MATHEMATICS

https://doi.org/10.26117/2079-6641-2023-42-1-69-79

Research Article

Full text in English

MSC 35K05, 35K15

Control Problem Concerned With the Process of Heating a Thin Plate

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Abstract. Previously, a mathematical model for the following problem was considered. On a part of the border of the right rectangle there is a heater with controlled temperature. It is required to find such a mode of its operation that the average temperature in some region reaches some given value. In this paper, we consider a boundary control problem associated with a parabolic equation on a right rectangle. On the part of the border of the considered domain, the value of the solution with control parameter is given. Restrictions on the control are given in such a way that the average value of the solution in some part of the considered domain gets a given value. The auxiliary problem is solved by the method of separation of variables, while the problem in consideration is reduced to the Volterra integral equation. In addition, the definition of the generalized solution of the given initial-boundary problem is given in the article and the existence of such a solution is proved. The solution of Volterra's integral equation was found by the Laplace transform method and the existence theorem for admissible control functions was proved. It is also shown that the initial value of the admissible control function is equal to zero using the change of variable in the integral equation. The proof of this comes from the fact that the kernels of the integral equations are positive and finite, and the system has a single-valued solution.

Key words: parabolic equation, system of integral equations, initial-boundary problem, admissible control, Laplace transform.

Received: 11.01.2023; Revised: 14.03.2023; Accepted: 20.03.2023; First online: 15.04.2023

For citation. Dekhkonov F. N. Control problem concerned with the process of heating a thin plate. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2023, **42**: 1, 69-79. EDN: DJZRAU. https://doi.org/10.26117/2079-6641-2023-42-1-69-79.

Funding. Not applicable.

Competing interests. There are no conflicts of interest regarding authorship and publication.

Contribution and Responsibility. The author participated in the writing of the article and is fully responsible for submitting the final version of the article to print.

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МАТЕМАТИКА
https://doi.org/10.26117/2079-6641-2023-42-1-69-79
Научная статья
Полный текст на английском языке
УДК 517.977.5

Задача управления процессом нагрева тонкой пластины

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Аннотация. Ранее была рассмотрена математическая модель следующей задачи. На части границы правого прямоугольника расположен нагреватель с регулируемой температурой. Требуется найти такой режим его работы, чтобы средняя температура в каком-либо районе достигала некоторого заданного значения. В данной работе рассматривается задача граничного управления, связанная с параболическим уравнением на правом прямоугольнике. На части границы рассматриваемой области указано значение решения с управляющим параметром. Ограничения на управление задаются таким образом, чтобы среднее значение решения в некоторой части рассматриваемой области принимало заданное значение. Вспомогательная задача решается методом разделения переменных, а рассматриваемая задача сводится к интегральному уравнению Вольтерра. Кроме того, в статье дается определение обобщенного решения данной начально-краевой задачи и доказывается существование такого решения. Методом преобразования Лапласа найдено решение интегрального уравнения Вольтерра и доказана теорема существования допустимых управляющих функций. Также показано, что начальное значение допустимой функции управления равно нулю с помощью замены переменной в интегральном уравнении. Доказательство этого исходит из того, что ядра интегральных уравнений положительны и конечны, а система имеет однозначное решение.

Ключевые слова: параболическое уравнение, система интегральных уравнений, начально-краевая задача, допустимое управление, преобразование Лапласа.

Получение: 11.01.2023; Исправление: 14.03.2023; Принятие: 20.03.2023; Публикация онлайн: 15.04.2023

Для цитирования. Dekhkonov F. N. Задача управления процессом нагрева тонкой пластины // *Вестник КРАУНЦ. Физ.-мат. науки.* 2023. Т. 42. № 1. C. 69-79. EDN: DJZRAU. https://doi.org/10.26117/2079-6641-2023-42-1-69-79.

Финансирование. Исследование выполнялось без финансовой поддержки фондов.

Конкурирующие интересы. Конфликтов интересов в отношении авторства и публикации нет.

