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Research Article

Note on the spectral theorem for unbounded non-selfadjoint operators


M. V. Kukushkin

Moscow State University of Civil Engineering, Yaroslavl highway, 26, 129337,
Moscow, Russia

E-mail: kukushkinmv@rambler.ru


In this paper, we deal with non-selfadjoint operators with the compact resolvent. Having been inspired by the Lidskii idea involving a notion of convergence of a series on the root vectors of the operator in a weaker – Abel-Lidskii sense, we proceed constructing theory in the direction. The main concept of the paper is a generalization of the spectral theorem for a non-selfadjoint operator. In this way, we come to the definition of the operator function of an unbounded non-selfadjoint operator. As an application, we notice some approaches allowing us to principally broaden conditions imposed on the right-hand side of the evolution equation in the abstract Hilbert space.

Key words: Spectral theorem; Abel-Lidskii basis property; Schatten-von Neumann class; operator function; evolution equation.

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Introduction

We should recall that the concept of the spectral theorem for a selfadjoint operator is based on the notion of a spectral family or the decomposition of the identical operator. Having constructed a spectral family, we can define a selfadjoint operator using the concept of the Riemann integral, it is the very statement of the spectral theorem for a selfadjoint operator. Using the same scheme, we come to a notion of the function of a selfadjoint operator. The idea can be clearly demonstrated if we consider the well-known representation of the compact selfadjoint operator as a series on its eigenvectors.

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The case corresponding to the non-selfadjoint operator is not so clear but we can adopt some notions and techniques to obtain similar results. Firstly, we should note that the question regarding decompositions of the operator on the series of eigenvectors (root vectors) is rather complicated and deserves to be considered itself. For this purpose, we need to involve some generalized notions of the series convergence, we are forced to understand it in one or another sense, we mean Bari, Riesz, Abel (Abel-Lidskii) senses of the series convergence [1],[2]. A reasonable question that appears is about minimal conditions that guaranty the desired result, for instance in the mentioned papers there considered a domain of the parabolic type containing the spectrum of the operator. In the paper [1], non-selfadjoint operators with the special condition imposed on the numerical range of values are considered. The main advantage of this result is a weaker condition imposed upon the numerical range of values comparatively with the sectorial condition (see definition of the sectorial operator). Thus, the convergence in the Abel-Lidskii sense was established for an operator class wider than a class of sectorial operators. Here, we make a comparison between results devoted to operators with the discrete spectra and operators with the compact resolvent, for they can be easily reformulated from one to the other realm. The central idea of this paper is to represent propositions analogous to the spectral theorem, the main obstacle that appears is how to define an analogue of a spectral family or decomposition of the identical operator. Thus, one of the paper challenges is to find a harmonious way of reformulating the main principles of the spectral theorem, taking into account the peculiarities of the convergence in the mentioned above senses. Here, we should make a brief digression and consider a theoretical background that allows us to obtain such exotic results.

The application part of the direction appeals to the theory of differential equations. In particular, the existence and uniqueness theorems for evolution equations with the right-hand side – a differential operator with a fractional derivative in final terms are covered by the invented abstract method. In this regard such operators as a Riemann-Liouville fractional differential operator, Kipriyanov operator, Riesz potential, difference operator are involved [9],[18],[19],[21]. The analysis of the conditions imposed upon the right-hand side of the evolution equations that are in the scope leads us to relevance of the central idea of the paper [10] which major contribution to the theory is a sufficient conditions of the Abel-Lidskii basis property of the root functions system for a sectorial non-selfadjoint operator of the special type. We ought to note a well-known fact [24],[7] that a particular interest appears in the case when a senior term of the operator is not selfadjoint, for in the contrary case there is a plenty of results devoted to the topic wherein the following papers are well-known [4],[13],[16],[17],[24]. The fact is that most of them deal with a decomposition of the operator on a sum, where the senior term must be either a selfadjoint or normal operator. In other cases, the methods of the papers [8], [7] become relevant and allow us to study spectral properties of operators whether we have the mentioned representation or not. In this regard, we should appeal to the paper [20] that presents an operator which in the origin of the abstract problem.

In its own turn, the application of results connected with the basis property [10] covers many problems in the framework of the theory of evolution equations [11]. The

peculiar contribution of this paper is the obtained formula for the solution of the evolution equation with the special conditions imposed upon the right-hand side, where the derivative at the left-hand side is supposed to be of the fractional order. Here, we should remind that involving the notion of the operator function, we broaden a great deal a class corresponding to the right-hand side. It gives us a prerequisite to consider the offered approach relevant.

Preliminaries

Let $C, C_i, i \in \mathbb{N}_0$ be real constants. We assume that a value of C is positive and can be different in various formulas but values of C_i are certain. Denote by $\text{int} M, \text{Fr} M$ the interior and the set of boundary points of the set M respectively. Everywhere further, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space \mathfrak{H} . Denote by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on \mathfrak{H} . Denote by \tilde{L} the closure of an operator L . We establish the following agreement on using symbols $\tilde{L}^i := (\tilde{L})^i$, where i is an arbitrary symbol. Denote by $D(L), R(L), N(L)$ the *domain of definition*, the *range*, and the *kernel* or *null space* of an operator L respectively. Let $P(L)$ be the resolvent set of an operator L and $R_L(\zeta), \zeta \in P(L), [R_L := R_L(0)]$ denotes the resolvent of an operator L . Denote by $\lambda_i(L), i \in \mathbb{N}$ the eigenvalues of an operator L . Suppose L is a compact operator and $N := (L^*L)^{1/2}, r(N) := \dim R(N)$; then the eigenvalues of the operator N are called the *singular numbers* (*s-numbers*) of the operator L and are denoted by $s_i(L), i = 1, 2, \dots, r(N)$. If $r(N) < \infty$, then we put by definition $s_i = 0, i = r(N) + 1, 2, \dots$. According to the terminology of the monograph [2] the dimension of the root vectors subspace corresponding to a certain eigenvalue λ_k is called the *algebraic multiplicity* of the eigenvalue λ_k . Denote by $n(r)$ a function equals to the number of the elements of the sequence $\{a_n\}_1^\infty, |a_n| \uparrow \infty$ within the circle $|z| < r$. Let A be a compact operator, denote by $n_A(r)$ *counting function* a function $n(r)$ corresponding to the sequence $\{s_i^{-1}(A)\}_1^\infty$. Let $\mathfrak{S}_p(\mathfrak{H}), 0 < p < \infty$ be a Schatten-von Neumann class and $\mathfrak{S}_\infty(\mathfrak{H})$ be the set of compact operators. Suppose L is an operator with a compact resolvent and $s_n(R_L) \leq Cn^{-\mu}, n \in \mathbb{N}, 0 \leq \mu < \infty$; then we denote by $\mu(L)$ order of the operator L in accordance with the definition given in the paper [24]. Denote by $\Re L := (L + L^*)/2, \Im L := (L - L^*)/2i$ the real and imaginary components of an operator L respectively. In accordance with the terminology of the monograph [3] the set $\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_{\mathfrak{H}}, f \in D(L), \|f\|_{\mathfrak{H}} = 1\}$ is called the *numerical range* of an operator L . An operator L is called *sectorial* if its numerical range belongs to a closed sector $\mathfrak{L}_\iota(\theta) := \{\zeta : |\arg(\zeta - \iota)| \leq \theta < \pi/2\}$, where ι is the vertex and θ is the semi-angle of the sector $\mathfrak{L}_\iota(\theta)$. If we want to stress the correspondence between ι and θ , then we will write θ_ι . By the *convergence exponent* ρ of the sequence $\{a_n\}_1^\infty \subset \mathbb{C}, a_n \neq 0, a_n \rightarrow \infty$ we mean the greatest lower bound for such numbers λ that the following series converges

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda} < \infty.$$

More detailed information can be found in [14]. Denote by $\tilde{\mathfrak{S}}_\rho(\mathfrak{H})$ the class of the operators such that

$$\tilde{\mathfrak{S}}_\rho(\mathfrak{H}) := \{T \in \mathfrak{S}_{\rho+\varepsilon}, T \bar{\in} \mathfrak{S}_{\rho-\varepsilon}, \forall \varepsilon > 0\},$$

we will call it *Schatten-von Neumann class of the convergence exponent*. Everywhere further, unless otherwise stated, we use notations of the papers [2], [3], [5], [6], [23].

