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On Some New Results in Large Area Nevanlinna Spaces in the Unit Disk

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Abstract. The study of various infinite products in various spaces of analytic functions in the unit disk is a well known and well studied problem of complex function theory in the unit disk. The goal of our paper is to study so-called Blashcke type products in new large, general analytic area Nevanlinna spaces in the unit disk. A new approach is suggested in this paper, namely we prove, use and apply various new embedding theorems which relate new general, large analytic area Nevanlinna spaces with less general well-studied and well-known such type analytic spaces in the unit disk. Our theorems can be applied or can be used even in more general situation, when we consider large, general analytic area Nevanlinna spaces not in the unit disk, but in the circular ring. In our paper, using same approach new parametric representations of mentioned large, general analytic area Nevanlinna spaces are presented. These results also can be applied or used in the future to obtain more general theorems on parametric representations of mentioned large, general area Nevanlinna type spaces not in the unit disk, but in more general circular domains.

Key words: Blaschke type infinite products, area Nevanlinna - type spaces, Nevanlinna characteristic, parametric representations, analytic function.

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О некоторых новых теоремах в классах типа Неванлинны в единичном круге

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Аннотация. Общая задача о принадлежности тех или иных бесконечных произведений тем или иным аналитическим классам функций хорошо известна в литературе. Цель исследования, в частности, рассмотреть и изучить вопрос о принадлежности бесконечных произведений типа Бляшке к общим новым широким классам типа Неванлинны в единичном круге. Авторы для этого применяют новый метод, а именно доказываются и приводятся в статье различные новые теоремы вложения,связывающие новые общие классы типа Неванлинны с уже хорошо изученными и известными менее общими классами типа Неванлинны в единичном круге. Результаты статьи могут быть обобщены или использованы в более общем случае, когда рассматриваются общие, широкие классы Неванлинны в круговом кольце. В статье тем же методом также получены новые параметрические представления указанных широких классов типа Неванлинны в единичном круге. Вывод: эти результаты также могут быть использованы для получения новых параметрических представлений общих классов типа Неванлинны в круговом кольце.

Ключевые слова: бесконечные произведения типа Бляшке, площадь пространств неванлинновского типа, характеристика Неванлинны, параметрические представления, аналитическая функция.

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Финансирование. Исследование выполнялось без финансовой поддержки фондов.

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Introduction, basic definitions and history of problems

Assuming that $D=\{z\in\mathbb{C}|:\ |z|<1\}$ is the unit disk of the finite complex plane $\mathbb{C}$, T is the boundary of D, $T=\{z\in\mathbb{C}:\ |z|=1\}$ and $H(D)$ is the space of all functions holomorphic in D we introduce the following classes of functions

$$N_{\alpha}^{\infty}(D)=\{f\in H(D):T(\tau,f)\le C_f(1-\tau)^{-\alpha},\ 0\le\tau<1,\ \alpha\ge 0\},$$

where $T(\tau,f)$ is classical Nevanlinna characteristic defined by $T(\tau,f)=\frac{1}{2\pi}\int_T\log^+|f(\tau\xi|d\xi$, where $a^+=\max\{0,a\}$, $a\in\mathbb{R}$, (see [5]). It is obvious that if $\alpha=0$ then $N_0^{\infty}(D)=N(D)$, where $N(D)$ is the well known classical Nevanlinna class (see [2], [3], [5]).

Let $f\in H(D)$, then we define

$$M_p(f,r)=\frac{1}{2\pi}\left(\int_T|f(r\xi)|^p dm(\xi)\right)^{\frac{1}{p}},\ r\in(0,1),\ p\in(0,\infty),$$

where by $m(\xi)$ we denote the normalized Lebesgue measure on T. Also, by $m_2(\xi)$ we denote standard normalized Lebesques area measure.
Everywhere below by $n_f(t)=n(t)$ we denote the quantity of zeros of an analytic function f in the unit disk $|z|\le t<1$ and by $Z(X)$ the zero set of an analytic class X, $X\subset H(D)$. By let $\{z_k\}_{k=1}^{\infty}$ be a sequence of numbers from D below we mean that $\{z_k\}_{k=1}^{\infty}$ is an arbitrary sequence from unit disk enumerated by it is growth $(|z_k|\le|z_{k+1}|\le\ldots)$ according to it is multiplicity.
By n_k we denote $n(1-2^{-k})$, i.e. $n_k=n(1-2^{-k})$, $k=1,2,\ldots$, (see [2]).

In all our assertions below we assume in advance that our functions are not identically zero or infinity.