Авторский вклад и ответственность. Автор участвовал в написании статьи и полностью несет ответственность за предоставление окончательной версии статьи в печать.

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Introduction

Consider the following mathematical model of the heat conduction process along the domain $\Omega = \{(x,y) \in R^2 : 0 < x < a, 0 < y < b\}$:

$$u_t = u_{xx} + u_{yy}, \ (x,y) \in \Omega, \quad t > 0 \tag{1}$$

with boundary value conditions

$$u|_{x=0} = \varphi(y)\mu_1(t), \quad u|_{x=a} = \psi(y)\mu_2(t), \quad 0 < x < a, \tag{2}$$

$$u|_{y=0} = 0, \quad u|_{y=b} = 0, \quad 0 < y < b, \quad t > 0 \tag{3}$$

and initial value condition

$$u(x,y,0) = 0, \quad 0 \le x \le a, \quad 0 \le y \le b. \tag{4}$$

Let $M_j > 0$ be some given constants. We say that the functions $\mu_j(t)$ are *admissible control*, if this functions are smooth on the half-line $t \ge 0$ and satisfies the constraints:

$$\mu_j(0) = 0, \quad |\mu_j(t)| \le M_j, \quad j = 1,2.$$

Set

$$\varphi_m = \frac{2}{b}\int_0^b \varphi(y)\sin\frac{m\pi y}{b}\,dy, \quad \psi_m = \frac{2}{b}\int_0^b \psi(y)\sin\frac{m\pi y}{b}\,dy, \quad m = 1,2,... \tag{5}$$

Assume that the functions $\varphi(y), \psi(y) \in W_2^2[0,b]$ are smooth and satisfies conditions

$$\varphi(0) = \varphi(b) = 0, \quad \psi(0) = \psi(b) = 0, \quad \varphi_1\psi_2 - \varphi_2\psi_1 \neq 0, \quad 0 \le y \le b. \tag{6}$$

Consider the following eigenvalue problem

$$\Delta X_{nm}(x,y) + \lambda_{mn} X_{nm}(x,y) = 0, \quad 0 < x < a, \quad 0 < y < b,$$

with boundary conditions

$$X_{nm}(x,y)\big|_{\partial\Omega} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b.$$

Then we have (see, [21])

$$\lambda_{nm} = (n\pi/a)^2 + (m\pi/b)^2, \quad X_{nm}(x,y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}, \quad n,m = 1,2,... \tag{7}$$

In the present work we consider the following problem:

Problem A. For the given functions $\theta_j(t)$ Problem A consists in looking for the admissible controls $\mu_j(t)$ such that the solution $u(x,y,t)$ of the initial-boundary value problem (1)-(4) exists and for all $t > 0$ satisfies the equations

$$\frac{4}{ab}\int_0^a\int_0^b X_{1j}(x,y)\,u(x,y,t)\,dx\,dy = \theta_j(t), \quad j = 1,2. \tag{8}$$

We recall that the time-optimal control problem for partial differential equations of parabolic type was first investigated in [7] and [8]. More recent results concerned with this problem were established in [1]- [4], [6], [11]- [16]. Detailed information on the problems of optimal control for distributed parameter systems is given in [9] and in the monographs [10], [17] and [19].

General numerical optimization and optimal boundary control have been studied in a great number of publications such as [5]. The practical approaches to optimal control of the heat equation are described in publications like [20].

System of the Integral Equations

Let B be the Banach space and $T > 0$. Denote by $C([0,T] \to B)$ the Banach space of all continuous maps $u : [0,T] \to B$ with the norm $\|u\| = \max_{0\leq t\leq T} |u(t)|$. By symbol $\widetilde{W}_2^1(\Omega)$ we denote the subspace of the Sobolev space $W_2^1(\Omega)$ formed by functions, whose trace is equal to zero. Note that since $\widetilde{W}_2^1(\Omega)$ is closed, then the sum of a series of functions from $\widetilde{W}_2^1(\Omega)$ converging in metric $W_2^1(\Omega)$ also belongs to $\widetilde{W}_2^1(\Omega)$.

