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Optimal quadrature formulas in the space $\widetilde{W_2}^{(m,m-1)}$ of periodic functions

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This paper is devoted to the process of finding the upper bound for the absolute error of the optimal quadrature formula in the space $\widetilde{W_2}^{(m,m-1)}$ of real-valued, periodic functions. For this the extremal function of the quadrature formula is used. In addition, it is shown that the norm of the error functional for the optimal quadrature formula constructed in the space $\widetilde{W_2}^{(m,m-1)}$ is less than the value of the norm of the error functional for the optimal quadrature formula in the Sobolev space $\widetilde{L_2}^{(m)}$.

Key words: optimal quadrature formula, optimal coefficients, error of quadrature formula, the Hilbert space, the error functional, Fourier transform.

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Introduction

Numerical calculation of integrals of the highly oscillating integrals is one of the more critical problems on numerical analysis because such integrals are widely used in science and technology. The following types of the Fourier integrals are also examples of strongly oscillating integrals for sufficiently large ω:

$$I[\varphi,\omega]=\int_0^1 e^{2\pi i\omega x}\varphi(x)dx. \tag{1}$$

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It is known that the Fourier transforms are very important tools and they are used widely, in particular, in the problems of Computed Tomography (see, for instance [9]). Since in practice we have finite discrete values of a function, we have to approximately calculate the Fourier transforms. Therefore, one has to consider the problem of approximate calculation of the integral with the weight function $\exp(2\pi i\omega x)$. Initially, a formula for the numerical integration of the strongly oscillatory functions one of the non standard numerical integration methods was given by Filon [19].

Subsequently for integrals with different types of highly oscillating functions many special effective methods such as the Filon-type method, Clenshaw-Curtis-Filon type method, modified Clenshaw-Curtis method, Levin type methods, generalized quadrature rule, and Gauss-Laguerre quadrature are developed (see [2, 14, 21, 26] for full details, for instance, [13, 24] and references therein).

Recently, in the works [6, 7, 8, 25] authors constructed optimal quadrature formulas for numerical calculation of Fourier integrals in the Sobolev space $L_2^{(m)}$ and in the Hilber space $W_2^{(m,m-1)}$, and these formulas were applied to approximate reconstruction of Computed Tomography images. Compared with the optimal quadrature formulas in non-periodic case constructed in [6], the approximation formula for the periodic case constructed in the works [7, 17, 18] is much simpler, therefore it is easy to implement and it costs less computation even though both provide similar performances.

In various Hilbert and Banach spaces of periodic and non-periodic functions, optimal quadrature formulas have been constructed by many researchers for integral (1) with $\omega = 0$. The results for this case can be found, for instance, in the books [12, 23, 22] and the literature in them. In particular, some recent results on optimal quadrature and interpolation formulas are obtained in the works [1, 3, 5, 15].

In the work [16, pp. 119–142] weighted optimal quadrature and cubature formulas in the Sobolev space of periodic functions were constructed. In particular, one can get from these results the optimal quadrature formulas for numerical calculation of the integral (1). For example, the practical use of these formulas which were constructed in the Hilbert spaces of non-periodic functions created difficulties in computational work.

Therefore, construction of new optimal quadrature formulas, which are simple in implementation, in various Hilbert spaces of periodic functions is very important. We note that we have constructed the optimal quadrature formula in the case $m = 1$ for integral (1) with $\omega = 0$ in the paper [10]. In the present work we get the results for $m \geq 2$.

Here, we solve the problem of construction of optimal quadrature formula in the Hilbert space $\widetilde{W}_2^{(m,m-1)}(0,1]$ of periodic functions. In the book [23], the optimal quadrature formulas were constructed in the general case in the space of periodic functions, and it was proved that the coefficients of the formulas have the form $\mathring{C}_k = h$. Nevertheless the estimation of the error for the optimal formulas was not given. The main goal of the present work is to find the sharp upper bound for the error of the optimal quadrature formula constructed in the space $\widetilde{W_2}^{(m,m-1)}(0,1]$ of periodic, real-valued functions.