The following statement holds by Nevanlinna's classical result on the parametric representation of $N(D)$ (see [2], [3], [5]). $N(D)$ class coincides with the set of functions representable in the form

$$f(z)=C_{\lambda}z^{\lambda}B(z,\{z_k\})\exp\left(\int_{-\pi}^{\pi}\frac{d\mu(\theta)}{1-ze^{-i\theta}}\right),\ z\in D,$$

where C_{λ} is a complex number, λ is a nonnegative integer, $B(z,\{z_k\})$ is the classical Blaschke product with zeros $\{z_k\}_{k=1}^{\infty}\subset D$ enumerated according to their multiplicities and satisfying the condition $\sum_{k=1}^{\infty}(1-|z_k|)<\infty$, and $\mu(\theta)$ is a function of bounded variation in $[-\pi,\pi]$.

In [2], [3] the following proposition is established (see also [5]) for sequences $\{z_k\}_{k=1}^{\infty}\subset D$ satisfying the greater density condition

$$\sum_{k=1}^{\infty}(1-|z_k|)^{t+2}<\infty,\ t>-1. \tag{1}$$

Proposition A: (see [2]) Let $\{z_k\}_{k=1}^{n}$ be a sequence in the unit disc satisfying the density condition (1) for some $t>-1$, then the Djrbashian infinite product

$$\Pi_t(z,\{z_k\})=\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left(\frac{-(t+1)}{\pi}\int_D\frac{(1-|\xi|^2)^t\ln\left|1-\frac{\xi}{z_k}\right|}{(1-\overline{\xi}z)^{t+2}}dm_2(\xi)\right),z\in D, \tag{2}$$

converges absolutely and uniformly inside D where it presents an analytic function with zeros $\{z_k\}_{k=1}^{\infty}$.

After the appearance of the classical Nevanlinna's parametric representation in Hayman's book (see [5]) which we mentioned above various new results of the same type appeared during past decades where Blaschke products were substituted by more general so called Djrbashian $\Pi_\alpha(z,\{z_k\})$ products (see [3]) and we will mention them partially below in Theorem A and Theorem B. In [9] it was shown that these $\Pi_\alpha(z,\{z_k\})$ products can be in their turn replaced by other infinite $B_\alpha(z,\{z_k\})$ products and some aspects of this last development will stand as one of the topics of this paper.

We denote by $B_\alpha^{p,q}(T)$, $0<p<\infty$, $0<q\le\infty$, $\alpha>0$, the classical Besov space on the unit circle T, (see [1]).

Theorem A: (see [9]) Let $\alpha>0$ and $\beta>\alpha-1$, then the $N_\alpha^\infty(D)$ class coincides with the set of functions representable in the form

$$f(z)=C_\lambda z^\lambda \Pi_\beta(z,\{z_k\})\exp\left(\int_{-\pi}^{\pi}\frac{\psi(e^{i\theta})d\theta}{(1-ze^{-i\theta})^{\beta+2}}\right),\ z\in D, \tag{3}$$

where C_λ is a complex number, λ is a nonnegative integer, $\Pi_\beta(z,\{z_k\})$ is the Djrbashian infinite product (2), $\{z_k\}_{k=1}^{\infty}\subset D$ is a sequence satisfying the condition

$$n(\tau)\le\frac{c}{(1-\tau)^{\alpha+1}},$$

where $c>0$ is a positive constant and $\psi(e^{i\theta})$ is a real function of $B_{\beta-\alpha+1}^{1,\infty}(T)$.

We also give below a theorem which is established in [8] and in a sense is similar to Theorem A.

Let $S_\alpha^p(D)$ be the class of analytic functions defined by

$$S_\alpha^p(D)=\{f\in H(D):\|f\|_{S_\alpha^p}^p=\int_0^1(1-\tau)^\alpha T^p(\tau,f)d\tau<\infty,\ 0<p<\infty,\ \alpha>-1\}.$$

Theorem B: (see [8]) For $p\ge0$, $\beta>\frac{\alpha+1}{p}$, $f\in S_\alpha^p(D)$ if and only if $f(z)$ admits representation $f(z)=C_\lambda z^\lambda\Pi_\beta(z,\{z_k\})\exp\left(\int_{-\pi}^{\pi}\frac{\psi(e^{i\theta})d\theta}{(1-ze^{-i\theta})^{\beta+1}}\right)$, $z\in D$, where C_λ is a complex number, λ is a nonnegative integer, $\{z_k\}_{k=1}^{\infty}\subset D$ is a sequence for which $\int_0^1(1-\tau)^{\alpha+p}[n(\tau)]^p d\tau<\infty$ and $\psi\in B_s^{1,p}(T)$, where $s=\beta-\frac{\alpha+1}{p}$.

