MSC 49J15, 49N05 Research Article

# Integral operators, embedding theorems, Taylor coefficients, isometries, boundary behaviour of Area-Nevanlinna type spaces in higher dimension and related problems

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This paper contains an overview of recent results of Area-Nevanlinna classes in higher dimension. We here consider various aspects of this new interesting research area of analytic function theory in higher dimension (integral operations, embedding theorems, Taylor coefficients). Previously in one dimension all these results were known. New open interesting Problems in this new research area will be also discussed and indicated.

*Keywords: polydisk, unit ball, Taylor coefficients, integral operators, analytic functions, analytic spaces, area Nevanlinna type spaces, tubular domain, pseudoconvex domain, isometries, boundary behaviour*

DOI: 10.26117/2079-6641-2021-36-3-40-64

Original article submitted: 01.04.2021 Revision submitted: 15.07.2021

**For citation.** Shamoyan R. F. Integral operators, embedding theorems, Taylor coefficients, isometries, boundary behaviour of Area-Nevanlinna type spaces in higher dimension and related problems. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2021, **36**: 3, 40-64. DOI: 10.26117/2079-6641-2021-36-3-40-64



## 1. Introduction

The goal of this expository paper to combine together some recent results on a new interesting area of analytic function theory so-called Area-Nevanlinna spaces in higher dimension. This paper is the second part of our notes on area Nevanlinna type spaces. We refer to [1] for the first part related with spaces of this type in one variable. Known various one dimensional results related with such spaces will not be included in this paper. We restrict ourselves only on functions of several complex variables. Probably the first important results in this direction of research were obtained by Dautov, Henkin and Skoda in [2, 3]. We mention [4, 5, 6] where some other results on this interesting topic can be seen. These results are not new and will not be discussed here. On the other hand some interesting open problems in this research area will be posed by us

**Funding.** The study was carried out without financial support from foundations.

and some recent new extensions of known one dimensional results will be discussed and provided or mentioned.

Some new basic ideas needed for proofs will also be indicated in this paper. Some new one dimensional results of our colleagues related with the topic of this paper published recently in local Russian journals (or by Russian experts) will also be given shortly in this work.(See [7, 8, 9] for another survey on analytic area Nevanlinna spaces in $\mathbb{C}$)

In this paper we discuss various new interesting problems in area Nevalinna spaces in $\mathbb{C}^n$. But we will not talk on vital issues related with zero sets and factorization in such spaces.

Let $U^n$ be the unit polydisk in $\mathbb{C}^n$, $U^n = \{z \in \mathbb{C}^n : |z_j| < 1, j = 1,\dots,n\}$. Let $H(U^n)$ be the space of all analytic functions in $U^n$. Let further $B_n$ be the unit ball in $\mathbb{C}^n$, $B_n = \{z \in \mathbb{C}^n : |z| < 1\}, S_n = \{|z| = 1\}$ be unit sphere. Let $H(B_n)$ be the space of all analytic functions in $B_n$.

The main goal of this paper is to present or discuss certain new results concerning Taylor coefficients, integral operators of differentiation and integration and some new embedding theorems in new analytic area Nevanlinna type spaces in the unit polydisk and in the unit ball. Note some other related problems, related with boundary behaviour and isometries for example for such spaces in higher dimension will be also discussed. We shortly also discuss extremal problems, diagonal mapping etc in such analytic function spaces in $C^n$ .

These new results will be provided mainly without proofs, some sketches of proofs however will be presented. We also discuss shortly some new open problems concerning area Nevanlinna spaces in general domains such as tubular domains over symmetric cones and bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$. These analytic Nevanlinna type spaces are new as far as we know and it will be nice to study them.

We in this paper as usual denote below by $c, C_1, C_\alpha, \dots$ various positive constants in various inequalities and various estimates.

## 2. On the action of differentiation operators in Nevanlinna type spaces in $\mathbb{C}^n$ and related problems

To formulate our results we need some definitions and more notations.

Let further $\omega$ be a certain special fixed positive weight from certain fixed S class (see [10, 11, 12, 13]) of slowly varying functions.