We consider the Hilbert space $W_2^{(m,m-1)}[0,1], m \geq 2$ of non-periodic, real-valued functions $\varphi(x), 0 \leq x \leq 1$, that $(m-1)^{st}$ order derivative is absolute continuous and m^{th} order derivative (in the generalized sense) are square integrable, with the inner product

$$\langle \varphi, \psi \rangle_{W_2{}^{(m,m-1)}} = \int_0^1 (\varphi^{(m)}(x) + \varphi^{(m-1)}(x))(\psi^{(m)}(x) + \psi^{(m-1)}(x))dx, \tag{2}$$

where $\varphi, \psi \in W_2^{(m,m-1)}[0,1]$, and the corresponding norm of the function φ is defined as follows

$$\|\varphi\|_{W_2^{(m,m-1)}} = \left(\int_0^1 \left(\varphi^{(m)}(x) + \varphi^{(m-1)}(x) \right)^2 dx \right)^{1/2}.$$

The last equality is a semi-norm and $\|\varphi\| = 0$ if and only if $\varphi(x) = P_{m-2}(x) + de^{-x}$, where $P_{m-2}(x)$ is a polynomial of degree $(m-2)$ and d is a constant. Every element of the space $W_2^{(m,m-1)}$ is a class of functions that are differ from each other by a linear combination of any polynomial of degree $(m-2)$ and e^{-x}. The space $W_2^{(m,m-1)}[0,1]$ is a quotient space.

Let we denote by $\widetilde{W}_2^{(m,m-1)}(0,1]$ the subspace of the space $W_2^{(m,m-1)}[0,1]$ consisting of real-valued, 1-periodic functions $\varphi(x),\ x \in (0,1]$.

In the present paper, we consider the Hilbert space $\widetilde{W}_2^{(m,m-1)}$ of 1-periodic, real-valued functions. Notice that every element of the space $\widetilde{W_2}^{(m,m-1)}$ satisfies the following condition of 1-periodicity

$$\varphi(x+\beta) = \varphi(x) \text{ for } x \in \mathbb{R} \text{ and } \beta \in \mathbb{Z}.$$

We consider a quadrature formula of the following form

$$\int_0^1 \varphi(x)dx \cong \sum_{k=1}^{N} C_k \varphi(x_k), \tag{3}$$

where $\varphi(x) \in \widetilde{W}_2^{(m,m-1)}$, C_k are the coefficients of the quadrature formula and N is the number of nodes, $h = 1/N$ and x_k $(0 < x_1 < x_2 < \cdots < x_N \leq 1)$ are *nodes*. Since quadrature formula of the form (1) with equidistant nodes is optimal for the periodic functions class, we choose the nodes as $x_k = hk$ (see [23]).
The error of the quadrature formula (1) is given as follows

$$\begin{aligned}(\ell, \varphi) &= \int_0^1 \varphi(x)dx - \sum_{k=1}^{N} C_k \varphi(hk) \\ &= \int_0^1 \left[\left(\varepsilon_{(0,1]}(x) - \sum_{k=1}^{N} C_k \delta(x-hk) \right) * \phi_0(x) \right] \varphi(x)dx, \end{aligned} \tag{4}$$

where $h=\frac{1}{N}$, $\varepsilon_{(0,1]}(x)$ is the characteristic function of the interval $(0,1]$, δ is the Dirac delta-function, $\phi_0(x)=\sum\limits_{\beta=-\infty}^{\infty}\delta(x-\beta)$, * is the convolution operation and

$$\ell(x)=\left(\varepsilon_{(0,1]}(x)-\sum_{k=1}^{N}C_k\delta(x-hk)\right)*\phi_0(x), \tag{5}$$

and it is the periodic error functional of the quadrature formula (1).

The error (2) of the quadrature formula (1) is a linear functional in $\widetilde{W_2}^{(m,m-1)*}$. It should be noted that $\widetilde{W_2}^{(m,m-1)*}$ is the conjugate space to the space $\widetilde{W_2}^{(m,m-1)}$ and the conjugate space consists of all periodic functionals which are orthogonal to the unity [22], i.e.,

$$(\ell,1)=0. \tag{6}$$

This condition means the exactness of the quadrature formula (1) for any constant and it can be written as follows

$$\sum_{k=1}^{N}C_k=1. \tag{7}$$

The absolute value of the error (2) is estimated by the Cauchy-Schwarz inequality as follows

$$|(\ell,\varphi)|\leq\|\ell\|_{\widetilde{W_2}^{(m,m-1)*}}\cdot\|\varphi\|_{\widetilde{W_2}^{(m,m-1)}},$$

where

$$\|\ell\|_{\widetilde{W_2}^{(m,m-1)*}}=\sup_{\|\varphi\|_{\widetilde{W_2}^{(m,m-1)}}=1}|(\ell,\varphi)| \tag{8}$$

is the norm of the error functional (3).