Note complete analogues of Theorem A and Theorem B were given also for $N_{\alpha,\beta}^p$ and $N_{\alpha,\beta}^{\infty,p}$. $N_{p,\gamma,\beta}^p$ area Nevanlinna spaces in the disk (see [12], [14], [11] and definitions below).

One can easily see that Theorem A gives the parametric representations of the spaces $N_\alpha^\infty(D)$ while Theorem B gives the parametric representations of $S_\alpha^p(D)$ analytic area Nevanlinna type spaces in the unit disk via same Djrbashian $\Pi_t(z,\{z_k\})$ infinite product.

One of the goals of this paper are parametric representations of the larger spaces via completely other infinite product.

Let further

$$N^{p}_{\alpha,\beta}(D)=\{f\in H(D):\int_0^1\left(\int_{|z|\le R}(\ln^+|f(z)|)(1-|z|)^{\alpha}dm_2(z)\right)^p(1-R)^{\beta}dR<\infty\};$$

$$N^{\infty,p}_{\alpha,\beta_1}(D)=\{f\in H(D):\sup_{0\le R<1}\left(\int_0^R\left(\int_T\ln^+|f(z)|d\xi\right)^p(1-|z|)^{\alpha}d|z|\right)(1-R)^{\beta_1}<\infty\},$$

where it is assumed that $\alpha>-1,\ \beta>-1,\ \beta_1\ge 0$ and $0<p<\infty$, and let

$$N_{p,\gamma,\beta}(D)=\{f\in H(D):\int_0^1(1-|z|)^{\beta}\left(\sup_{0<\tau<|z|}T(\tau,f)(1-\tau)^{\gamma}\right)^p d|z|<\infty\},$$

where $\gamma\ge 0,\ \beta>-1$.

We refer for basic properties of these new large are Nevanlinna spaces to [11] - [11]. We note in these papers various results on zero sets and parametric representations can also be seen. Note similar, but less general results can be seen in various papers of various authors, we refer, for example, to [2], [3], [8], [9].

Note that various properties of $N^{\infty,p}_{\alpha,0}(D)$ are studied in [8]. In particular, the works [2], [8] give complete descriptions of zero sets and parametric representations of $N^{\infty,p}_{\alpha,0}(D)$ (in [2] for $p=1$). Thus it is natural to consider the problem of extension of these important results to all $N^{\infty,p}_{\alpha,\beta_1}(D)$ analytic classes.

We remark that these analytic classes of area Nevanlinna type in the unit disk was considered by us in our recent paper (see [11], [11], [14]).

It is not difficult to verify that all the above mentioned area Nevanlinna analytic classes are topological vector spaces with complete invariant metric. We note that the mentioned problem of parametric representation have various applications and are important in function theory (see [2], [3], [4]).

Solution of many problems for example the existence of radial limits is based also on parametric representations. Parametric representations are used also in spectral theory of linear operators (see [3], [4]).

The next section will be devoted to study of certain infinite Blashcke type products $B_{\alpha}(z,\{z_k\})$ in new analytic area Nevanlinna classes we introduced above, then partially based on these results we will turn to the main topic of paper we mentioned above and we will provide some new parametric representations via these infinite Blaschke type products $B_{\alpha}(z,\{z_k\})$.

The main idea to get new results on infinite B_{α} products and parametric representations via such products in our large new Nevanlinna type spaces to use a group of new embeddings relating them to known Nevanlinna type spaces for which such results were provided by other authors, then apply these known results.

For this we use a simple idea. Namely, we analyze various known embeddings between mixed norm and Bergman spaces and then simply replace in these estimates $|f|^p$ by $(\log^+|f|)^p$, since both are subharmonic functions for $p\ge 1$, and since the already known proof is based only on this fact.

Throughout the paper we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables discussed.

The notation $A \asymp B$ means that there is a positive constant C, such that $\frac{B}{C} \leq A \leq CB$. We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

We formulated certian assertions below on Nevanlinna spaces after careful analysis of some already known proofs for mixed norm spaces, their proofs will be given below in a sketchy form, since new proofs are almost the same. We leave some arguments in proofs below to readers since they are easy to recover.

On some new theorems on canonical infinite products of Blaschke type in $N^p_{\alpha,\beta}$, $N^{\infty,p}_{\alpha,\beta}$ and $N_{p,\gamma,\beta}(D)$ classes in the unit disk

First we introduce a new Blaschke type canonical product and list some of it is properties, then based on these properties we will find conditions on $\{z_k\}_{k=1}^{\infty}$ sequence from D such that our product belongs to mentioned above new analytic Nevanlinna classes in the unit disk. We remark that the $\Pi_t(z,\{z_k\})$ defined above and the product we are going to consider act as same kind of substitution for Blaschke product in classes with $\log^+$. The problem of finding conditions on $\{z_k\}_{k=1}^{\infty}$ sequences so that the classical Blaschke product belongs to analytic Bergman or Hardy or other spaces is well known and studied by many authors before (see, for example, [2], [3], [16] and the references there).

