2+\beta_2\} < 0$, then the solution of the problem (7)-(9) is blow-up.

In [7, 8] nonlinear parabolic equations in approximate derivatives with nonlinear boundary conditions are used to describe mathematical models of nonlinear cross diffusion. There are very few explicit analytical solutions to these nonlinearly coupled partial differential equation systems, so several numerical methods have been used to obtain approximations. Self-similar analysis and the standard equation approach are used to study the qualitative characteristics considering a cross-diffusion system that is nonlinear and has nonlocal boundary conditions. For the slow diffusion scenario, a number of solutions of the cross-diffusion problem have been developed that are self-similar. It is shown that a nonlinear cross-diffusion system of parabolic equations, connected by nonlinear boundary conditions may not have global solutions for some values of numerical parameters. The Fujita-type critical exponent and the global solvability critical exponent are established using self-similar analysis and the comparison principle. Upper limits are achieved for global solutions while lower limits are found for blow-up solutions using the comparison theorem [1].

This article was inspired by the aforementioned works and has two goals. In order to determine the (1)-(3) system's critical global existence curve, firstly one should to build the self-similar super- and subsolutions. On the other hand, with the help of some fresh findings, the critical curve of the Fujita type is hypothesized. Being faced with a system as opposed to a single equation compels us to create some novel strategies.

Degenerate equations don't necessarily need to have classical solutions, as is common knowledge. As a result, its solution is comprehended in a broad way, see [1, 2].

Definition 1. The function $u(x,t)$ is seen as the inadequate remedy for the problem (1) - (3) in $\Omega=\{(0,+\infty)\times(0,T)\}$, if $0\le u_i(x,t)\in C(\Omega)$, $\left|\frac{\partial u_i^k}{\partial x}\right|^{m-1}\frac{\partial u_i^k}{\partial x}\in C(\Omega)$, $i=1,2$ and if it complies with (1)-(3) in the sense of distribution in Ω, where $T>0$ is the longest possible time, see [6].

Main results

This paragraph is intended to build self-similar sub- and super-solutions to (1)-(3), to establish the theorems of Global existence and nonexistence solutions. The first theorem is about the conditions in which the problem (1)-(3) has the global solution. The global existence of solution draws conclusions from the comparison principle.

Theorem 1. *If* $q_1q_2\le\left(\frac{m}{m+1}\right)^2(k+1-p_1)(k+1-p_2)$, *then every nonnegative solution of the problem* (1)-(3) *is global in time.*

Proof. To prove the theorem self-similar sub-solution has been constructed and showed that it is limited for any $t>0$. For this purpose, it has been looked for strict supersolutions of self-similar form

$$\bar{u}_i(t,x)=e^{h_{2i-1}}\left(M+e^{-L_ixe^{-h_{2i}t}}\right)^{\frac{1}{k}} \tag{10}$$

where $L_i > 0$, $h_{2i-1,2i} > 0$, $M = \max\left\{\|\bar{u}_i\|_\infty^k + 1\right\}$; $i = 1,2$. Substituting (10) into (1)-(2) and using comparison principles it has been obtained:

$$\frac{\partial \bar{u}_i}{\partial t} = h_{2i-1}\cdot e^{h_{2i-1}t}\cdot\left(M+e^{-L_i x e^{-h_{2i}t}}\right)^{\frac{1}{k}} + e^{(h_{2i-1}-h_{2i})t}\cdot\frac{1}{k}\cdot L_i\cdot x\cdot h_{2i}\times$$
$$\times\left(M+e^{-L_i x e^{-h_{2i}t}}\right)^{\frac{1}{k}-1} \geq h_{2i-1}e^{h_{2i-1}t}\left(M+e^{-L_i x e^{-h_{2i}t}}\right)^{\frac{1}{k}} \geq h_{2i-1}e^{h_{2i-1}t}M^{\frac{1}{k}}$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}_i^k}{\partial x}\right|^{m-1}\frac{\partial \bar{u}_i^k}{\partial x}\right) = mL_i^{m+1}e^{[h_{2i-1}km-(m+1)h_{2i}]t}e^{-mL_i x e^{-h_{2i}t}} \leq mL_i^{m+1}e^{[h_{2i-1}km-(m+1)h_{2i}]t}$$