The problem of constructing an optimal quadrature formula of the form (1) is as follows.

Problem 1. *Find the coefficients* $\overset{\circ}{C}_k$ *that give the minimum value to the quantity* $\|\ell\|_{\widetilde{W_2}^{(m,m-1)*}}$, *and calculate the following*

$$\left\|\overset{\circ}{\ell}\right\|_{\widetilde{W_2}^{(m,m-1)*}}=\inf_{C_k}\|\ell\|_{\widetilde{W_2}^{(m,m-1)*}}.$$

We note that the coefficients $\overset{\circ}{C}_k$ which are the solution for Problem 1 are called *the optimal coefficients* and the quadrature formula (1) with these coefficients is said to be *the optimal quadrature formula in the sense of Sard* [11].

Further, in the next sections we solve Problem 1.

The rest of the paper is organized as follows. Section 2 is devoted to calculation the norm of the error functional and to obtain the system of linear equations for optimal coefficients which give the minimum value to the norm of the error function. In section 3 this system is solved and explicit expressions for the coefficients of the optimal quadrature formula (1) are found. Finally, in Section 4 we calculate the quantity of the norm of the error functional (3) that is the sharp upper bound for the error of the optimal quadrature formula (1).

The norm for the error functional of the quadrature formula

To calculate the norm (5), we use *the extremal function* ψ_ℓ for the error functional ℓ (see [22]) that satisfies the following equality:

$$(\ell, \psi_\ell) = \|\ell\|_{\widetilde{W_2}^{(m,m-1)*}} \cdot \|\psi_\ell\|_{\widetilde{W_2}^{(m,m-1)}}. \tag{9}$$

Since $\widetilde{W_2}^{(m,m-1)}$ is the Hilbert space by the Riesz theorem for the error functional ℓ for any φ from $\widetilde{W}_2^{(m,m-1)}$ there exists an element $\psi_\ell \in \widetilde{W}_2^{(m,m-1)}$ that satisfies the equality

$$(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle_{\widetilde{W_2}^{(m,m-1)}}, \tag{10}$$

where $\langle \psi_\ell, \varphi \rangle_{\widetilde{W_2}^{(m,m-1)}}$ is the inner product of the functions ψ_ℓ and φ defined by the formula (4). In addition, the equality $\|\ell\|_{\widetilde{W_2}^{(m,m-1)*}} = \|\psi_\ell\|_{\widetilde{W_2}^{(m,m-1)}}$ is fulfilled. So, taking into account equality (9), we derive

$$(\ell, \psi_\ell) = \|\ell\|^2_{\widetilde{W_2}^{(m,m-1)*}}. \tag{11}$$

Integrating by parts the right-hand side of (10), keeping in mind periodicity of functions φ, for ψ_ℓ we have

$$\psi_\ell^{(2m)}(x) - \psi_\ell^{(2m-2)}(x) = (-1)^m \cdot \ell(x). \tag{12}$$

Further, we give the main results of this work.

Lemma 1. *The solution of equation (12) is the extremal function ψ_ℓ of the error functional ℓ and it is expressed as*

$$\psi_\ell(x) = d_0 - \sum_{k=1}^{N} C_k G_m(x - hk), \tag{13}$$

where d_0 is a constant and

$$G_m(x) = (-1)^m \sum_{\beta \neq 0} \frac{e^{-2\pi i \beta x}}{(2\pi i \beta)^{2m} - (2\pi i \beta)^{2m-2}}. \tag{14}$$

Proof. The idea of the proof is as follows. For this, we use equation (10). Integrating by parts its right-hand side, and taking into account that φ and ψ are 1-periodic functions, we obtain the following

$$(\ell, \varphi) = \int_0^1 \left(\psi_\ell^{(2m)}(x) - \psi_\ell^{(2m-2)}(x) \right) \varphi(x) dx. \tag{15}$$