We would like to note that our results can be considered as analogues of mentioned assertions concerning Blaschke products. Note that similar results for $\Pi_t(z,\{z_k\})$ products are well known (see [2], [3], [8]).

We give one such type example.

Theorem C: (see [2]) Let $\{z_k\}_{k=1}^{\infty}$ be a sequence from unit disk D. Then if $\sum_{k=1}^{\infty}(1-|z_k|)^{\alpha+2} < \infty$, $\alpha > -1$, then $\Pi_t(z,\{z_k\}) \in S^1_{\alpha}(D)$ for $t > \alpha$ that is

$$\int_D \log^+|\Pi_t(z,\{z_k\})|(1-|z|)^{\alpha}dm_2(z) < \infty,$$

and the reverse is also true. Let (1) holds, then $\Pi_{\alpha}(z,\{z_k\}) \in S^1_t(D)$ if $\alpha > t$.

We introduce now another infinite product which is the main object of this note. It is known that (see [3]) the following assertion is true. The infinite Blaschke type product $B_{\alpha}(z,\{z_k\})$, $\alpha > -1$

$$B_{\alpha}(z,\{z_k\}) = \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left(-W_{\alpha}(z,z_k)\right), \text{ and}$$

$$W_{\alpha}(z,\xi) = \sum_{k=1}^{\infty}\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+2)\Gamma(k+1)}\left((\overline{\xi}z)^k\int_{|\xi|}^1\frac{(1-x)^{\alpha}dx}{x^{k+1}} - \left(\frac{z}{\xi}\right)^k\int_0^{|\xi|}(1-x)^{\alpha}x^{k-1}dx\right), z,\xi \in D,$$

is converges uniformly within D if and only if $\sum_{k=1}^{\infty}(1-|z_k|)^{\alpha+1} < \infty$. Moreover it represents an analytic function in D.

Remark 1: An interesting generalization of this product can be found in [4].

Our intention is to find conditions on $\{z_k\}_{k=1}^{\infty}$ of this product, so that $B_\alpha(z,\{z_k\}) \in N^p_{\alpha,\beta}(D)$ or $N^{\infty,p}_{\alpha,\beta_1}(D)$ or $N_{p,\gamma,\beta}(D)$. We mention that the following result were given before. It puts in particular direct condition on $\{z_k\}_{k=1}^{\infty}$ sequences so that $B_\alpha(z,\{z_k\}) \in S^1_\alpha(D)$.

Let Ω be a set of positive on $(0,1)$ measurable functions w such that $m_w \le \dfrac{w(\lambda x)}{w(r)} \le M_w$, for all $x \in (0,1)$, $\lambda \in [q_w, 1]$ and some fixed M_w, m_w, q_w such that $m_w, q_w \in (0,1)$, $M_w > 0$, (see [2]).

We define general S^p_ω area Nevanlinna spaces similarly as a S^p_α spaces by replacing $(1-r)^\alpha$ by $w(r)$.

Theorem D: (see [9])Let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence from unit disk D. Let $w \in \Omega$, $\alpha_w = \dfrac{\ln m_w}{\ln q_w}$, $\beta_w = \dfrac{\frac{1}{\ln M_w}}{\ln \frac{1}{q_w}}$, $0 < \beta_w < 1$ and one of the following two conditions holds:

$$0 < p \le 1, \qquad p > \frac{\alpha_w + \beta_w}{2}, \ \frac{\alpha_w + 1}{p} < \alpha < \frac{1-\beta_w}{p} + 2;$$

or

$$1 < p < \infty, \qquad p > \alpha_w + \beta_w - 1, \ \frac{\alpha_w}{p} + 1 \le \alpha < \frac{1-\beta_w}{p} + 2.$$

Then if $\int_0^1 w(1-r)n^p(r)(1-r)^p dr < \infty$, then $B_\alpha(r\xi,\{\lambda_k\})$ uniformly converges within D and belongs to S^p_ω that is

$$\int_0^1 \left(\int_T \log^+ |B_\alpha(r\xi,\{\lambda_k\})| dm(\xi)\right)^p w(1-r)dr < \infty.$$