Convergence in the Abel-Lidskii sense

In this subsection, we reformulate results obtained by Lidskii [15] in a more convenient form applicable to the reasonings of this paper. However, let us begin our narrative. In accordance with the Hilbert theorem (see [22], [2, p.32]) the spectrum of an arbitrary compact operator B consists of the so-called normal eigenvalues, it gives us an opportunity to consider a decomposition to a direct sum of subspaces

$$\mathfrak{H} = \mathfrak{N}_q \oplus \mathfrak{M}_q, \tag{1}$$

where both summands are invariant subspaces regarding the operator B , the first one is a finite dimensional root subspace corresponding to the eigenvalue μ_q and the second one is a subspace wherein the operator $B - \mu_q I$ is invertible. Let n_q is a dimension of \mathfrak{N}_q and let B_q is the operator induced in \mathfrak{N}_q . We can choose a basis (Jordan basis) in \mathfrak{N}_q that consists of Jordan chains of eigenvectors and root vectors of the operator B_q . Each chain $e_{q_\xi}, e_{q_{\xi+1}}, \dots, e_{q_{\xi+k}}, k \in \mathbb{N}_0$, where $e_{q_\xi}, \xi = 1, 2, \dots, m$ are the eigenvectors corresponding to the eigenvalue μ_q and other terms are root vectors, can be transformed by the operator B in accordance with the following formulas

$$B e_{q_\xi} = \mu_q e_{q_\xi}, B e_{q_{\xi+1}} = \mu_q e_{q_{\xi+1}} + e_{q_\xi}, \dots, B e_{q_{\xi+k}} = \mu_q e_{q_{\xi+k}} + e_{q_{\xi+k-1}}. \tag{2}$$

Considering the sequence $\{\mu_q\}_1^\infty$ of the eigenvalues of the operator B and choosing a Jordan basis in each corresponding space \mathfrak{N}_q , we can arrange a system of vectors $\{e_i\}_1^\infty$ which we will call a system of the root vectors or following Lidskii a system of the major vectors of the operator B . Assume that e_1, e_2, \dots, e_{n_q} is the Jordan basis in the subspace \mathfrak{N}_q . We can prove easily (see [15, p.14]) that there exists a corresponding biorthogonal basis g_1, g_2, \dots, g_{n_q} in the subspace \mathfrak{M}_q^\perp .

Using the reasonings [10], we conclude that $\{g_i\}_1^{n_q}$ consists of the Jordan chains of the operator B^* which correspond to the Jordan chains (2) due to the following formula

$$B^* g_{q_\xi+k} = \bar{\mu}_q g_{q_\xi+k}, B^* g_{q_\xi+k-1} = \bar{\mu}_q g_{q_\xi+k-1} + g_{q_\xi+k}, \dots, B^* g_{q_\xi} = \bar{\mu}_q g_{q_\xi} + g_{q_{\xi+1}}.$$

It is not hard to prove that the set $\{g_\nu\}_1^{n_j}, j \neq i$ is orthogonal to the set $\{e_\nu\}_1^{n_i}$ (see [10]). Gathering the sets $\{g_\nu\}_1^{n_j}, j = 1, 2, \dots$, we can obviously create a biorthogonal system $\{g_n\}_1^\infty$ with respect to the system of the major vectors of the operator B . It is rather reasonable to call it as a system of the major vectors of the operator B^* . Note that if an element $f \in \mathfrak{H}$ allows a decomposition in the strong sense

$$f = \sum_{n=1}^{\infty} e_n c_n, c_n \in \mathbb{C},$$

then by virtue of the biorthogonal system existing, we can claim that such a representation is unique. Further, let us come to the previously made agreement that the vectors in each Jordan chain are arranged in the same order as in (2) i.e. at the first place there stands an eigenvector. It is clear that under such an assumption we have

$$c_{q_\xi+i} = \frac{(f, g_{q_\xi+k-i})}{(e_{q_\xi+i}, g_{q_\xi+k-i})}, 0 \leq i \leq k(q_\xi),$$

where $k(q_\xi) + 1$ is a number of elements in the q_ξ -th Jourdan chain. In particular, if the vector e_{q_ξ} is included to the major system solo, there does not exist a root vector corresponding to the same eigenvalue, then

$$c_{q_\xi} = \frac{(f, g_{q_\xi})}{(e_{q_\xi}, g_{q_\xi})}.$$

Note that in accordance with the property of the biorthogonal sequences, we can expect that the denominators equal to one in the previous two relations. Consider a formal series corresponding to a decomposition on the major vectors of the operator B

$$f \sim \sum_{n=1}^{\infty} e_n c_n,$$

where each number n corresponds to a number $q_\xi + i$ (thus, the coefficients c_n are defined in accordance with the above and numerated in a simplest way). Consider a set of functions with respect to a real parameter t

$$Q_m(\varphi, \zeta^{-1}, t) := e^{\zeta^{-1}t} \frac{1}{m!} \frac{d^m}{d\zeta^m} \left\{ \varphi(1/\zeta) e^{-\zeta^{-1}t} \right\}, m = 0, 1, 2, \dots,$$

Here we should note that if $\varphi = \text{const}$, then we have a set of polynomials, what is in the origin of the concept, see [15]. Consider a series

$$\sum_{n=1}^{\infty} c_n(t) e_n, \quad (3)$$

where the coefficients $c_n(t)$ are defined in accordance with the correspondence between the indexes n and $q_\xi + i$ in the following way

$$c_{q_\xi+i}(t) = e^{-\lambda_{q_\xi} t} \sum_{m=0}^{k(q_\xi)-i} Q_m(\varphi, \lambda_{q_\xi}, t) c_{q_\xi+i+m}, i = 0, 1, 2, \dots, k(q_\xi), \quad (4)$$

here $\lambda_q = 1/\mu_q$ is a characteristic number corresponding to e_{q_ξ} . It is clear that in any case, we have a limit $c_n(t) \rightarrow \tilde{c}_n, t \rightarrow +0$, where a value \tilde{c}_n can be calculated directly due to the formula (4). For instance in the case $\varphi = 1$, we have $\tilde{c}_n = c_n$. Generalizing the definition given in [15, p.17], we will say that series (3) converges to the element f in the sense $(B, \varphi, 1)$, if there exists a sequence of the natural numbers $\{N_j\}_1^\infty$ such that

$$f = \lim_{t \rightarrow +0} \lim_{j \rightarrow \infty} \sum_{n=1}^{N_j} c_n(t) e_n.$$

Note that sums of the latter relation forms a subsequence of the partial sums of the series (3).