From equality (15) we get the differential equation (12). We use the Fourier transform to find a periodic solution of the differential equation (12). To do this, we use the following properties of the Fourier transforms

$$\begin{aligned} F[\varphi] &= \int_{-\infty}^{\infty} \varphi(x) e^{2\pi i p x} dx, \\ F^{-1}[\varphi] &= \int_{-\infty}^{\infty} \varphi(p) e^{-2\pi i p x} dp, \\ F[\varphi^{(\alpha)}] &= (-2\pi i p)^{\alpha} F[\varphi], \ (\alpha \in \mathbb{N}), \\ F[\varphi * g] &= F[\varphi] \cdot F[g], \\ F[\varphi \cdot g] &= F[\varphi] * F[g], \\ F[\phi_0(x)] &= \phi_0(p), \end{aligned}$$

where $\phi_0(x) = \sum\limits_{\beta=-\infty}^{\infty} \delta(x-\beta)$.

Applying the Fourier transform to both sides of equation (12) we get

$$F[\psi_\ell^{(2m)} - \psi_\ell^{(2m-2)}] = (-1)^m F[\ell].$$

Since, the Fourier transform is a linear operator, we have

$$\left((2\pi i p)^{2m} - (2\pi i p)^{2m-2}\right) F[\psi_\ell] = (-1)^m F\left[\left(\varepsilon_{(0,1]}(x) - \sum_{k=1}^{N} C_k \delta(x-hk)\right) * \phi_0(x)\right]$$

or

$$\left((2\pi i p)^{2m} - (2\pi i p)^{2m-2}\right) F[\psi_\ell] = (-1)^m \left(F\left[\varepsilon_{(0,1]}(x)\right] - \sum_{k=1}^{N} C_k e^{2\pi i p h k}\right) \cdot \phi_0(p). \quad (16)$$

Consequently, we can divide both sides of equation (16) by $(2\pi i p)^{2m} - (2\pi i p)^{2m-2}$. This division is not uniquely defined. From equation (16) the function $F[\psi_\ell]$ is defined up to the term of the form $\delta(p)$. Taking into account the above said and the properties of the delta-function, we get

$$F[\psi_\ell(x)] = (-1)^m \left[d_0 \delta(p) + \frac{F\left[\varepsilon_{(0,1]}(x)\right] - \sum\limits_{k=1}^{N} C_k e^{2\pi i p h k}}{(2\pi i p)^{2m} - (2\pi i p)^{2m-2}} \cdot \sum_{\beta \neq 0} \delta(p-\beta) \right].$$

Using the property $f(x)\delta(x-a) = f(a)\delta(x-a)$ of the delta-function, we have the following

$$\begin{aligned} F[\psi_\ell(x)] &= (-1)^m \Bigg[d_0 \delta(p) + F\left[\varepsilon_{(0,1]}(x)\right] \cdot \sum_{\beta \neq 0} \frac{\delta(p-\beta)}{(2\pi i \beta)^{2m} - (2\pi i \beta)^{2m-2}} \\ &+ \sum_{k=1}^{N} C_k \sum_{\beta \neq 0} \frac{e^{2\pi i \beta h k} \cdot \delta(p-\beta)}{(2\pi i \beta)^{2m} - (2\pi i \beta)^{2m-2}} \Bigg]. \end{aligned}$$

Now, applying the inverse Fourier transform to both sides of equation (29) and using some properties of Fourier transform, after some calculations we have

$$\begin{aligned}
\psi_\ell(x) &= d_0+\varepsilon_{[0,1]}(x)*\left((-1)^m\sum_{\beta\neq 0}\frac{e^{-2\pi i\beta x}}{(2\pi i\beta)^{2m}-(2\pi i\beta)^{2m-2}}\right)\\
&- \sum_{k=1}^{N}C_k\left((-1)^m\sum_{\beta\neq 0}\frac{e^{-2\pi i\beta (x-hk)}}{(2\pi i\beta)^{2m}-(2\pi i\beta)^{2m-2}}\right)\\
&= d_0+\varepsilon_{[0,1]}(x)*G_m(x)-\sum_{k=1}^{N}C_kG_m(x-hk)\\
&= d_0+\int_{-\infty}^{\infty}\varepsilon_{[0,1]}(y)G_m(x-y)dy-\sum_{k=1}^{N}C_kG_m(x-hk)\\
&= d_0+\int_0^1 G_m(x-y)dy-\sum_{k=1}^{N}C_kG_m(x-hk)\\
&= d_0-\sum_{k=1}^{N}C_kG_m(x-hk),
\end{aligned}$$

where $G_m(x)$ is defined by equality (14) and it is easy to show that

$$\int_0^1 G_m(x-y)dy=0.$$

And thus, Lemma 1 is completely proved.