Corollary 1: Let $0 < p < \infty$, $\alpha > -1$, $\{z_k\}_{k=1}^{\infty} \in D$, $0 < |z_k| \le |z_{k+1}|$, $k = 1,2,\dots$ and let also $\int_0^1 (1-r)^{\alpha+p} n^p(r) dr < \infty$, then $B_\beta(z,\{z_k\}) \in S^p_\alpha$ that is

$$\int_0^1 \left(\int_T \log^+ |B_\beta(r\xi,\{z_k\})| dm(\xi)\right)^p (1-r)^\alpha dr < \infty$$

$$\text{if } \ 0 < p \le 1, \quad \frac{\alpha+1}{p} < \beta < 2 + \frac{\alpha+1}{p}$$
$$\text{if } \ 1 < p < \infty, \quad 1 + \frac{\alpha}{p} \le \beta < 2 + \frac{\alpha+1}{p}.$$

We formulate below in theorems 1 and 2 new such type results on B_α infinite products in our new large area Nevanlinna type spaces.

Theorem 1: Let $\alpha > -1$, $\beta > -1$ and $p \in (0,\infty)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequences of complex numbers in the unit disk such that

$$\int_0^1 (1-r)^{\alpha+p} n^p(r)\,dr < \infty. \tag{4}$$

Then there is an interval (t_0,t_1) for $t_0,t_1 \in \mathbb{R}\cup\{\infty\}$ and for $p \le 1$ and for $p \ge 1$ for which the canonical product $B_t(z,\{\lambda_k\})$, $t \in (t_0,t_1)$ converges absolutely and uniformly within D and belongs to $N^p_{\alpha,\beta}(D)$ class.

Theorem 2: Let $\alpha > -1$, $\beta > -1$ and $p \in (0,\infty)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequences of complex numbers in the unit disk such that

$$\int_0^1 (1-r)^{\alpha+p} n^p(r)\,dr < \infty. \tag{5}$$

Then there is an interval (t_0',t_1') for $t_0',t_1' \in \mathbb{R}\cup\{\infty\}$ and for $p \le 1$ and for $p \ge 1$ for which the canonical product $B_t(z,\{\lambda_k\})$, $t \in (t_0',t_1')$ converges absolutely and uniformly within D and belongs to $N^{\infty,p}_{\alpha,\beta}(D)$ class.

Remark 2: It is not difficult to extend the statements and the proofs of Theorems 1 and 2 to more general, slowly varying weights $w(1-\tau)$ from S class (see [2]).

The very similar result is also valid for $N_{p,\gamma,\nu}$ area Nevanlinna type spaces, with some restrictions on parameters.

The proof of Theorems 1 and 2 will be given below.

The goal of this section to provide ways to get some new parametric representations for $N^p_{\alpha,\beta}(D)$, $N^{\infty,p}_{\alpha,\beta}(D)$ and $N_{p,\gamma,\beta}(D)$ classes in the unit disk via $B_\alpha(z_k,z)$ products. The following theorem provides complete parametric representations for $N^p_{\alpha,\beta}(D)$ spaces via Djrbashian products from theorems A and B, (see [12], [13]).

Theorem E: (see [12], [13]) If $0 < p < \infty$, $\alpha > -1$ and $\beta > -1$, then the class $N^p_{\alpha,\beta}$ coincides with the set of functions representable for $z \in D$ as

$$f(z) = C_\lambda z^\lambda \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left\{\frac{t+1}{\pi}\int_0^1\int_{-\pi}^{\pi}\frac{(1-\rho^2)\ln\left|1-\frac{\rho e^{i\varphi}}{z_k}\right|}{(1-\rho e^{-i\varphi}z)^{t+2}}\rho\,d\rho\,d\varphi\right\}\exp\{h(z)\},$$

where $t > \max\left\{\left(\alpha+\beta/p\right)+\max\{1,1/p\},\,(\alpha+1)\right\}$, C_λ is a complex number, $\lambda \ge 0$,

$$\sum_{k=1}^{\infty}\frac{n_k^p}{2^{k(\beta+1+2p+\alpha p)}} < \infty,$$

and $h \in H(D)$ is a function satisfying the condition

$$\int_0^1\left(\int_0^R\left(\int_{-\pi}^{\pi}|h(\tau e^{i\varphi})|\,d\varphi\right)(1-\tau)^\alpha d\tau\right)^p (1-R)^\beta\,dR < \infty.$$

Similar results holds for $N^{\infty,p}_{\alpha,\beta}(D)$ and $N_{p,\gamma,\beta}(D)$ classes (see [11]).