We need the following lemmas [15], in the adopted form, also see [10]. Further, considering an arbitrary compact operator $B : \mathfrak{H} \rightarrow \mathfrak{H}$ such that $\Theta(B) \subset \mathfrak{L}_0(\theta)$, $\theta < \pi$, we put the following contour in correspondence to the operator

$$\gamma(B) := \{\lambda : |\lambda| = r > 0, |\arg\lambda| \leq \theta + \varepsilon\} \cup \{\lambda : |\lambda| > r, |\arg\lambda| = \theta + \varepsilon\},$$

where $\varepsilon > 0$ is an arbitrary small number, the number r is chosen so that the operator $(I - \lambda B)^{-1}$ is regular within the corresponding closed circle. Here, we should note that the compactness property of B gives us the fact $(I - \lambda B)^{-1} \in \mathcal{B}(\mathfrak{H})$, $\lambda \in \mathbb{C} \setminus \text{int}\gamma(B)$. It can be proved easily if we note that in accordance with the Corollary 3.3 [3, p.268], we have $P(B) \subset \mathbb{C} \setminus \overline{\Theta(B)}$.

Lemma 1. *Assume that B is a compact operator, $\Theta(B) \subset \mathfrak{L}_0(\theta)$, $\theta < \pi$, then on each ray ζ containing the point zero and not belonging to the sector $\mathfrak{L}_0(\theta)$ as well as the real axis, we have*

$$\|(I - \lambda B)^{-1}\| \leq \frac{1}{\sin \psi_0}, \lambda \in \zeta,$$

where $\psi_0 = \min\{|\arg\zeta - \theta|, |\arg\zeta + \theta|\}$.

Lemma 2. *Assume that a compact operator B satisfies the condition $B \in \tilde{\mathfrak{E}}_\rho$, then for arbitrary numbers R, κ such that $R > 0, 0 < \kappa < 1$, there exists a circle $|\lambda| = \tilde{R}$, $(1 - \kappa)R < \tilde{R} < R$, so that the following estimate holds*

$$\|(I - \lambda B)^{-1}\|_{\mathfrak{H}} \leq e^{\gamma(|\lambda|)|\lambda|^\rho} |\lambda|^m, |\lambda| = \tilde{R}, m = [\rho], \rho \geq \rho,$$

where

$$\gamma(|\lambda|) = \beta(|\lambda|^{m+1}) + C\beta(|C\lambda|^{m+1}), \beta(r) = r^{-\frac{\rho}{m+1}} \left(\int_0^r \frac{n_{B^{m+1}}(t)dt}{t} + r \int_r^\infty \frac{n_{B^{m+1}}(t)dt}{t^2} \right).$$

Lemma 3. *Assume that the operator B satisfies conditions of Lemma 1, $f \in R(B)$, then*

$$\lim_{t \rightarrow +0} \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = f.$$

Main results

In this section, we consider an adopted technique allowing us to reformulate results [10] in accordance with our challenge to make a harmonious generalization of a notion – a function of an unbounded non-selfadjoint operator. Firstly, we consider statements with the necessary refinement caused by the involved functions, here we should note that a particular case corresponding to a constant function was considered by Lidskii

[15]. The interesting fact is that involving a non-constant function we are still able to consider a partial sums of the series on the root vectors corresponding to some groups of eigenvalues. Secondly, we find conditions that guarantee convergence of the involved integral construction and formulate a main theorem giving us a tool for further study. We consider an analogue of the spectral family having involved the operators similar to Riesz projectors (see [2, p.20]) and using a notion of the convergence in the Abel-Lidskii sense. Finally, we discuss an approach that we can realize for applying the abstract theoretical results to the evolution equations.

Lemma 4. *Assume that B is a compact operator, φ is an analytical function inside $\gamma(B)$, then in the pole λ_q of the operator $(I - \lambda B)^{-1}$, the residue of the vector function $\varphi(\lambda)e^{-\lambda t}B(I - \lambda B)^{-1}f$, ($f \in \mathfrak{H}$), equals to*

$$- \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t),$$

where $m(q)$ is a geometrical multiplicity of the q -th eigenvalue, $k(q_\xi) + 1$ is a number of elements in the q_ξ -th Jourdan chain,

$$c_{q_\xi+j}(t) := e^{-\lambda_q t} \sum_{m=0}^{k(q_\xi)-j} c_{q_\xi+j+m} Q_m(\varphi, \lambda_q, t).$$

Proof. Consider an integral

$$\mathfrak{J} = \frac{1}{2\pi i} \oint_{\gamma_q} \varphi(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda, \quad f \in \mathfrak{R}(B),$$

where the interior of the contour γ_q does not contain any poles of the operator $(I - \lambda B)^{-1}$, except of λ_q . Assume that \mathfrak{N}_q is a root space corresponding to λ_q and consider a Jordan basis $\{e_{q_\xi+i}\}$, $i = 0, 1, \dots, k(q_\xi)$, $\xi = 1, 2, \dots, m(q)$ in \mathfrak{N}_q . Using decomposition of the Hilbert space in the direct sum (1), we can represent an element

$$f = f_1 + f_2,$$

where $f_1 \in \mathfrak{N}_q$, $f_2 \in \mathfrak{M}_q$. Note that the operator function $\varphi(\lambda)e^{-\lambda t}B(I - \lambda B)^{-1}f_2$ is regular in the interior of the contour γ_q , it follows from the fact that λ_q is a normal eigenvalue (see the supplementary information). Hence, we have

$$\mathfrak{J} = \frac{1}{2\pi i} \oint_{\gamma_q} \varphi(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} f_1 d\lambda.$$

Using the formula

$$B(I - \lambda B)^{-1} = \frac{1}{\lambda} \left\{ (I - \lambda B)^{-1} - I \right\} = \frac{1}{\lambda^2} \left\{ \left(\frac{1}{\lambda} I - B \right)^{-1} - \lambda I \right\},$$

we obtain

$$\mathcal{J} = -\frac{1}{2\pi i} \oint_{\tilde{\gamma}_q} \varphi(1/\zeta) e^{-\zeta^{-1}t} B(\zeta I - B)^{-1} f_1 d\zeta, \zeta = 1/\lambda.$$

Now, let us decompose the element f_1 on the corresponding Jordan basis, we have

$$f_1 = \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}. \tag{5}$$

In accordance with the relation (2), we get

$$B e_{q_\xi} = \mu_q e_{q_\xi}, B e_{q_\xi+1} = \mu_q e_{q_\xi+1} + e_{q_\xi}, \dots, B e_{q_\xi+k} = \mu_q e_{q_\xi+k} + e_{q_\xi+k-1}.$$

Using this formula, we can prove the following relation

$$(\zeta I - B)^{-1} e_{q_\xi+i} = \sum_{j=0}^i \frac{e_{q_\xi+j}}{(\zeta - \mu_q)^{i-j+1}}. \tag{6}$$

Note that the case $i = 0$ is trivial. Consider a case, when $i > 0$, we have

$$\begin{aligned} \frac{(\zeta I - B) e_{q_\xi+j}}{(\zeta - \mu_q)^{i-j+1}} &= \frac{\zeta e_{q_\xi+j} - B e_{q_\xi+j}}{(\zeta - \mu_q)^{i-j+1}} = \frac{e_{q_\xi+j}}{(\zeta - \mu_q)^{i-j}} - \frac{e_{q_\xi+j-1}}{(\zeta - \mu_q)^{i-j+1}}, j > 0, \\ \frac{(\zeta I - B) e_{q_\xi}}{(\zeta - \mu_q)^{i+1}} &= \frac{e_{q_\xi}}{(\zeta - \mu_q)^i}. \end{aligned}$$