Definition. When we say a solution of problem (1)–(4), we mean function $u(x,y,t)$ represented in the form

$$u(x,y,t) = \psi(y)\,\mu_2(t) + \frac{a-x}{a}\Big(\varphi(y)\,\mu_1(t) - \psi(y)\,\mu_2(t)\Big) - \nu(x,y,t),$$

where the function $\nu(x,y,t)$ is a generalized solution from the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ of the problem

$$\nu_t(x,y,t) - \Delta\nu(x,y,t) = \psi(y)\,\mu_2'(t) + \frac{a-x}{a}\Big(\varphi(y)\,\mu_1'(t) - \psi(y)\,\mu_2'(t)\Big) -$$

$$-\left(\psi''(y)\mu_2(t) + \frac{a-x}{a}\Big(\varphi''(y)\mu_1(t) - \psi''(y)\mu_2(t)\Big)\right)$$

with boundary value conditions

$$\nu(x,y,t)\,|_{\partial\Omega} = 0$$

and initial value condition

$$\nu(x,y,0) = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b.$$

Consequently, we have (see, [21], [22])

$$\nu(x,y,t) \;=\; \int_0^t \sum_{n,m=1}^{\infty} e^{-\lambda_{nm}(t-s)}\,\frac{2X_{nm}(x,y)}{n\pi} \times$$

$$\times\left(\varphi_m\,\mu_1'(s) - (-1)^n\psi_m\,\mu_2'(s) + \Big(\frac{m\pi}{b}\Big)^2\big[\varphi_m\,\mu_1(s) - (-1)^n\psi_m\,\mu_2(s)\big]\right) ds,$$

where λ_{nm}, X_{mn} defined by (7).

Note that the class $C([0,T]\to \widetilde{W}_2^1(\Omega))$ is a subset of the class $W_2^1(\Omega)$ considered in the monograph [18] in order to define a problem with homogeneous boundary conditions. Thus, the generalized solution introduced above is also a generalized solution in the sense of monograph [18]. However, unlike a solution from the class $W_2^1(\Omega)$, which is guaranteed to have a trace of almost all $t\in[0,T]$, a solution from the class $C([0,T]\to \widetilde{W}_2^1(\Omega))$ continuously depends of $t\in[0,T]$ in the metric $L_2(\Omega)$.

Proposition. Let $\mu_1(t)$, $\mu_2(t)$ are smooth functions on the half-line $t\geq 0$. Then the function

$$u(x,y,t) = \frac{2\pi}{a^2}\int_0^t \mu_1(s)\sum_{n,m=1}^{\infty} n\,\varphi_m X_{nm}(x,y)\,e^{-\lambda_{nm}(t-s)}\,ds-$$

$$-\frac{2\pi}{a^2}\int_0^t \mu_2(s)\sum_{n,m=1}^{\infty}(-1)^n n\,\psi_m X_{nm}(x,y)\,e^{-\lambda_{nm}(t-s)}\,ds, \qquad (9)$$

is the solution of the initial-boundary value problem (1)-(4).

Proof. We rewrite the solution to the problem in the form

$$u(x,y,t)=\psi(y)\,\mu_2(t)+\frac{a-x}{a}\big(\varphi(y)\,\mu_1(t)-\psi(y)\,\mu_2(t)\big)-\int_0^t\sum_{n,m=1}^{\infty}\frac{2X_{nm}(x,y)}{n\pi}\,e^{-\lambda_{nm}(t-s)}\times$$

$$\times\left(\varphi_m\,\mu_1'(s)-(-1)^n\psi_m\,\mu_2'(s)+\big(\frac{m\pi}{b}\big)^2\big[\varphi_m\,\mu_1(s)-(-1)^n\psi_m\,\mu_2(s)\big]\right)ds.$$