To obtain parametric representations of $N^{p}_{\alpha,\beta}(D)$, $N^{\infty,p}_{\alpha,\beta}(D)$ and $N_{p,\gamma,\beta}(D)$ classes via $B_\alpha(z,\{z_k\})$ infinite Blaschke type products we can use some embeddings and known parametric representations for analytic classes of area Nevanlinna type with quazinorms $\int_0^1 \left(\int_T \log^+|f(|z|\xi)|d\xi\right)^p (1-|z|)^\alpha dm_2(z) < \infty$, for certain $0<p<\infty$, $\alpha>-1$, that were obtained earlier by other authors.

First we formulate a result that will be used by us via B_t product.

Theorem F: Let $0<p<\infty$, $\alpha>0$. Then $S^p_\alpha(D)$ coinciding with the class of f functions such that

$$f(z)=e^{i\alpha+mK_\beta}z^m B_\beta(z,\{a_k\})\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{2}{(1-e^{-i\varphi}z)^{\beta+1}}-1\right)\psi(e^{i\varphi})d\varphi\right), z\in D,$$

$\{a_k\}_{k=1}^\infty$ and $0<|a_k|\le|a_{k+1}|$, $k=1,2,\dots$, is arbitrary sequences of points from D, so that

$$\int_0^1 n^p(r,f)(1-r)^{\alpha+p}dr<\infty,$$

where $\beta\in\left(\frac{\alpha+1}{p},\ \frac{\alpha+1}{p}+2\right)$, $\psi\in B^s_{1,p}(T)$, $s=\beta-\frac{\alpha+1}{p}$, $\psi(e^{i\theta})=\lim_{r\to1-0}\frac{1}{\Gamma(\beta)}\int_0^r(r-t)^{\beta-1}\ln|f(te^{i\varphi})|dt$ and $K_\beta=\beta\sum_{k=1}^\infty\frac{1}{k(k+\beta)}$.

Now it is clear that obtain a parametric representations of classes we study in this paper via $B_t(z,\{z_k\})$ all we have to do is to show, for example, that if $f\in X$, $X=N^p_{\alpha,\beta}(D)$ or $X=N^{\infty,p}_{\alpha,\beta}(D)$ or $X=N_{p,\gamma,\beta}(D)$, then $f\in S^1_\tau(D)$ for some big enough $\tau>0$ then apply Theorem F we just formulated above. To do that we formulate the following propositions.

Note that we collect several such propositions below and they can be interesting also as separate statements and relate various analytic area Nevanlinna type spaces to each other.

Proposition 1: Let $f\in H(D)$. Let $\beta>-1$, $\gamma\ge0$, $0<q<\infty$. Then

$$\left(\int_0^1(1-\tau)^{\beta+(\gamma+1)q}\left(\int_T\log^+|f(\tau\xi)|dm(\xi)\right)^q d\tau\right)^{\frac1q}$$

$$\le c_1\left(\int_0^1(1-\tau)^{\beta}\left(\int_{|z|<\tau}\log^+|f(z)|(1-|z|)^\gamma dm_2(z)\right)^q d\tau\right)^{\frac1q}.$$

Even more general result is valid with the same proof (see Proposition 2 below). We will consider for simplicity only this case. Similar estimate can be proved for $N^{\infty,p}_{\alpha,\beta}(D)$.

Let now

$$L(A^{p,q}_\gamma)(D)=\{f\in H(D):\|f\|_{L(A^{p,q}_\gamma)}=\int_0^1\left(\int_{-\pi}^{\pi}\left(\ln^+|f(re^{i\varphi})|\right)^p d\varphi\right)^{q/p}(1-r)^\gamma dr<\infty\},$$

where $0 < p < \infty$, $0 < q < \infty$, $\gamma > -1$ and

$$L(F^{p,q}_{\gamma})(D) = \{f \in H(D) : \|f\|_{L(F^{p,q}_{\gamma})} = \int_{-\pi}^{\pi}\left(\int_0^1 \left(\ln^+|f(re^{i\varphi})|\right)^q (1-r)^{\gamma} dr\right)^{p/q} d\varphi < \infty\},$$

where $0 < p < \infty$, $0 < q < \infty$, $\gamma > -1$.

Proofs of Proposition 2 and Proposition 3 as it follows from [6] and [10] are based on arguments from [6] and [10] and their are valid for subharmonic function $\left(\log^+|f(z)|\right)^s$ for any $s \geq 1$.

Note that Proposition 2 extends Proposition 1 and Proposition ??.