Using these formulas, we obtain

$$\begin{aligned} \sum_{j=0}^i \frac{(\zeta I - B) e_{q_\xi+j}}{(\zeta - \mu_q)^{i-j+1}} &= \frac{e_{q_\xi}}{(\zeta - \mu_q)^i} + \frac{e_{q_\xi+1}}{(\zeta - \mu_q)^{i-1}} - \frac{e_{q_\xi}}{(\zeta - \mu_q)^i} + \dots \\ &+ \frac{e_{q_\xi+i}}{(\zeta - \mu_q)^{i-i}} - \frac{e_{q_\xi+i-1}}{(\zeta - \mu_q)^{i-i+1}} = \frac{e_{q_\xi+i}}{(\zeta - \mu_q)^{i-i}}, \end{aligned}$$

what gives us the desired result. Now, substituting (5),(6), we get

$$\mathcal{J} = -\frac{1}{2\pi i} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} c_{q_\xi+i} \sum_{j=0}^i e_{q_\xi+j} \oint_{\tilde{\gamma}_q} \frac{\varphi(1/\zeta) e^{-\zeta^{-1}t}}{(\zeta - \mu_q)^{i-j+1}} d\zeta.$$

Note that the function $\varphi(1/\zeta)$ is analytic inside the interior of $\tilde{\gamma}_q$, hence

$$\frac{1}{2\pi i} \oint_{\tilde{\gamma}_q} \frac{\varphi(1/\zeta) e^{-\zeta^{-1}t}}{(\zeta - \mu_q)^{i-j+1}} d\zeta = \frac{1}{(i-j)!} \lim_{\zeta \rightarrow \mu_q} \frac{d^{i-j}}{d\zeta^{i-j}} \left\{ \varphi(1/\zeta) e^{-\zeta^{-1}t} \right\} =: e^{-\lambda_q t} Q_{i-j}(\varphi, \lambda_q, t).$$

Changing the indexes, we have

$$\mathcal{J} = -\sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} c_{q_\xi+i} e^{-\lambda_q t} \sum_{j=0}^i e_{q_\xi+j} Q_{i-j}(\varphi, \lambda_q, t) =$$

$$= - \sum_{\xi=1}^{m(q)} \sum_{j=0}^{k(q_\xi)} e_{q_\xi+j} e^{-\lambda_q t} \sum_{m=0}^{k(q_\xi)-j} c_{q_\xi+j+m} Q_m(\varphi, \lambda_q, t) = - \sum_{\xi=1}^{m(q)} \sum_{j=0}^{k(q_\xi)} e_{q_\xi+j} c_{q_\xi+j}(t),$$

where

$$c_{q_\xi+j}(t) := e^{-\lambda_q t} \sum_{m=0}^{k(q_\xi)-j} c_{q_\xi+j+m} Q_m(\varphi, \lambda_q, t).$$

The proof is complete. \square

Note that using the reasonings of the last lemma, it is not hard to prove that

$$c_{q_\xi+j}(t) \longrightarrow \sum_{m=0}^{k(q_\xi)-j} c_{q_\xi+j+m} Q_m(\varphi, \lambda_q, 0), \quad t \rightarrow +0.$$

Bellow, we consider an invertible operator B and use a notation $W := B^{-1}$. Consider a function φ that can be represented by a Laurent series about the point zero. Denote by

$$\varphi(W) := \sum_{n=-\infty}^{\infty} c_n W^n \tag{7}$$

a formal construction called by a function of the operator, where c_n are coefficients corresponding to the function φ . The lemmas given bellow are devoted to the study of the conditions under which being imposed the series (7) converges on some elements of the Hilbert space \mathfrak{H} , thus the operator $\varphi(W)$ is defined. We will also consider a closure $\tilde{\varphi}(W)$ of the operator $\varphi(W)$.

Lemma 5. *Assume that the operator B satisfies conditions of Lemma 1,*

$$\varphi(z) = \sum_{n=-\infty}^k c_n z^n, \quad z \in \mathbb{C}, \quad k \in \mathbb{N},$$

then

$$\lim_{t \rightarrow +0} \frac{1}{2\pi i} \int_{\gamma(B)} \varphi(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \sum_{n=-\infty}^k c_n W^n f, \quad f \in D(W^k).$$

Proof. Consider a decomposition of the Laurent series on two terms

$$\varphi_1(z) = \sum_{n=0}^k c_n z^n; \quad \varphi_2(z) = \sum_{n=-\infty}^{-1} c_n z^n.$$

Consider an obvious relation

$$\lambda^n B^n (E - \lambda B)^{-1} = (E - \lambda B)^{-1} - (E + \lambda B + \dots + \lambda^{n-1} B^{n-1}), \quad n \in \mathbb{N}. \tag{8}$$

It gives us the following representation

$$\frac{1}{2\pi i} \int_{\gamma(B)} \lambda^n e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} \lambda^n B^{n+2} (I - \lambda B)^{-1} W^{n+1} f d\lambda =$$

$$= \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} \lambda^{-2} (I - \lambda B)^{-1} W^{n+1} f d\lambda + \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} \lambda^{-2} \sum_{k=0}^{n+1} \lambda^k B^k W^{n+1} f d\lambda = I_1(t) + I_2(t).$$

Let us show that $I_2(t) = 0$, define a contour $\gamma_R(B) := \text{Fr}\{\text{int } \gamma(B) \cap \{\lambda : r < |\lambda| < R\}\}$ and let us prove that

$$I_R(t) := \frac{1}{2\pi i} \oint_{\gamma_R(B)} e^{-\lambda t} \lambda^{-2} \sum_{k=0}^{n+1} \lambda^k B^k W^{n+1} f d\lambda \rightarrow I_2(t), R \rightarrow \infty. \tag{9}$$

Consider a decomposition of the contour $\gamma_R(B)$ on terms $\tilde{\gamma}_R := \{\lambda : |\lambda| = R, |\arg \lambda| \leq \theta + \varepsilon\}$, $\gamma_{R+} := \{\lambda : r \leq |\lambda| < R, \arg \lambda = \theta + \varepsilon\}$, $\gamma_{R-} := \{\lambda : r < |\lambda| < R, \arg \lambda = -\theta - \varepsilon\}$. It is clear that

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\gamma_R(B)} e^{-\lambda t} \lambda^{-2} \sum_{k=0}^{n+1} \lambda^k B^k W^{n+1} f d\lambda &= \frac{1}{2\pi i} \int_{\tilde{\gamma}_R} e^{-\lambda t} \lambda^{-2} \sum_{k=0}^{n+1} \lambda^k B^k W^{n+1} f d\lambda + \\ &+ \frac{1}{2\pi i} \int_{\gamma_{R+}} e^{-\lambda t} \lambda^{-2} \sum_{k=0}^{n+1} \lambda^k B^k W^{n+1} f d\lambda + \frac{1}{2\pi i} \int_{\gamma_{R-}} e^{-\lambda t} \lambda^{-2} \sum_{k=0}^{n+1} \lambda^k B^k W^{n+1} f d\lambda. \end{aligned}$$

Having noticed that $I_R(t) = 0$, since the operator function under the integral is analytic inside the contour, we come to the conclusion that to obtain the desired result, we should show

$$\frac{1}{2\pi i} \int_{\tilde{\gamma}_R} e^{-\lambda t} \lambda^{-2} \sum_{k=0}^{n+1} \lambda^k B^k W^{n+1} f d\lambda \rightarrow 0, R \rightarrow \infty. \tag{10}$$