Simplifying the error functional of the form (3), we can rewrite it in the following form

$$\begin{aligned}
\ell(x) &= \sum_{\beta=-\infty}^{\infty}\varepsilon_{[0,1]}(x)*\delta(x-\beta)-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk)*\delta(x-\beta)\\
&= 1-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-\beta-hk). \qquad (17)
\end{aligned}$$

The square of the norm for ℓ of the quadrature formula (1) is expressed in terms of the bilinear form of the coefficients C_k. Indeed, since the space $\widetilde{W}_2^{(m,m-1)}$ is the Hilbert space, then using equalities (11) and (13) for the square of norm (5) we have

$$\begin{aligned}\|\ell\|^2_{\widetilde{W_2}^{(m,m-1)*}} &= (\ell,\psi_\ell)=\int_0^1 \ell(x)\psi_\ell(x)dx\\ &= \int_0^1\left(1-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta)\right)\\ &\times\left(d_0-\sum_{k'=1}^{N}C_{k'}G_m(x-hk')\right)dx.\end{aligned}$$

Simplifying the right-hand side of the above equality and taking into account the condition (6) we obtain the following analytical expression

$$\|\ell\|^2_{\widetilde{W_2}^{(m,m-1)*}}=(-1)^m\sum_{k=1}^{N}\sum_{k'=1}^{N}C_kC_{k'}\sum_{\beta\neq 0}\frac{e^{-2\pi i\beta h(k-k')}}{(2\pi i\beta)^{2m}-(2\pi i\beta)^{2m-2}}. \tag{18}$$

Now, we apply the method of Lagrange unknown multipliers to solve Problem 1. For this, consider the following function

$$L(C,d_0)=\|\ell\|^2_{\widetilde{W_2}^{(m,m-1)*}}-2(-1)^m d_0\,(\ell,1),$$

where $C=(C_1,C_2,...,C_N)$.

Equating all partial derivatives of the function $L(C,d_0)$ by C_k and d_0 we have

$$\frac{\partial L}{\partial C_k}=0,\ \text{for } k=1,2,...,N,$$
$$\frac{\partial L}{\partial d_0}=0.$$

They give the following system of linear equations with respect to $C_k, k=\overline{1,N}$ and d_0:

$$\sum_{k'=1}^{N}C_{k'}\sum_{\beta\neq 0}\frac{e^{-2\pi i\beta h(k-k')}}{(2\pi i\beta)^{2m}-(2\pi i\beta)^{2m-2}}=d_0,\ \text{for } k=1,...,N, \tag{19}$$

$$\sum_{k'=1}^{N}C_{k'}=1. \tag{20}$$

The last system is a system Wiener-Hopf type. The solution of the system (19)–(20) gives the minimum to the square of the norm (18) for the error functional (3) in certain values of $C_k=\mathring{C}_k$ $(k=1,2,...,N)$ and $\mathring{C}_k$ are called *the optimal coefficients*, and d_0 is a stationary point for the function $L(C,d_0)$.

Its solvability follows from the general theory of the Lagrange multipliers. But, as shown in calculations, the matrix of the system (19) and (20) coincides with the matrix of the system considered in the work [16, p. 127] in the construction of optimal cubature formulas in the Sobolev space $\widetilde{L_2}^{(m)}$ of periodic functions, and it was proved the uniqueness of the set of the optimal coefficients. Hence follows that the system (19) and (20) has a unique solution.

The optimal coefficients of the quadrature formula

In this section, we show how to find the optimal coefficients of the quadrature formula.
Let, we seek the solution of the system (19)–(20) in the form

$$\mathring{C}_k = C(h), \text{ for } k = 1,2,...,N, \tag{21}$$

where $C(h)$ is an unknown function of h.