Proposition 2: Let $p \geq 1$, $q \in (0,\infty)$, $\alpha > -1$, $\beta > -1$, $\tau = \beta + \frac{q}{p}(\alpha+1)$ then

$$\int_0^1 \left(\int_{|z|<R} \left(\log^+|f(z)|\right)^p (1-|z|)^{\alpha} dm_2(z)\right)^{q/p} (1-R)^{\beta} dR < \infty$$

if and only if

$$\int_0^1 \left(\int_T \left(\log^+|f(r\xi)|\right)^p dm(\xi)\right)^{q/p} (1-|z|)^{\tau} dr < \infty.$$

Proposition 3: Let $1 \leq \min(p,q) \leq s$ and $\gamma > -1$. Then we have

$$\left(\int_D \left(\log^+|f(w)|\right)^s (1-|w|)^{s(\gamma+1)/q+s/p-2} dm_2(w)\right)^{1/s} \leq c_3 \|f\|_{L(A^{p,q}_{\gamma})},\ f \in L(A^{p,q}_{\gamma})(D), \tag{6}$$

$$\left(\int_D \left(\log^+|f(w)|\right)^s (1-|w|)^{s(\gamma+1)/q+s/p-2} dm_2(w)\right)^{1/s} \leq c_4 \|f\|_{L(F^{p,q}_{\gamma})},\ f \in L(F^{p,q}_{\gamma})(D). \tag{7}$$

Proposition 4: Let $q \geq 1$ and $p \leq s$. Then

$$\left(\int_0^1 T^s_q(r,f)(1-|z|)^{\alpha} d|z|\right)^{p/s} \leq c_5 \int_0^1 (1-r)^{\tau} \left(\sup_{0<\rho<r} T_q(\rho,f)(1-|\rho|)^{\gamma}\right)^p dr,$$

for the following values of indexes:
$\alpha > -1$, $p,\ q,\ s \in (0,\infty)$, $\gamma \geq 0$, $\tau = (\alpha+1)(p/s) - \gamma p - 1$.

The easy proof of Proposition 4 follows from dyadic decomposition of unit interval and growing properties of $T_q(r,f)$ function immediately.

We show only particular case of estimate in Proposition 4, the general case is the same.

Let $\tau_n = 1 - \frac{1}{2^n}$, $n \in \mathbb{N}$, $p \leq 1$, $\widetilde{f}(z) = \log^+|f(z)|$. Then we have based on basic properties of Nevanlinna characteristics and dyadic decomposition of the unit interval

$$\left(\int_D \widetilde{f}(z)(1-|z|)^{\alpha} dm_2(z)\right)^p \lesssim \sum_{k=1}^{\infty} 2^{-kp(\alpha+2)} \left(M_1(\tau_k, \widetilde{f})\right)^p$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-kp(\alpha+1)} \sup_{0<\rho\leq\tau_k} \left(M_1(\rho,\widetilde{f})(1-\rho)^{\gamma}\right)^p 2^{k\gamma p}$$

$$\lesssim \sum_{k=1}^{\infty} \int_{1-2^{-k-2}}^{1-2^{-k-3}} (1-|z|)^{(\alpha+1)p-\gamma p-1} \sup_{0<\rho\leq|z|} \left(M_1(\rho,\widetilde{f})(1-\rho)^{\gamma}\right)^p d|z|$$

$$\leq C\int_0^1 (1-|z|)^{(\alpha+1)p-\gamma p-1} \left(\sup_{0<\rho\leq|z|} T(\tau,f)(1-\tau)^{\gamma}\right)^p d|z|.$$

Let us show assertions in Proposition 1.

$$\int_0^1 (1-r)^{\beta+(\gamma+1)q} \left(\int_T \widetilde{f}(r\xi)dm(\xi)\right)^q d\tau$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k(\beta+(\gamma+1)q+1)} \left(M_1(\tau_k,\widetilde{f})\right)^q$$

$$\lesssim \sum_{k=1}^{\infty} \left(\int_{\tau_k<|z|<\tau_{k+1}} \widetilde{f}(z)(1-|z|)^{\gamma} dm_2(z)\right)^q 2^{-k(\beta+1)}$$

$$\lesssim \sum_{k=1}^{\infty} \int_{\tau_{k+1}}^{\tau_{k+2}} (1-\tau)^{\beta} \left(\int_{|z|<\tau} \widetilde{f}(z)(1-|z|)^{\gamma} dm_2(z)\right)^q d\tau$$

$$\lesssim \int_0^1 (1-\tau)^{\beta} \left(\int_{|z|<\tau} \widetilde{f}(z)(1-|z|)^{\gamma} dm_2(z)\right)^q d\tau.$$

Remark 3: Classes of analytic functions of area Nevanlinna type with quasinorms that can be seen in the first part of Proposition 4 studied in [11]. There complete descriptions od zeros and parametric representations via other $\Pi_t(z,\{z_k\})$ infinite products are given.