We have

$$\left\| \int_{\tilde{\gamma}_R} e^{-\lambda t} \lambda^{n-1} W f d\lambda \right\|_{\mathcal{H}} \leq R^{n-1} \|W f\|_{\mathcal{H}} \int_{\tilde{\gamma}_R} |e^{-\lambda t}| d|\lambda| \leq R^n \|W f\|_{\mathcal{H}} \int_{-\theta-\varepsilon}^{\theta+\varepsilon} e^{-t \text{Re} \lambda} d \arg \lambda.$$

Using the condition $|\arg \lambda| < \pi/2$, we have $\text{Re} \lambda \geq |\lambda| \cos(\pi/2 - \delta) = |\lambda| \sin \delta$, where δ is a sufficiently small number. Substituting this estimate to the last integral, we obtain (10). Therefore (9) holds and as a result, we have $I_2(t) = 0$. It is also clear that

$$I_1(t) \rightarrow \frac{1}{2\pi i} \int_{\gamma(B)} \lambda^{-2} (I - \lambda B)^{-1} W^{n+1} f d\lambda, t \rightarrow +0,$$

since in consequence of Lemma 1 the integral is uniformly convergent with respect to the parameter t . Note that the last integral can be calculated as a residue, we have

$$\frac{1}{2\pi i} \int_{\gamma(B)} \lambda^{-2} (I - \lambda B)^{-1} W^{n+1} f d\lambda = \lim_{\lambda \rightarrow 0} \frac{d(I - \lambda B)^{-1}}{d\lambda} W^{n+1} f = W^n f.$$

Thus, we obtain

$$\lim_{t \rightarrow +0} \frac{1}{2\pi i} \int_{\gamma(B)} \varphi_1(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \sum_{n=0}^k c_n W^n f, f \in D(W^k).$$

Consider a principal part of the Laurent series. Using (8), analogously to the above, we get for values $n \in \mathbb{N}$

$$\frac{1}{2\pi i} \int_{\gamma(B)} \lambda^{-n} e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \frac{1}{2\pi i} \int_{\gamma(B)} \lambda^{-n-1} e^{-\lambda t} (I - \lambda B)^{-1} f d\lambda.$$

Applying Lemma 1, we obtain

$$\frac{1}{2\pi i} \int_{\gamma(B)} \lambda^{-n-1} e^{-\lambda t} (I - \lambda B)^{-1} f d\lambda \rightarrow \frac{1}{2\pi i} \int_{\gamma(B)} \lambda^{-n-1} (I - \lambda B)^{-1} f d\lambda.$$

Having noticed that the integral equals to the residue of the operator function under the integral, we have

$$\frac{1}{2\pi i} \int_{\gamma(B)} \lambda^{-n-1} (I - \lambda B)^{-1} f d\lambda = \frac{1}{n!} \lim_{\lambda \rightarrow 0} \frac{d^n (I - \lambda B)^{-1}}{d\lambda^n} f = B^n f.$$

The following reasonings are mostly the same, we get

$$\frac{1}{2\pi i} \int_{\gamma(B)} \varphi_2(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \frac{1}{2\pi i} \int_{\gamma(B)} \varphi_2(\lambda) e^{-\lambda t} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda.$$

Applying Lemma 1, we can get easily

$$\frac{1}{2\pi i} \int_{\gamma(B)} \varphi_2(\lambda) e^{-\lambda t} \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda \rightarrow \frac{1}{2\pi i} \int_{\gamma(B)} \varphi_2(\lambda) \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda, t \rightarrow +0.$$

Note that the principal part of the Laurent series is uniformly convergent outside a circle with a sufficiently large radius. Taking into account this fact, using the inversion, it is not hard to prove that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma(B)} \varphi_2(\lambda) \lambda^{-2} (I - \lambda B)^{-1} W f d\lambda &= \sum_{n=-\infty}^{-1} \frac{c_n}{2\pi i} \int_{\gamma(B)} \lambda^{n-2} (I - \lambda B)^{-1} W f d\lambda = \\ &= \sum_{n=1}^{\infty} \frac{c_{-n}}{(n+1)!} \lim_{\lambda \rightarrow 0} \frac{d^{n+1} (I - \lambda B)^{-1}}{d\lambda^{n+1}} W f = \sum_{n=1}^{\infty} c_{-n} B^n f = \sum_{n=-\infty}^{-1} c_n W^n f. \end{aligned}$$

Thus, we obtain the desired result. \square

Lemma 6. Assume that B is a compact operator, $\Theta(B) \subset \mathfrak{L}_0(\theta)$, $\theta < \pi/2$, the entire function φ is of the order less than one and of the minimal or normal type, moreover the series

$$\sum_{n=0}^{\infty} c_n n! z^n, z \in \mathbb{C}$$

is absolutely convergent, where c_n are the Taylor coefficients of the function φ . Then

$$\frac{1}{2\pi i} \int_{\gamma(B)} \varphi(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \varphi(W)u(t),$$

where

$$u(t) := \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda, f \in D(W^n), n \in \mathbb{N}.$$

Moreover, if there exists a limit

$$\varphi(W)u(t) \rightarrow g, t \rightarrow +0, \tag{11}$$

then $g = \tilde{\varphi}(W)f$.

Proof. Firstly, we should note that the conditions imposed upon the order of the function φ allow us to claim that the latter integral converges for a fixed value of the parameter t . Let us establish the formula

$$\frac{1}{2\pi i} \int_{\gamma(B)} \varphi(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda = \sum_{n=0}^{\infty} c_n \cdot \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} \lambda^n B(I - \lambda B)^{-1} f d\lambda.$$

To prove this fact, we should show that the series

$$\sum_{n=0}^{\infty} c_n \cdot \frac{1}{2\pi i} \int_{\gamma_j(B)} e^{-\lambda t} \lambda^n B(I - \lambda B)^{-1} f d\lambda$$

is uniformly convergent with respect to $j \in \mathbb{N}$, where

$$\gamma_j(B) := \{\lambda : |\lambda| = r > 0, |\arg \lambda| \leq \theta + \varepsilon\} \cup \{\lambda : r < |\lambda| < r_j, r_j \uparrow \infty, |\arg \lambda| = \theta + \varepsilon\}.$$

Using lemma 3, we get a trivial inequality

$$\left\| \int_{\gamma_j(B)} e^{-\lambda t} \lambda^n B(I - \lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq C \|f\|_{\mathfrak{H}} \int_{\gamma_j(B)} e^{-|\lambda|t \cos(\theta + \varepsilon)} |\lambda|^n |d\lambda| = C \|f\|_{\mathfrak{H}} \int_r^{r_j} e^{-xt \cos(\theta + \varepsilon)} x^n dx.$$

It is clear that

$$\int_r^{r_j} e^{-xt \cos(\theta + \varepsilon)} x^n dx \leq \frac{\Gamma(n + 1)}{t^{n+1} \cos^{n+1}(\theta + \varepsilon)}.$$

Thus, we get

$$\left\| \int_{\gamma_j(B)} e^{-\lambda t} \lambda^n B(I - \lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq \frac{n!}{t^{n+1} \cos^{n+1}(\theta + \varepsilon)}.$$

The latter relation gives us the desired result. Using formula (8), we get

$$\frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} \lambda^n B(I - \lambda B)^{-1} f d\lambda =$$

$$= \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} B(I - \lambda B)^{-1} W^n f d\lambda + \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} \sum_{k=0}^{n-1} \lambda^k B^{k+1} W^n f d\lambda = I_1(t) + I_2(t).$$

Since the operators W^n and $B(I - \lambda B)^{-1}$ commute, this fact can be obtained by direct calculation, we get $I_1(t) = W^n u(t)$. Analogously to the reasonings of Lemma 5, we get $I_2(t) = 0$. Hence

$$\frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} \lambda^n B(I - \lambda B)^{-1} f d\lambda = W^n u(t).$$