$$\bar{u}_i^{p_i} = e^{p_i h_{2i-1}t}\left(M+e^{-L_i x e^{-h_{2i}t}}\right)^{\frac{p_i}{k}} \leq e^{p_i h_{2i-1}t}(M+1)^{\frac{p_i}{k}}$$

$$-\left|\frac{\partial \bar{u}_i^k}{\partial x}\right|^{m-1}\frac{\partial \bar{u}_i^k}{\partial x}\Bigg|_{x=0} = L_i^m e^{(h_{2i-1}k-h_{2i})mt},\ \bar{u}_{3-i}^{q_i}\Big|_{x=0} = e^{q_i h_{5-2i}t}(M+1)^{\frac{q_i}{k}}$$

The solution $\bar{u}_i$ are considered to be global if inequalities

$$\frac{\partial \bar{u}_i}{\partial t} \geq \frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}_i^k}{\partial x}\right|^{m-1}\frac{\partial \bar{u}_i^k}{\partial x}\right) + \bar{u}_i^{p_i}, \qquad i = 1,2 \tag{11}$$

hold for any $x \in R_+$, $t > 0$. That is why using the computations above in (11) following expressions have been achieved:

$$h_{2i-1}e^{h_{2i-1}t}M^{\frac{1}{k}} \geq mL_i^{m+1}e^{[h_{2i-1}km-(m+1)h_{2i}]t} + e^{p_i h_{2i-1}t}(M+1)^{\frac{p_i}{k}}$$

$$L_i^m e^{(h_{2i-1}k-h_{2i})mt} = e^{q_i h_{5-2i}t}(M+1)^{\frac{q_i}{k}}$$

$$L_i = (M+1)^{\frac{q_i}{km}},\ q_i h_{5-2i} = (h_{2i-1}k-h_{2i})\,m,\ i = 1,2$$

$$h_{2i-1} \geq h_{2i-1}km-(m+1)h_{2i}+p_i h_{2i-1}, h_1 k - h_2 = \frac{q_1}{m}h_3, h_3 k - h_4 = \frac{q_2}{m}h_1$$

$$h_{2i} \geq \frac{(km+p_i-1)\,h_{2i-1}}{m+1},$$

$$h_1 k - \frac{q_1}{m}h_3 \geq \frac{(km+p_1-1)\,h_1}{m+1}, h_3 k - \frac{q_1}{m}h_1 \geq \frac{(km+p_2-1)\,h_3}{m+1}$$

computing the last expressions, it can be seen that the last inequality should be always hold for any $m > 1$, $k \geq 1$ in order to the solution of the problem (1)-(3) was global in time. Theorem is proved.

Remark 1. Theorem demonstrates that $q_1 q_2 = \left(\frac{m}{m+1}\right)^2 (k+1-p_1)(k+1-p_2)$ is critical global existence of the problem (1)-(3).

Theorem 2. *If* $0 < p_i \le 1$ *and* $q_i \ge \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ *or* $p_i > 1$ *and* $q_i \le \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ *then, each of the solutions to* (1)-(3) *blows up.*

Proof. To prove the theorem the subsolutions of the problem (1)-(3) have been looked for in the next form:

$$\underline{u}_i(t,x) = t^{\alpha_i} f_i(\xi_i), \qquad \xi_i = x t^{-\beta_i}, \tag{12}$$

where $\alpha_i = \dfrac{1}{1-p_i}$, $\beta_i = \dfrac{p_i - km}{(p_i-1)(m+1)}$, $i = 1,2$.