Note, for example, obviously $B_t(z,z_k)$ belongs to spaces with quasinorms

$$\int_0^1 \left(\int_{|z|\leq R} \ln^+|f(z)|(1-|z|)^{\alpha} dm_2(z)\right)^q \sup_R (1-R)^{\beta} dR$$

by Proposition 2, $0<p<\infty$, $\tau=\beta+\dfrac{q}{p}(\alpha+1)$ and Theorem D for some values of t parameter.

Note to use Theorem F we have to apply reverse embedding in Proposition 2. Estimates of Proposition 1 and Proposition 2 give many new results on parametric representation via $B_t(z,z_k)$ product we give such examples below.

To get immediately new assertions of type $B_t(z,z_k)\in X$, where X is a certain new large area Nevanlinna type space we simply will use the following elementar embeddings:

(A) $(1-R)^{\beta}\int_0^R \left(\int_T \ln^+|f(|z|\xi)|dm(\xi)\right)^p (1-|z|)^{\alpha} d|z| \leq$
$\leq C_1\int_0^1 \left(\int_T \ln^+|f(|z|\xi)|dm(\xi)\right)^p (1-|z|)^{\alpha} d|z| = \|f\|_{S^p_\alpha}$, $0<p<\infty, \alpha>-1, \beta\geq 0$;

(B) $\int_0^1 \left(\int_{|z|\leq R} \ln^+|f(z)|(1-|z|)^{\alpha} dm_2(z)\right)^p (1-R)^{\beta} dR \leq C_2\|f\|_{S^1_\alpha}$,
$0<p<\infty,\ \alpha>-1,\ \beta>-1$;

(C) $\|f\|_{NA_{p,\gamma,\nu}} \le C_3\|f\|_{S^1_{\gamma-1}}$, $0 < p < \infty$, $\gamma > 0$, $\nu > -1$.

Indeed area Nevanlinna type spaces with quasinorms on the right side were studied and assertions of the following type $B_t \in S^p_\alpha$ were given by us above. It remains to apply (A) − (C).

We have the following result from Corollary 1 and (A) − (C).

Theorem 3: Let $\alpha > -1$, $p \in (0,\infty)$ and $\{z_k\}_{k=1}^{\infty}$ be a sequences of complex numbers in the unit disk D such that $0 < |z_k| \le 1$, $k = 1,\dots$

1) $\int_0^1 (1-r)^{\alpha+p} n^p(r)dr < \infty$

then we have $B_t(z,\{z_k\}) \in N^{\infty,p}_{\alpha,\beta}$, $\alpha > -1$, $\beta \ge 0$

if $0 < p \le 1$, $\dfrac{\alpha+1}{p} < t < 2 + \dfrac{\alpha+1}{p}$

if $1 < p < \infty$, $1 + \dfrac{\alpha}{p} < t < 2 + \dfrac{\alpha+1}{p}$.

2) $\int_0^1 (1-r)^{\alpha+1} n(r)dr < \infty$

then we have $B_t(z,\{z_k\}) \in N^p_{\alpha,\beta}$, $\alpha > -1$, $\beta > -1$, $t \in (\alpha+1, \alpha+3)$

3) $\int_0^1 (1-r)^{\gamma} n(r)dr < \infty$

then we have $B_t(z,\{z_k\}) \in N_{p,\gamma,\nu}$, $\nu > -1$, $\gamma > 0$, $t \in (\gamma, \gamma+2)$.

In our next theorem we provide new interesting parametric representations of our large $N_{\alpha,\beta,\gamma}$, $N^p_{\alpha,\beta}$, $N^{\infty,p}_{\alpha,\beta}$ area Nevanlinna type spaces in the unit disk via $B_t(z, z_k)$ infinite products based on Theorem F and embeddings in Propositions 1 - 4.

Theorem 4: 1) Let $f \in N_{p,\gamma,\tau}$, $\alpha > 0$, $p < 1$, $\alpha > \gamma - 1$, $\tau = (\alpha+1)p - \gamma p - 1$. Then assertions of Theorem F (parametric representations) are valid for $p = 1$.

2) Let $f \in N^p_{\gamma,\beta}$, $\gamma \ge 0$, $0 < p < \infty$, $\beta > -1$. Then assertions of Theorem F are valid for $\alpha = \beta + (\gamma+1)p > 0$.

Various other assertions similar to those in our last two theorems can be proven also based on estimates above which relate many area Nevanlinna spaces in the unit disk with each other.

Conclusion

Our theorems can be probably used to obtain new results on parametric representations and Blashcke type products in similar type large area Nevanlinna type spaces which we introduced in this paper but in more general situation namely in so-called large area Nevanlinna type spaces in the circular ring.

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