Putting (21) into (20), and taking into account that the quadrature formulas with equal coefficients are optimal in the space of periodic functions (see [23] for full details), we obtain $C(h) = h$. Now, figuring on the system has a unique solution, putting the value of $C(h)$ into equation (19), we find d_0:

$$\sum_{k'=1}^{N} h \sum_{\beta\neq 0} \frac{e^{-2\pi i\beta h(k-k')}}{(2\pi i\beta)^{2m}-(2\pi i\beta)^{2m-2}} = d_0, \text{ for } k = 1,2,...,N.$$

Since the infinite series in (19) is convergent, we can rewrite the last system as follows

$$d_0 = h\sum_{\beta\neq 0} \frac{e^{-2\pi i\beta hk}}{(2\pi i\beta)^{2m}-(2\pi i\beta)^{2m-2}} \sum_{k'=1}^{N} e^{2\pi i\beta hk'}, \text{ for } k = 1,2,...,N. \tag{22}$$

It is known that

$$\sum_{k'=1}^{N} e^{2\pi i\beta hk'} = \frac{e^{2\pi i\beta h}(1-e^{2\pi i\beta})}{1-e^{2\pi i\beta h}} = \begin{cases} 0, & \text{if } \beta \neq \gamma N, \\ N, & \text{if } \beta = \gamma N, \end{cases} \quad \gamma \in \mathbb{Z}. \tag{23}$$

Taking into account (23), we rewrite equation (22) as follows

$$d_0 = h\sum_{\gamma\neq 0} \frac{e^{-2\pi i\gamma Nhk}}{(2\pi i\gamma N)^{2m}-(2\pi i\gamma N)^{2m-2}} \cdot N.$$

From the last equation and the well-known equality $e^{-2\pi i\gamma Nhk} = 1$ ($k = 1,2,...,N$ and $\gamma \in \mathbb{Z}$) we have

$$\begin{aligned} d_0 &= (-1)^m \left(\sum_{\gamma\neq 0} \frac{1}{(2\pi\gamma N)^{2m} - i^2(2\pi\gamma N)^{2m-2}} \right) \\ &= (-1)^m \cdot 2 \left(\sum_{\gamma=1}^{\infty} \frac{1}{(2\pi\gamma N)^{2m} + (2\pi\gamma N)^{2m-2}} \right). \end{aligned} \tag{24}$$

Now we calculate the following infinite series to find d_0

$$\begin{aligned} \sum_{\gamma=1}^{\infty} \frac{1}{(2\pi\gamma N)^{2m} + (2\pi\gamma N)^{2m-2}} &= \left(\frac{h}{2\pi}\right)^{2m} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^{2m} + \left(\frac{h}{2\pi}\right)^2 \gamma^{2m-2}} \\ &= \left(\frac{h}{2\pi}\right)^{2m-2} \cdot (s_1 - s_2), \end{aligned} \tag{25}$$

where $s_1=\sum\limits_{\gamma=1}^{\infty}\frac{1}{\gamma^{2m-2}}$ and $s_2=\sum\limits_{\gamma=1}^{\infty}\frac{\gamma^{4-2m}}{\gamma^2+\left(\frac{h}{2\pi}\right)^2}$ for $m\geq 2$.

The problem of calculating the above infinite series came to the problem of calculating s_1 and s_2.

Firstly, we calculate s_1. To do this, we use the following Riemann zeta function

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}.$$

The value of the Riemann zeta function is equal to the following (see [20, p. 57])

$$\zeta(2m)=\frac{(-1)^{m+1}(2\pi)^{2m}}{2\cdot(2m)!}\cdot B_{2m},\ \text{for } m=1,2,...,$$

where B_{2m} is Burnoulli number.

From the last equality we obtain the following

$$s_1=\zeta(2m-2)=\frac{(-1)^{m}(2\pi)^{2m-2}}{2\cdot(2m-2)!}\cdot B_{2m-2}. \tag{26}$$

Now, we use the following equality to calculate s_2 (see [20, p. 64])

$$\beta(t)=P_M(t)+(-1)^M t^{2M}\cdot\mu_M(t), \tag{27}$$

where

$$\beta(t)=\frac{1}{t^2}\left(\frac{t}{2}\coth\left(\frac{t}{2}\right)-1\right), \tag{28}$$

$$P_M(t)=\sum_{n=2}^{2M}\frac{B_n}{n!}\cdot t^{n-2},$$

$$\mu_M(t)=\sum_{k=1}^{\infty}\frac{2}{(4\pi^2k^2+t^2)\cdot(2\pi k)^{2M}},\ t\neq 2\pi ik.$$