We show that function $v(x,y,t)$ belongs to class $C([0,T]\to \widetilde{W}_2^1(\Omega))$. For this, it is enough to prove that the gradient of this function, taken in $(x,y)\in\Omega$, continuously depends on $t\in[0,T]$ in the norm of the space $L_2(\Omega)$. According to Parseval's equality, the norm of this gradient is

$$\|\nabla v\|^2_{L_2(\Omega)}=\sum_{n,m=1}^{\infty}\frac{1}{n^2}\lambda_{nm}\,b^2_{nm}(t)$$

where

$$b_{nm}(t)=\int_0^t e^{-\lambda_{nm}(t-s)}\,\varphi_m\left(\mu_1'(s)+\big(\frac{m\pi}{b}\big)^2\mu_1(s)\right)ds+$$

$$+\int_0^t e^{-\lambda_{nm}(t-s)}\,(-1)^{n+1}\psi_m\left(\mu_2'(s)+\big(\frac{m\pi}{b}\big)^2\mu_2(s)\right)ds.$$

From the Cauchy-Bunyakovsky inequality, we obtain the following estimate

$$|b_{nm}(t)|\leq|\varphi_m|\left(\frac{C_1}{\sqrt{\lambda_{nm}}}+C_2\frac{m^2}{\lambda_{nm}}\right)+|\psi_m|\left(\frac{C_3}{\sqrt{\lambda_{nm}}}+C_4\frac{m^2}{\lambda_{nm}}\right)\leq$$

$$\leq C_5\frac{m\,(|\varphi_m|+|\psi_m|)}{\sqrt{\lambda_{nm}}},\quad t\geq 0.$$

From (5), we write

$$\varphi_m = \frac{2}{b}\int_0^b \varphi(y)\sin\frac{m\pi y}{b}\,dy = -\frac{2}{b}\,\varphi(y)\frac{b}{m\pi}\cos\frac{m\pi y}{b}\Big|_{y=0}^{y=b} +$$

$$+\frac{2}{m\pi}\int_0^b \varphi'(y)\cos\frac{m\pi y}{b}\,dy = \frac{2}{m\pi}\int_0^b \varphi'(y)\cos\frac{m\pi y}{b}\,dy = \frac{b}{m\pi}\varphi'_m,$$

and

$$\psi_m = \frac{2}{b}\int_0^b \psi(y)\sin\frac{m\pi y}{b}\,dy = \frac{b}{m\pi}\psi'_m.$$

Consequently, we have

$$\|\nabla v\|^2_{L_2(\Omega)} \le C_0 \sum_{n,m=1}^{\infty} \frac{m^2(|\varphi_m|+|\psi_m|)^2}{n^2} \le$$

$$\le 2C_0\frac{\pi^2}{6}\frac{b^2}{\pi^2}\sum_{m=1}^{\infty}\left(|\varphi'_m|^2+|\psi'_m|^2\right) = C\left(\|\varphi'\|^2_{L_2[0,b]}+\|\psi'\|^2_{L_2[0,b]}\right).$$

□

From the condition (8) and the solution of the problem (1)-(4), we may write

$$\theta_j(t) = \frac{4}{ab}\int_0^a\int_0^b X_{1j}(x,y)\,u(x,y,t)\,dx\,dy =$$

$$\frac{4}{ab}\frac{2\pi}{a^2}\int_0^t \mu_1(s)ds\sum_{n,m=1}^{\infty} n\,\varphi_m\,e^{-\lambda_{nm}(t-s)}\int_0^a\int_0^b X_{1j}(x,y)X_{nm}(x,y)\,dx\,dy-$$

$$-\frac{4}{ab}\frac{2\pi}{a^2}\int_0^t \mu_2(s)\,ds\sum_{n,m=1}^{\infty}(-1)^n n\,\psi_m\,e^{-\lambda_{nm}(t-s)}\int_0^a\int_0^b X_{1j}(x,y)X_{nm}(x,y)\,dx\,dy =$$

$$= \frac{2\pi}{a^2}\int_0^t \varphi_j\,e^{-\lambda_{1j}(t-s)}\mu_1(s)ds + \frac{2\pi}{a^2}\int_0^t \psi_j\,e^{-\lambda_{1j}(t-s)}\mu_2(s)\,ds. \quad (10)$$

Set

$$A_j(t) = \frac{2\pi}{a^2}\varphi_j\,e^{-\lambda_{1j}t}, \quad B_j(t) = \frac{2\pi}{a^2}\psi_j\,e^{-\lambda_{1j}t}, \quad j=1,2, \quad (11)$$

where φ_j, ψ_j defined by (5).