After substitution (12) into (1)-(3) it has been reached the next self-similar inequalities and boundary conditions that should be hold for any $u_i(t,x)$ that treated as blow up solutions:

$$\frac{d}{d\xi_i}\left(\left|\frac{df_i^k}{d\xi_i}\right|^{m-1}\frac{df_i^k}{d\xi_i}\right) + \beta_i \xi_i \frac{df}{d\xi_i} - \alpha_i f_i + f_i^{p_i} \ge 0 \tag{13}$$

$$-\left|\frac{\partial \underline{u}_i^k}{\partial x}\right|^{m-1}\frac{\partial \underline{u}_i^k}{\partial x}\Bigg|_{x=0} \le \underline{u}_{3-i}^{q_i}(0,t) \tag{14}$$

Let

$$f_i(\xi_i) = A_i\left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}}\right)^{\frac{m}{mk-1}} \tag{15}$$

Substitution (15) into (13), (14) lead us to the following conditions that show (14) always takes place:

$$\left(\frac{k(m+1)}{mk-1}\right)^m \left(\frac{m+1}{mk-1}\right) A_i^{mk} \ge \beta_i \frac{m+1}{mk-1} A_i,$$

$$A_i \ge \left[\beta_i \left(\frac{mk-1}{k(m+1)}\right)^m\right]^{\frac{1}{mk-1}},$$

$$f_i^{p_i} = A_i^{p_i}\left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}}\right)_+^{\frac{m}{mk-1}} \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}}\right)_+^{\frac{m}{mk-1}(p_i-1)} \le$$
$$\le A_i^{p_i} a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \left(a_i^{\frac{m+1}{m}} - \xi_i^{\frac{m+1}{m}}\right)_+^{\frac{m}{mk-1}}$$

$$A_i^{p_i} a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \ge \alpha_i A_i + A_i^{mk}\left(\frac{k(m+1)}{mk-1}\right)^m,$$

By taking

$$a_i^{\frac{(m+1)(p_i-1)}{mk-1}} \geq \alpha_i A_i^{1-p_i} + A_i^{mk-p_i}\left(\frac{k(m+1)}{mk-1}\right)^m$$

$0 < p_i \leq 1$ and $q_i \geq \dfrac{m(p_{3-i}-1)(p_i+k)}{(p_i-1)(m+1)}$ can be easily checked and ensure that A_1 and A_2 can be taken sufficient to prevent inequalities (13) and (14) are valid. Because of this, if the initial data $u_1(x,0)$, $u_2(x,0)$ are large enough that $u_{10}(x) \geq \underline{u}_1(x,0)$, $u_{20}(x) \geq \underline{u}_2(x,0)$ then $\underline{u}_i(t,x),\ i=1,2$ is a subsolution to (1)-(3). According to the comparison principle, for enormous beginning data, the solutions of (1)-(3) blow up in a finite amount of time. The proof is finished.

Theorem 3. *If* $q_1q_2 < \left(\dfrac{m(k+1)}{m+1}\right)^2$ *and* $p_i > \left(1+\dfrac{1}{k}\right)m + \dfrac{1}{k}$, *then every solution of the problem* (1)-(3) *is blow-up in finite time.*

Proof. It is efficiently enough to show that the problem (1)-(3) without sources fulfilled out conditions. Let, construct

$$u_{ib}(t,x) = t^{\mu_i} g_i(\xi_i), \qquad \xi_i = xt^{-\gamma_i} \tag{16}$$

where g_i are two compactly supported functions,

$$\mu_i = \frac{m[m(k+1)+(m+1)q_i]}{(m(k+1))^2-(m+1)^2q_iq_{3-i}}, \gamma_i = \frac{m[mk(k+1)+(mk-1)q_i]-(m+1)q_1q_2}{(m(k+1))^2-(m+1)^2q_iq_{3-i}}.$$

Now substituting (16) into (1)-(3) and obtain the next:

$$\frac{d}{d\xi_i}\left(\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\right) + \gamma_i\xi_i\frac{dg_i}{d\xi_i} - \mu_i g_i \geq 0 \tag{17}$$

$$-\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\Bigg|_{\xi_i=0} \leq g_{3-i}^{q_i}(0) \tag{18}$$

Now it is time to find self-similar solutions of the problem (17), (18).