Thus, we obtain the first statement of the lemma. Let us show that the operator $\varphi(W)$ is closeable on the set of elements

$$u(t) = \frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda, f \in D(W^n), n \in \mathbb{N}.$$

In accordance with the definition (see (5.6) [3, p. 165]), we need prove that if there exist simultaneous limits $u_k^{(j)}(t) \rightarrow u^{(0)}$, $\varphi(W)u_k^{(j)}(t) \rightarrow u^{(j)}$, $k \rightarrow \infty, t \rightarrow +0, j = 1, 2$, then $u^{(1)} = u^{(2)}$. Note that in accordance with Lemma 3, for each $k \in \mathbb{N}$, we get $u_k^{(j)}(t) \rightarrow u_k^{(j)}$, $t \rightarrow +0$. Applying the theorem which gives the connection between simultaneous limits and repeated limits, we get $u_k^{(j)} \rightarrow u^{(0)}$, $k \rightarrow \infty$. Using the simple estimating, we get

$$\begin{aligned} & \left\| \int_{\gamma(B)} \varphi(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} \{u_k^{(j)} - u^{(0)}\} d\lambda \right\|_{\mathfrak{H}} \\ & \leq C \|u_k^{(j)} - u^{(0)}\|_{\mathfrak{H}} \int_{\gamma(B)} |\varphi(\lambda) e^{-\lambda t}| \cdot |d\lambda| \leq C \|u_k^{(j)} - u^{(0)}\|_{\mathfrak{H}}. \end{aligned}$$

Therefore, there exist coincident limits

$$\varphi(W)u_k^{(j)}(t) \rightarrow \frac{1}{2\pi i} \int_{\gamma(B)} \varphi(\lambda) e^{-\lambda t} B(I - \lambda B)^{-1} u^{(0)} d\lambda, k \rightarrow \infty.$$

Applying the theorem on the connection between simultaneous limits and repeated limits, we get $u^{(1)} = u^{(2)}$. Thus, we have proved that the operator $\varphi(W)$ is closeable. Applying Lemma 3, we get $u(t) \rightarrow f, t \rightarrow +0$, using condition (11), we obtain

$$\varphi(W)u(t) \rightarrow g, t \rightarrow +0.$$

Hence $f \in D\{\tilde{\varphi}(W)\}$ and we obtain the second statement of the lemma. The proof is complete. \square For convenience, we will use the following notations

$$J = \sum_{\nu=0}^{\infty} J_{\nu}; J^+ = \sum_{\nu=0}^{\infty} J_{\nu}^+; J^- = \sum_{\nu=0}^{\infty} J_{\nu}^-.$$

Theorem 1. Assume that B is a compact operator, $\Theta(B) \subset \mathfrak{L}_0(\theta)$, $\theta < \pi/2$, $B \in \tilde{\mathfrak{S}}_\rho$, moreover in the case $B \in \tilde{\mathfrak{S}}_\rho \setminus \mathfrak{S}_\rho$ the additional condition holds

$$\frac{n_{B^{m+1}}(r^{m+1})}{r^\rho} \rightarrow 0, m = [\rho], \tag{12}$$

the function φ is an entire function of the minimal or normal type and of the order ξ , the following condition holds $\rho < \xi < 1$. Then a sequence of natural numbers $\{N_\nu\}_0^\infty$ can be chosen so that

$$\frac{1}{2\pi i} \int_{\gamma(B)} e^{-\lambda t} \varphi(\lambda) B(I - \lambda B)^{-1} f d\lambda = \sum_{\nu=0}^\infty \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)k(q_\xi)} \sum_{i=0}^{m(q)k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t),$$

moreover

$$\sum_{\nu=0}^\infty \left\| \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)k(q_\xi)} \sum_{i=0}^{m(q)k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t) \right\|_{\mathfrak{H}} < \infty. \tag{13}$$

Proof. Consider a contour $\gamma(B)$. Having fixed $R > 0, 0 < \kappa < 1$, so that $R(1 - \kappa) = r$, consider a monotonically increasing sequence $\{R_\nu\}_0^\infty, R_\nu = R(1 - \kappa)^{-\nu+1}$. Using Lemma 3, we get

$$\|(I - \lambda B)^{-1}\|_{\mathfrak{H}} \leq e^{\gamma(|\lambda|)|\lambda|^\rho} |\lambda|^m, m = [\rho], |\lambda| = \tilde{R}_\nu, R_\nu < \tilde{R}_\nu < R_{\nu+1},$$

where the function $\gamma(r)$ is defined in Lemma 2,

$$\beta(r) = r^{-\frac{\rho}{m+1}} \left(\int_0^r \frac{n_{B^{m+1}}(t)}{t} dt + r \int_r^\infty \frac{n_{B^{m+1}}(t)}{t^2} dt \right).$$

Note that in accordance with Lemma 3 [15] the following relation holds

$$\sum_{i=1}^\infty \lambda_i^{\frac{\rho+\varepsilon}{m+1}}(\tilde{B}) \leq \sum_{i=1}^\infty s_i^{\rho+\varepsilon}(B) < \infty, \varepsilon > 0, \tag{14}$$

where $\tilde{B} := (B^{*m+1} A^{m+1})^{1/2}$. It is clear that $\tilde{B} \in \tilde{\mathfrak{S}}_\nu, \nu \leq \rho/(m+1)$. Denote by γ_ν a bound of the intersection of the ring $\tilde{R}_\nu < |\lambda| < \tilde{R}_{\nu+1}$ with the interior of the contour $\gamma(B)$, denote by N_ν a number of poles being contained in the set $\text{int } \gamma(B) \cap \{\lambda : r < |\lambda| < \tilde{R}_\nu\}$. In accordance with Lemma 4, we get

$$\frac{1}{2\pi i} \oint_{\gamma_\nu} e^{-\lambda t} \varphi(\lambda) B(I - \lambda B)^{-1} f d\lambda = \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)k(q_\xi)} \sum_{i=0}^{m(q)k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t).$$

Let us estimate the above integral, for this purpose split the contour γ_ν on terms $\tilde{\gamma}_\nu := \{\lambda : |\lambda| = \tilde{R}_\nu, |\arg \lambda| \leq \theta + \varepsilon\}, \tilde{\gamma}_{\nu+1}, \gamma_{\nu+} := \{\lambda : \tilde{R}_\nu < |\lambda| < \tilde{R}_{\nu+1}, \arg \lambda = \theta + \varepsilon\}, \gamma_{\nu-} := \{\lambda : \tilde{R}_\nu < |\lambda| < \tilde{R}_{\nu+1}, \arg \lambda = -\theta - \varepsilon\}$. In accordance with Lemma 2, we have

$$J_\nu := \left\| \int_{\tilde{\gamma}_\nu} e^{-\lambda t} \varphi(\lambda) B(I - \lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq \int_{\tilde{\gamma}_\nu} e^{-\lambda t} |\varphi(\lambda)| \|B(I - \lambda B)^{-1} f\|_{\mathfrak{H}} |d\lambda| \leq$$

$$\leq e^{\gamma(|\lambda|)|\lambda|^\rho} |\lambda|^{m+1} \max |\varphi(\lambda)| \int_{-\theta-\varepsilon}^{\theta+\varepsilon} e^{-t\operatorname{Re}\lambda} d\arg\lambda, |\lambda| = \tilde{R}_\nu.$$