Then we get system of the integral equations

$$\int_0^t A_j(t-s)\,\mu_1(s)ds + \int_0^t B_j(t-s)\,\mu_2(s)ds = \theta_j(t), \quad t>0, \quad j=1,2. \quad (12)$$

Denote by $W(M_0)$ the set of function $\theta \in W_2^2(-\infty,+\infty)$, $\theta(t)=0$ for $t \leq 0$ which satisfies the condition

$$\|\theta\|_{W_2^2(R_+)} \leq M_0. \tag{13}$$

Theorem. *There exists* $M_0>0$ *such that for any functions* $\theta_j \in W(M_0)$ *the solutions* $\mu_j(t)$ *of the system* (12) *exists and satisfies conditions*

$$|\mu_j(t)| \leq M_j, \quad j=1,2.$$

Proof of the Theorem

For solve the system (12), we use the Laplace transform method. We introduce the notation

$$\widetilde{\mu}_j(p)=\int\limits_0^\infty e^{-pt}\,\mu_j(t)\,dt, \quad p=\beta+i\,\xi, \quad \beta>0.$$

Then, we use Laplace transform

$$\widetilde{\theta}_j(p)=\int\limits_0^\infty e^{-pt}\,dt\int\limits_0^t A_j(t-s)\mu_1(s)ds+\int\limits_0^\infty e^{-pt}\,dt\int\limits_0^t B_j(t-s)\mu_2(s)ds=$$

$$=\widetilde{A}_j(p)\,\widetilde{\mu}_1(p)+\widetilde{B}_j(p)\,\widetilde{\mu}_2(p). \tag{14}$$

According to (11), we get

$$\widetilde{A}_j(p)=\int\limits_0^\infty A_j(t)e^{-pt}\,dt=\frac{2\pi}{a^2}\frac{\varphi_j}{p+\lambda_{1j}}, \tag{15}$$

and

$$\widetilde{B}_j(p)=\int\limits_0^\infty B_j(t)e^{-pt}\,dt=\frac{2\pi}{a^2}\frac{\psi_j}{p+\lambda_{1j}}, \quad j=1,2, \tag{16}$$

where α_j,β_j defined by (5).

According to the condition (6) $\varphi_1\psi_2-\varphi_2\psi_1\neq 0$. Consequently, from the system (14) and (15), (16), we can obtain

$$\widetilde{\mu}_1(p)=\frac{a^2}{2\pi}\frac{\psi_1\,(\lambda_{12}+p)}{\varphi_2\psi_1-\varphi_1\psi_2}\widetilde{\theta}_2(p)-\frac{a^2}{2\pi}\frac{\psi_2\,(\lambda_{11}+p)}{\varphi_2\psi_1-\varphi_1\psi_2}\widetilde{\theta}_1(p), \tag{17}$$

and

$$\widetilde{\mu}_2(p)=\frac{a^2}{2\pi}\frac{\varphi_1\,(\lambda_{12}+p)}{\varphi_1\psi_2-\varphi_2\psi_1}\widetilde{\theta}_2(p)-\frac{a^2}{2\pi}\frac{\varphi_2\,(\lambda_{11}+p)}{\varphi_1\psi_2-\varphi_2\psi_1}\widetilde{\theta}_1(p), \tag{18}$$