Using the well-known formula $\frac{t}{e^t-1}=\sum\limits_{n=0}^{\infty}\frac{B_n}{n!}t^n$, $|t|<2\pi$, we can rewrite equality (28) as follows

$$\beta(t)=\sum_{n=2}^{\infty}\frac{B_n}{n!}t^{n-2}.$$

Now, using values of functions $\beta(t), P_M(t)$ and $\mu_M(t)$ at $M=m-2$, we can write equality (27) as follows

$$\sum_{n=2}^{\infty}\frac{B_n}{n!}t^{n-2}=\sum_{n=2}^{2m-4}\frac{B_n}{n!}t^{n-2}+(-1)^m t^{2m-4}\cdot\frac{2}{(2\pi)^{2m-2}}\cdot T_m(t),$$

where

$$T_m(t)=\sum_{n=1}^{\infty}\frac{n^{4-2m}}{n^2+\left(\frac{t}{2\pi}\right)^2}.$$

From the last expression and taking account that $B_{2k+1} = 0$, $k \geq 1$, we obtain the following

$$T_m(t) = \frac{(-1)^m (2\pi)^{2m-2}}{2t^{2m-4}} \sum_{n=2m-2}^{\infty} \frac{B_n}{n!} t^{n-2}.$$

And so

$$s_2 = T_m(h) = \frac{(-1)^m (2\pi)^{2m-2}}{2h^{2m-4}} \sum_{n=2m-2}^{\infty} \frac{B_n}{n!} h^{n-2}. \quad (29)$$

Taking into account equalities (25), (26) and (29), we have the following

$$\sum_{\gamma=1}^{\infty} \frac{1}{(2\pi\gamma N)^{2m} + (2\pi\gamma N)^{2m-2}} = \left(\frac{h}{2\pi}\right)^{2m-2} \left[\frac{(-1)^m (2\pi)^{2m-2}}{2(2m-2)!} B_{2m-2}\right.$$
$$\left. - \frac{(-1)^m (2\pi)^{2m-2}}{2h^{2m-4}} \sum_{n=2m-2}^{\infty} \frac{B_n}{n!} h^{n-2}\right] = (-1)^{m+1} \frac{1}{2} \sum_{n=2m}^{\infty} \frac{B_n}{n!} h^n. \quad (30)$$

From equalities (24) and (30), we have

$$d_0 = -\sum_{n=2m}^{\infty} \frac{B_n}{n!} h^n. \quad (31)$$

Thereby, we have found $(\mathring{C}_k, d_0)$ which the solution of the system (19) and (20). Now, in the next section, using this solution, we give the main theorem of our work.

Calculation of the norm for the error functional of the optimal quadrature formula (1)

The following theorem is valid for the error functional norm of the optimal quadrature formula.

Theorem 1. *The norm of the error functional* (3) *for the optimal quadrature formula (1) has the form*

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}} = (-1)^{m+1} \sum_{n=2m}^{\infty} \frac{B_n}{n!} h^n, \ \textit{for } m \geq 2, \quad (32)$$

where B_n *are Bernoulli numbers.*

Proof. We use equations (18) and (19) and then we have

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W}_2^{(m,m-1)*}} = (-1)^m \sum_{k=1}^{N} \sum_{k'=1}^{N} \mathring{C}_k \mathring{C}_{k'} \sum_{\beta \neq 0} \frac{e^{-2\pi i\beta h(k-k')}}{(2\pi i\beta)^{2m} - (2\pi i\beta)^{2m-2}}$$
$$= \sum_{k=1}^{N} \mathring{C}_k \left[\sum_{k'=1}^{N} \mathring{C}_{k'} \sum_{\beta \neq 0} \frac{e^{-2\pi i\beta h(k-k')}}{(2\pi i\beta)^{2m} - (2\pi i\beta)^{2m-2}}\right].$$

Hence, taking into account (19) for the square of $\|\mathring{\ell}\|$, we obtain the following

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(m,m-1)*}} = (-1)^m \sum_{k=1}^{N} \mathring{C}_k \cdot d_0.$$

Hence, taking into account equalities (20) and (31), we obtain

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(m,m-1)*}} = (-1)^{m+1} \sum_{n=2m}^{\infty} \frac{B_n}{n!} h^n, \text{ for } m \geq 2.$$

Thus, Theorem 1 is completely proved.