Let

$$\bar{g}_i(\xi_i) = B_i(b_i-\xi_i)^{\frac{m}{mk-1}} \tag{19}$$

then

$$\frac{d\bar{g}_i}{d\xi_i} = -\frac{B_i m}{mk-1}(b_i-\xi_i)^{\frac{m}{mk-1}-1}.$$

$$\gamma_i\xi_i\frac{dg_i}{d\xi_i} - \mu_i g_i = -\frac{B_i m}{mk-1}\xi_i(b_i-\xi_i)^{\frac{m}{mk-1}-1} - \mu_i B_i(b_i-\xi_i)^{\frac{m}{mk-1}} =$$

$$= -\frac{B_i m}{mk-1}\xi_i(b_i-\xi_i)^{\frac{m}{mk-1}-1} - \mu_i B_i(b_i-\xi_i)^{\frac{m}{mk-1}-1}(b_i-\xi_i) \geq$$

$$\geq -\left(\frac{b_iB_im}{mk-1} - \mu_i b_i B_i\right)(b_i-\xi_i)_+^{\frac{m}{mk-1}-1}.$$

$$\frac{d}{d\xi_i}\left(\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\right)=B_i^{mk}\left(\frac{m}{mk-1}\right)^{m+1}k^m\left(b_i-\xi_i\right)_+^{\frac{m}{mk-1}-1}\geq$$

$$\geq b_iB_i\left(\mu_i+\frac{m}{mk-1}\right)\left(b_i-\xi_i\right)_+^{\frac{m}{mk-1}-1}.$$

$$B_i^{mk-1}\geq\frac{b_i}{k^m}\left(\frac{mk-1}{m}\right)^{m+1}\left(\mu_i+\frac{m}{mk-1}\right),-\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\Bigg|_{\xi_i=0}\leq\bar{g}_{3-i}^{q_i}(0)$$

Applying comparison principles to the expressions above it is obtained:

$$-\left|\frac{dg_i^k}{d\xi_i}\right|^{m-1}\frac{dg_i^k}{d\xi_i}\Bigg|_{\xi_i=0}=\left|B_i^k\left(b_i-\xi_i\right)_+^{\frac{m}{mk-1}-1}\right|^{m-1}\cdot\left(B_i^k\left(b_i-\xi_i\right)_+^{\frac{m}{mk-1}-1}\right)\Bigg|_{\xi_i=0}=$$

$$=B_i^{mk}\left(b_i-\xi_i\right)_+^{\left(\frac{m}{mk-1}-1\right)m}\Bigg|_{\xi_i=0}=B_i^{mk}b_i^{\frac{m}{mk-1}}\leq B_{3-i}^{q_i}b_{3-i}^{\frac{q_im}{mk-1}}.$$

And this show that when $p_i>\left(1+\frac{1}{k}\right)m+\frac{1}{k}$, (17) and (18) are valid. It results from the comparison concept that (1)-(3) have solutions blowing up in a finite time.

Theorem 4. *If* $q_1q_2\leq(m(k+1))^2$ *and* $p_i>1$*, then every solution of the problem* (1)-(3) *is blow-up in finite time.*

Proof. Theorems can be proved in the same manner as it was done in [8, 14].

Conclusion

It is accomplished to acquire the diffusive system's solution of the type Zeldovich-Barenblatt. It is demonstrated that the nonlinear diffusion issue characterized that global solutions may not exist for degenerate parabolic systems coupled via nonlinear boundary conditions for specific values of numerical parameters. Using the comparison approach, it is possible to study the finite speed properties of the problem diffusion with a source.

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