Then, when $\beta\to 0$ from (17) and (18), we obtain the following equalities

$$\mu_1(t)=\frac{a^2}{4\pi^2}\int\limits_{-\infty}^{+\infty}\left(\frac{\psi_1\,(\lambda_{12}+i\xi)}{\varphi_2\psi_1-\varphi_1\psi_2}\widetilde{\theta}_2(i\xi)-\frac{\psi_2\,(\lambda_{11}+i\xi)}{\varphi_2\psi_1-\varphi_1\psi_2}\widetilde{\theta}_1(i\xi)\right)e^{i\xi t}d\xi, \tag{19}$$

and

$$\mu_2(t) = \frac{a^2}{4\pi^2} \int\limits_{-\infty}^{+\infty} \left(\frac{\varphi_1(\lambda_{12}+i\xi)}{\varphi_1\psi_2-\varphi_2\psi_1}\, \widetilde{\theta}_2(i\xi) - \frac{a^2}{2\pi} \frac{\varphi_2(\lambda_{11}+i\xi)}{\varphi_1\psi_2-\varphi_2\psi_1}\, \widetilde{\theta}_1(i\xi) \right) e^{i\xi t} d\xi. \tag{20}$$

Lemma. *Let* $\theta(t) \in W(M_0)$. *Then for the image of the function* $\theta(t)$ *the following inequality*

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)| \sqrt{1+\xi^2}\, d\xi \le C\, \|\theta\|_{W_2^2(R_+)},$$

is valid.

Proof. We calculate the Laplace transform of a function $\theta(t)$ as follows

$$\widetilde{\theta}(\beta+i\xi) = \int\limits_0^\infty e^{-(\beta+i\xi)t} \theta(t)\, dt = -\theta(t)\, \frac{e^{-(\beta+i\xi)t}}{\beta+i\xi} \Big|_{t=0}^{t=\infty} + \frac{1}{\beta+i\xi} \int\limits_0^\infty e^{-(\beta+i\xi)t} \theta'(t)\, dt,$$

then, we get

$$(\beta+i\xi)\, \widetilde{\theta}(\beta+i\xi) = \int\limits_0^\infty e^{-(\beta+i\xi)t} \theta'(t)\, dt,$$

and for $\beta \to 0$ we have

$$i\xi \widetilde{\theta}(i\xi) = \int\limits_0^\infty e^{-i\xi t} \theta'(t)\, dt.$$

Also, we can write the following equality

$$(i\xi)^2 \widetilde{\theta}(i\xi) = \int\limits_0^\infty e^{-i\xi t} \theta''(t)\, dt.$$

Then we have

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2 (1+\xi^2)^2 d\xi \le C_1\, \|\theta\|^2_{W_2^2(R_+)}. \tag{21}$$

Consequently, according to (21) we get the following estimate

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)| \sqrt{1+\xi^2}\, d\xi = \int\limits_{-\infty}^{+\infty} \frac{|\widetilde{\theta}(i\xi)|(1+\xi^2)}{\sqrt{1+\xi^2}} \le$$

$$\le \left(\int\limits_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2 (1+\xi^2)^2 d\xi \right)^{1/2} \left(\int\limits_{-\infty}^{+\infty} \frac{1}{1+\xi^2} d\xi \right)^{1/2} \le C\, \|\theta\|_{W_2^2(R_+)}. \tag{22}$$

□

Proof of the Theorem. Note that

$$|\lambda_{nm} + i\,\xi| = \sqrt{\lambda_{nm}^2 + \xi^2} \le (1+\lambda_{nm})\, \sqrt{1+\xi^2}, \quad n, m \in N.$$