Remark 1. It should be noted that from equality (32) and keeping in mind that $B_{2n+1} = 0$ $(n \geq 1)$ we have

$$\begin{aligned} \left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(m,m-1)*}} &= (-1)^{m+1} \frac{B_{2m}}{(2m)!} h^{2m} + (-1)^{m+1} \frac{B_{2m+2}}{(2m+2)!} h^{2m+2} \\ &+ (-1)^{m+1} \sum_{n=2m+4}^{\infty} \frac{B_n}{n!} h^n, \text{ for } m \geq 2, \end{aligned} \tag{33}$$

i.e., the order of convergence of the optimal quadrature formula of the form (1) is $O(h^m)$.

According to the property of the Bernoulli numbers, if m is an even number then $B_{2m} = -|B_{2m}|$ is appropriate, otherwise $B_{2m} = |B_{2m}|$.

Taking into account the above properties of Bernoulli numbers, the following corollary follow.

Corollary 1. *If* m *is an even number we can rewrite equality* (33) *as follows*

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(m,m-1)*}} = \frac{|B_{2m}|}{(2m)!} h^{2m} - \frac{B_{2m+2}}{(2m+2)!} h^{2m+2} + O(h^{2m+4}), \tag{34}$$

where $B_{2m+2} > 0$, *and if* m *is an odd number* $(m \geq 2)$

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(m,m-1)*}} = \frac{B_{2m}}{(2m)!} h^{2m} - \frac{|B_{2m+2}|}{(2m+2)!} h^{2m+2} + O(h^{2m+4}). \tag{35}$$

where $B_{2m+2} < 0$.

Remark 2. It should be noted that for any $m \geq 2$ from (34) and (35) we obtain

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(m,m-1)*}} = \frac{|B_{2m}|}{(2m)!} h^{2m} - \frac{|B_{2m+2}|}{(2m+2)!} h^{2m+2} + O(h^{2m+4}).$$

This is less than the sharp error bound

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{L_2}^{(m)*}} = \frac{|B_{2m}|}{(2m)!} h^{2m}$$

of the optimal quadrature of the form (3) in the space $\widetilde{L_2}^{(m)}(0,1]$ (see [4, Theorem 4.5, page 205]).

Corollary 2. *In the case* $m = 2$ *expression* (32) *can be written as follows*

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(2,1)*}} = \frac{1}{12}h^2 - \frac{h}{2}\cdot\frac{e^h+1}{e^h-1} + 1. \qquad (36)$$

Corollary 3. *In the case* $m = 3$ *expression* (32) *can be written as follows*

$$\left\|\mathring{\ell}\right\|^2_{\widetilde{W_2}^{(3,2)*}} = \frac{h^4}{720} - \frac{h^2}{12} + \frac{h}{2}\cdot\frac{e^h+1}{e^h-1} - 1. \qquad (37)$$

Conclusion

In the present paper, the optimal quadrature formula in the sense of Sard is constructed in the space $\widetilde{W}_2^{(m,m-1)}(0,1]$ of periodic, real-valued functions for approximation of the Fourier integrals (1) with $\omega = 0$. Here, we found analytical forms for coefficients of the constructed optimal quadrature formula. In addition, we calculated the norm of the error functional for the optimal quadrature formula and obtained that this norm is less than the norm of the error functional for the optimal quadrature formula in the space $\widetilde{L_2}^{(m)}(0,1]$ of periodic, real valued functions.

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Оптимальные квадратурные формулы в пространстве $\widetilde{W_2}^{(m,m-1)}$ периодических функций

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Данная статья посвящена процессу нахождения верхней оценки абсолютной погрешности оптимальной квадратурной формулы в пространстве $\widetilde{W_2}^{(m,m-1)}$ вещественнозначных периодических функций. Для этого используется экстремальная функция квадратурной формулы. Кроме того, показано, что норма функционала ошибки для оптимальной квадратурной формулы, построенной в пространстве $\widetilde{W_2}^{(m,m-1)}$, меньше значения нормы ошибки функционал для оптимальной квадратурной формулы в пространстве Соболева $\widetilde{L_2}^{(m)}$.

Ключевые слова: оптимальная квадратурная формула, оптимальные коэффициенты, погрешность квадратурной формулы, гильбертово пространство, функционал погрешности, преобразование Фурье.

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