Using the theorem conditions, we get $|\arg\lambda| < \pi/2, \lambda \in \tilde{\gamma}_\nu, \nu = 0, 1, 2, \dots$. It follows that

$$\operatorname{Re}\lambda \geq |\lambda| \cos(\pi/2 - \delta) = |\lambda| \sin \delta,$$

where δ is a sufficiently small number. Thus, we get

$$J_\nu \leq C e^{\gamma(|\lambda|)|\lambda|^\rho - t|\lambda| \sin \delta} |\lambda|^{m+1} \max |\varphi(\lambda)| \leq C e^{|\lambda|^\rho \{\gamma(|\lambda|) - |\lambda|^{\xi-\rho} (|\lambda|^{1-\xi} t \sin \delta - C)\}} |\lambda|^{m+1},$$

where $m = [\rho], |\lambda| = \tilde{R}_\nu$. Let us show that for a fixed t and a sufficiently large $|\lambda|$, we have $\gamma(|\lambda|) - |\lambda|^{\xi-\rho} (|\lambda|^{1-\xi} t \sin \delta - C) < 0$. It follows directly from Lemma 2 [10], we should consider (14), in the case when $B \in \mathfrak{S}_\rho$ as well as in the case $B \in \tilde{\mathfrak{S}}_\rho \setminus \mathfrak{S}_\rho$ but here we must involve the additional condition (12). Therefore, the series J converges. Using the analogous estimates, applying Lemma 1, we get

$$\begin{aligned} J_\nu^+ &:= \left\| \int_{\gamma_{\nu+}} e^{-\lambda t} \varphi(\lambda) B(I - \lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq C \|f\|_{\mathfrak{H}} \cdot \max_{|\lambda|=\tilde{R}_{\nu+1}} |\varphi(\lambda)| \int_{\tilde{R}_\nu}^{\tilde{R}_{\nu+1}} |e^{-t\lambda}| d\lambda \leq \\ &\leq C e^{-t\tilde{R}_\nu \sin \delta} e^{C\tilde{R}_{\nu+1}^\xi} \int_{\tilde{R}_\nu}^{\tilde{R}_{\nu+1}} |d\lambda| = C e^{-t\tilde{R}_\nu \sin \delta + C\tilde{R}_{\nu+1}^\xi} \{\tilde{R}_{\nu+1} - \tilde{R}_\nu\} = \\ &= C e^{-t\tilde{R}_\nu \sin \delta + C\tilde{R}_\nu^\xi (1-\kappa)^{-\xi}} \{\tilde{R}_{\nu+1} - \tilde{R}_\nu\}; \\ J_\nu^- &:= \left\| \int_{\gamma_{\nu-}} e^{-\lambda t} B(I - \lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq C e^{-t\tilde{R}_\nu \sin \delta + C\tilde{R}_\nu^\xi (1-\kappa)^{-\xi}} \{\tilde{R}_{\nu+1} - \tilde{R}_\nu\}. \end{aligned}$$

The obtained results allow us to claim (the proof is left to the reader) that the series J^+, J^- are convergent. Thus, we obtain relation (13), from what follows the rest part of the theorem statement. \square

Corollary 1. *Under the Lemma 6, Theorem 1 assumptions, we get*

$$\begin{aligned} \varphi(W)u(t) &= \sum_{\nu=0}^{\infty} \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t), f \in D(W^n), n \in \mathbb{N}; \\ \tilde{\varphi}(W)f &= \lim_{t \rightarrow +0} \sum_{\nu=0}^{\infty} \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t). \end{aligned}$$

Note that Corollary 1 establishes the analog of the function of a selfadjoint compact operator. The series at the right-hand side of the last formula converges to the element at the left-hand side in the $(B, \varphi, 1)$ sense. The analog of Theorem 1 corresponding to the function with a polynomial regular part of the Laurent series considered in Lemma 5 can be obtained in a simpler way.

Introduction to evolution equations in the abstract Hilbert space

In this paragraph, we consider a Hilbert space \mathfrak{H} consists of element-functions $u : \mathbb{R}_+ \rightarrow \mathfrak{H}$, $u := u(t)$, $t \geq 0$ and we will assume that if u belongs to \mathfrak{H} then the fact holds for all values of the variable t . Notice that under such an assumption all standard topological properties as completeness, compactness e.t.c. remain correctly defined. We understand such operations as differentiation and integration in the generalized sense that is caused by the topology of the Hilbert space \mathfrak{H} . The derivative is understood as the following limit

$$\frac{u(t + \Delta t) - u(t)}{\Delta t} \xrightarrow{\mathfrak{H}} \frac{du}{dt}, \Delta t \rightarrow 0.$$

Let $t \in \Omega := [a, b]$, $0 < a < b < \infty$. The following integral is understood in the Riemann sense as a limit of partial sums

$$\sum_{i=0}^n u(\xi_i) \Delta t_i \xrightarrow{\mathfrak{H}} \int_{\Omega} u(t) dt, \lambda \rightarrow 0,$$

where $(a = t_0 < t_1 < \dots < t_n = b)$ is an arbitrary splitting of the segment Ω , $\lambda := \max_i (t_{i+1} - t_i)$, ξ_i is an arbitrary point belonging to $[t_i, t_{i+1}]$. The sufficient condition of the last integral existence is a continuous property (see [12, p.248]) i.e. $u(t) \xrightarrow{\mathfrak{H}} u(t_0)$, $t \rightarrow t_0$, $\forall t_0 \in \Omega$. The improper integral is understood as a limit

$$\int_a^b u(t) dt \xrightarrow{\mathfrak{H}} \int_a^c u(t) dt, b \rightarrow c, c \in [0, \infty].$$

Combining these operations, we can consider a fractional differential operator in the Riemann-Liouville sense (see [23]), i.e. in the formal form, we have

$$\mathfrak{D}_-^\alpha f(t) := -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^\infty f(t+x)x^{-\alpha} dx, 0 < \alpha < 1.$$

The auxiliary facts given above allow us to consider the following Cauchy problems as a prerequisite of a rather wide area of applications

$$\frac{du}{dt} = \varphi(W)u(t), u(0) = h \in D(W), \mathfrak{D}_-^{1/\alpha} u = \varphi(W)u(t), u(0) = h \in D(W), \quad (15)$$

the first problem was considered in [11] under the assumption $\varphi(z) = z^n$, $n \in \mathbb{N}$ the second one was considered in [10], where we suppose $\varphi(z) = z$. If we analyze the proofs of the corresponding theorems establishing the existence and uniqueness of the problems (15), then taking into account theoretical results of this paper, we will see that the corresponding generalizations can be obtained. Certainly, we should impose such conditions on the function φ that guarantee convergence of the obtained integral constructions. Here, we want to end our narrative and leave the idea for a further more detailed and deep study.

Conclusions

In this paper, we invented a technique to study evolution equations with the right-hand side a function of the non-selfadjoint unbounded operator. Having inspired by the Lidskii idea involving a notion of convergence of a series on the root vectors of the operator in a weaker – Abel-Lidskii sense, we continue constructing the abstract theory in the direction.

The main issue of the paper is how to generalize the spectral theorem for the non-selfadjoint operator. In this way, we come to the definition of a function of the unbounded non-selfadjoint operator. The main highlights of this paper are propositions analogous to the spectral theorem, the main obstacle that appears is how to define an analogue of a spectral family or decomposition of the identical operator. We should admit that an analogous way of reformulating the main principles of the spectral theorem, considering the peculiarities of the convergence, has been found.