According to (19), (20) and Lemma, we can write

$$|\mu_1(t)| \leq \frac{a^2}{4\pi^2}\int\limits_{-\infty}^{+\infty}\left|\frac{\psi_1}{\varphi_2\psi_1-\varphi_1\psi_2}\right||\lambda_{12}+i\xi||\widetilde{\theta}_2(i\xi)|d\xi+$$

$$+\frac{a^2}{4\pi^2}\int\limits_{-\infty}^{+\infty}\left|\frac{\psi_2}{\varphi_2\psi_1-\varphi_1\psi_2}\right||\lambda_{11}+i\xi||\widetilde{\theta}_1(i\xi)|d\xi \leq$$

$$\leq \frac{a^2 C_1(1+\lambda_{12})}{4\pi^2}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}|\widetilde{\theta}_2(i\xi)|d\xi+\frac{a^2 C_2(1+\lambda_{11})}{4\pi^2}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}|\widetilde{\theta}_1(i\xi)|d\xi \leq$$

$$\leq \frac{a^2 C_1 C(1+\lambda_{12})}{4\pi^2}\|\theta_2\|_{W_2^2(R_+)}+\frac{a^2 C_2 C(1+\lambda_{11})}{4\pi^2}\|\theta_1\|_{W_2^2(R_+)} \leq$$

$$\leq \frac{a^2 C_1 C(1+\lambda_{12})}{4\pi^2}M_0+\frac{a^2 C_2 C(1+\lambda_{11})}{4\pi^2}M_0 = M_1,$$

and also, we obtain

$$|\mu_2(t)| \leq \frac{a^2}{4\pi^2}\int\limits_{-\infty}^{+\infty}\left|\frac{\varphi_1}{\varphi_1\psi_2-\varphi_2\psi_1}\right||\lambda_{12}+i\xi||\widetilde{\theta}_2(i\xi)|d\xi+$$

$$+\frac{a^2}{4\pi^2}\int\limits_{-\infty}^{+\infty}\left|\frac{\varphi_2}{\varphi_1\psi_2-\varphi_2\psi_1}\right||\lambda_{11}+i\xi||\widetilde{\theta}_1(i\xi)|d\xi \leq$$

$$\leq \frac{a^2 C_3(1+\lambda_{12})}{4\pi^2}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}|\widetilde{\theta}_2(i\xi)|d\xi+\frac{a^2 C_4(1+\lambda_{11})}{4\pi^2}\int\limits_{-\infty}^{+\infty}\sqrt{1+\xi^2}|\widetilde{\theta}_1(i\xi)|d\xi \leq$$

$$\leq \frac{a^2 C_3 C(1+\lambda_{12})}{4\pi^2}\|\theta_2\|_{W_2^2(R_+)}+\frac{a^2 C_4 C(1+\lambda_{11})}{4\pi^2}\|\theta_1\|_{W_2^2(R_+)} \leq$$

$$\leq \frac{a^2 C_3 C(1+\lambda_{12})}{4\pi^2}M_0+\frac{a^2 C_4 C(1+\lambda_{11})}{4\pi^2}M_0 = M_2.$$

It remains to verify the fulfillment of condition $\mu_j(0)=0$. For that we rewrite system (12) as follows:

$$\int\limits_0^t A_j(s)\,\mu_1(t-s)ds+\int\limits_0^t B_j(s)\,\mu_2(t-s)ds = \theta_j(t), \quad j=1,2.$$

By differentiating this system, we have

$$A_j(t)\,\mu_1(0)+B_j(t)\,\mu_2(0)+\int\limits_0^t A_j(s)\,\mu_1'(t-s)ds+\int\limits_0^t B_j(s)\,\mu_2'(t-s)ds = \theta_j'(t), \quad j=1,2.$$

Let us tend $t \to 0$ in this correlation. Then, taking into account the conditions imposed on the functions θ_j, and the fact the functions $A_j(t)$ and $B_j(t)$ are bounded at the point zero, we obtain the desired equality $\mu_j(0)=0$. □

Conclusion

In the theory of boundary control of processes described by partial differential equations, the main problem is to prove the existence of an admissible control parameter. We considered the problem of boundary management in a rectangular area. The difference between this problem and the previous works is that 2 different control functions are considered at the boundary. The existence of such control functions was proved using the Laplace transform method. In our work, we have chosen eigenfunctions as weight functions. This made it much easier to find control functions. Based on the results of the work on the basis of control theory, we can conclude that under certain conditions for these problems, it is possible to implement boundary control for processes associated with parabolic type equations.

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