As a prerequisite to applications, we noticed some approaches allowing us to principally broaden conditions imposed on the right-hand side of the evolution equation in the abstract Hilbert space. The application part of the paper appeals to the theory of differential equations. In particular, the existence and uniqueness theorems for evolution equations with the right-hand side – a differential operator with a fractional derivative in final terms are covered by the invented abstract method. In connection with this, such operators as a Riemann-Liouville fractional differential operator, Kipriyanov operator, Riesz potential, difference operator can be considered. Moreover, we can consider the artificially constructed normal operators for which the clarification of the Lidskii results relevantly works. Here, we should explain that we can construct a normal operator with a compact resolvent in the artificial way, having known its eigenvalues.

If we analyze the proofs of the corresponding theorems establishing the existence and uniqueness of the solution of the evolution equation and take into account theoretical results of this paper, then we can see that relevant generalizations can be obtain.

Apparently, we should study thoroughly conditions that should be imposed upon the function of the operator at the right-hand side of the evolution equation. This is where we want to finish in the hope that the concept will be further developed.

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References

- [1] Agranovich M. S. On series with respect to root vectors of operators associated with forms having symmetric principal part, *Functional Analysis and its applications*, 1994. vol. 28, pp. 151-167.
- [2] Gohberg I. C., Krein M. G. *[Introduction to the theory of linear non-selfadjoint operators in a Hilbert space]*. Nauka: Moscow, 1965 (In Russian).

- [3] Kato T. *Perturbation theory for linear operators*. Berlin, Heidelberg, New York: Springer-Verlag, 1980.
- [4] Katsnelson V. E. Conditions under which systems of eigenvectors of some classes of operators form a basis, *Funct. Anal. Appl.*, 1967. vol. 1, no. 2, pp. 122-132.
- [5] Kipriyanov I. A. On spaces of fractionally differentiable functions, *Izv. Akad. Nauk SSSR Ser. Mat.*, 1960. vol. 24, no. 6, pp. 865-882 (In Russian).
- [6] Kipriyanov I. A. The operator of fractional differentiation and powers of the elliptic operators, *Dokl. Akad. Nauk SSSR*, 1960. vol. 131, no. 2, pp. 238-241.
- [7] Kukushkin M. V. On One Method of Studying Spectral Properties of Non-selfadjoint Operators, *Abstract and Applied Analysis*, 2020. vol. 2020, 1461647 DOI:10.1155/2020/1461647.
- [8] Kukushkin M. V. Asymptotics of eigenvalues for differential operators of fractional order, *Fract. Calc. Appl. Anal.*, 2019. vol. 22, no. 3, pp. 658-681 DOI:10.1515/fca-2019-0037.
- [9] Kukushkin M. V. Abstract fractional calculus for m-accretive operators, *International Journal of Applied Mathematics*, 2021 DOI:10.12732/ijam.v34i1.1. vol. 34, no. 1.
- [10] Kukushkin M. V. Natural lacunae method and Schatten-von Neumann classes of the convergence exponent, *Mathematics*, 2022. vol. 10, no. 13, 2237 DOI:10.3390/math10132237.
- [11] Kukushkin M. V. Evolution Equations in Hilbert Spaces via the Lacunae Method, *Fractal Fract.*, 2022. vol. 6, no. 5, 229 DOI:10.3390/fractalfract6050229.
- [12] Krasnoselskii M. A., Zabreiko P. P., Pustynnik E. I., Sobolevskii P. E. *Integral'nyye operatory v prostranstvakh summiruyemykh funktsiy [Integral operators in the spaces of summable functions]*. Moscow: Nauka, FIZMATLIT, 1966.
- [13] Krein M. G. Criteria for completeness of the system of root vectors of a dissipative operator, *Amer. Math. Soc. Transl. Ser., Amer. Math. Soc., Providence, RI*, 1963. vol. 26, no. 2, pp. 221-229.
- [14] Levin B. Ja. *Distribution of Zeros of Entire Functions*, Translations of Mathematical Monographs, vol. 5. Providence, Rhode Island: American Mathematical Soc., 1964. 523 pp.
- [15] Lidskii V. B. Summability of series in terms of the principal vectors of non-selfadjoint operators, *Tr. Mosk. Mat. Obs.*, 1962. vol. 11, pp. 3-35 (In Russian).
- [16] Markus A. S., Matsaev V. I. Operators generated by sesquilinear forms and their spectral asymptotics. Linear operators and integral equations, *Mat. Issled.*, 1981. vol. 61, pp. 86-103 (In Russian).
- [17] Markus A. S. Expansion in root vectors of a slightly perturbed selfadjoint operator, *Soviet Math. Dokl.*, 1962. vol. 3, pp. 104-108 (In Russian).
- [18] Mamchuev M. O. Solutions of the main boundary value problems for the time-fractional telegraph equation by the Green function method, *Fractional Calculus and Applied Analysis*, 2017. vol. 20, no. 1, pp. 190-211 DOI: 10.1515/fca-2017-0010.
- [19] Moroz L., Maslovskaya A. G. Hybrid stochastic fractal-based approach to modeling the switching kinetics of ferroelectrics in the injection mode, *Mathematical Models and Computer Simulations*, 2020. vol. 12, pp. 348-356.
- [20] Nakhushev A. M. The Sturm-Liouville problem for an ordinary differential equation of the second order with fractional derivatives in lower terms, *Dokl. Akad. Nauk SSSR USSR*, 1977. vol. 234, no. 2, pp. 308-311 (In Russian).
- [21] Pskhu A. V. The fundamental solution of a diffusion-wave equation of fractional order, *Izvestiya: Mathematics*, 2009. vol. 73, no. 2, pp. 351-392.
- [22] Riesz F., Nagy B. Sz. *Functional Analysis*. Ungar, New York, 1955.
- [23] Samko S. G., Kilbas A. A., Marichev O. I. *Fractional Integrals and Derivatives: Theory and Applications*. Philadelphia, PA, USA: Gordon and Breach Science Publishers, 1993.
- [24] Shkalikov A. A. Perturbations of selfadjoint and normal operators with a discrete spectrum, *Russian Mathematical Surveys*, 2016. vol. 71, no. 5(431), pp. 113-174.

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Научная статья


Замечание о спектральной теореме для неограниченных несамосопряженных операторов

М. В. Кукушкин

Московский государственный строительный университет,
Ярославское ш., 26, 129337, Москва, Россия
E-mail: kukushkinmv@rambler.ru

В данной работе, дав определение сходимости ряда по корневым векторам в смысле Абеля-Лидского, мы представляем актуальное приложение в теории эволюционных уравнений. Основной целью является подход, позволяющий нам принципиально расширить условия, налагаемые на правую часть эволюционного уравнения в абстрактном гильбертовом пространстве. Таким образом, мы приходим к определению функции неограниченного не самосопряженного оператора. Между тем, мы вовлекаем дополнительную концепцию, которая является обобщением спектральной теоремы для не самосопряженного оператора.


Key words: базисное свойство Абеля-Лидского; класс Шаттена-фон Неймана; оператор функции; эволюционное уравнение.

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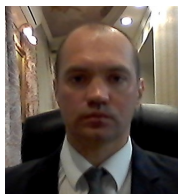
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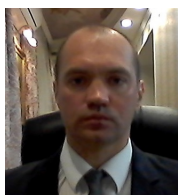
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Kukushkin Maksim Vladimirovich ✉ – PhD (Phys. & Math.), Senior Lecturer, Department of Higher Mathematics, Moscow State University of Civil Engineering, Moscow, Russia,  ORCID 0000-0003-0598-032X.



Кукущкин Максим Владимирович ✉ – кандидат физико-математических наук, старший преподаватель кафедры высшей математики, Московский государственный строительный университет, г. Москва, Россия,  ORCID 0000-0003-0598-032X.