We denote various positive constants in this paper as usual by $C, C_1, C_2, C_3, C_4, C_\alpha, \dots$

Let further

$$(N^{p,q}_{\omega})(U^n) = \{f \in H(U^n) :$$

$$= \int_{T^n} \left( \int_{I^n} \left( ln^+ |f(\tau_1\xi_1, ..., \tau_n\xi_n)| \right)^p \prod_{k=1}^{n} \omega(1-\tau_k) d\tau_k \right)^{\frac{q}{p}} d\xi_1 ... d\xi_n < +\infty\},$$

$$(\widetilde{N}^{p,q}_{\omega})(U^n) = \{f \in H(U^n) :$$

$$: \int_{I^n} \left( \int_{T^n} (ln^+ |f(\tau_1\xi_1, \dots, \tau_n\xi_n)|)^p d\xi_1 \dots d\xi_n \right)^{\frac{q}{p}} \prod_{k=1}^{n} \omega(1-\tau_k) d\tau_k < +\infty\},$$

where as usual

$T^n = \{z \in \mathbb{C}^n : |z_j| = 1, j = 1,\dots,n\}, I^n = [0,1]^n$, and $0 < p, q \le \infty$, and where $u^+ = max(u, 0)$. Our weight can be also radial $(1-r)^\alpha$. In this case we also define $N^{p,q}_{\vec{\omega}}, N^{p,q}_{\alpha} = N^p_\alpha$ if $\omega(r) = r^\alpha, \alpha > -1, p = q$.

These spaces are Banach spaces for $min(p,q) \geq 1$ and complete metric spaces for other values of $p,q$. Same type spaces $N^{p,q}_{\omega}(B^n); \tilde{N}^{p,q}_{\omega}(B^n)$ can be easily defined in $B^n$. We leave this to readers.

Let $\Lambda$ be a bounded (or unbounded) domain with $\mathbb{C}^2$ boundary, let $H(\Lambda)$ be the space of all analytic spaces in $\Lambda$. We denote by $dV_{\alpha}$ the weighted Lebegues measure on $\Lambda$. Let $\partial\Lambda$ be boundary of $\Lambda$. Let further

$$dV_{\alpha}(z) = (dist(z, \partial\Lambda))^{\alpha} dV(z), z \in \Lambda, \alpha > -1,$$

where $dV$ is a normalized Lebegues measure on $\Lambda$.

Further we considered also area Nevanlinna spaces on $\Lambda$.

$$N^p_{\alpha}(\Lambda) = \{f \in H(\Lambda) : ||f||^p_{N^p_{\alpha}(\Lambda)} = \int_{\Lambda} (log^+|f(z)|)^p dV_{\alpha}(z) < \infty\}; 0 < p \leq \infty$$

and these are Banach spaces for $p \geq 1$ and complete metric spaces for other values of $p$. If $\Lambda = B_n$ then we have analytic area Nevanlinna spaces in the unit ball and similarly in the polydisk. To study such spaces is a nice problem.

Area Nevanlinna type spaces similarly can be also considered in difficult general unbounded domains like tube domains over symmetric cones and more general Siegel domains of second type. Moreover they can be defined also on products of such type domains. We refer the reader for example to [13] and [14] for new results on other spaces of analytic functions on product domains.

Product domains and various problems on analytic spaces on them also were considered recently in a series of papers of author (see [13, 14] and various references there).

We formulate some problems for area Nevanlinna type spaces in the unit polydisk and in the unit ball.

The problem of coefficient multipliers can be formulated in polydisk as follows. Let $X,Y \subset H(U^n)$ be quazinormed subspaces of $H(U^n)$. We say $\{c_{k_1,\dots,k_n}\}$, $k_j \geq 0, j = 1,\dots,n$ is a multiplier from $X$ to $Y$ if for every $f, f \in X$ :

$$f(z) = \sum_{k_1 \geq 0} \dots \sum_{k_n \geq 0} a_{k_1,\dots,k_n} z_1^{k_1} \dots z_n^{k_n}, z \in U^n$$

we have that

$$\sum_{k_1 \geq 0} \dots \sum_{k_n \geq 0} c_{k_1,\dots,k_n} a_{k_1,\dots,k_n} z_1^{k_1} \dots z_n^{k_n} \in Y.$$

We denote this space as usual by $M_T(X,Y)$. Here $X$ and $Y$ are various spaces of area Nevanlinna type in the polydisk $U^n$ (see [8, 15, 16] for $n = 1$). For other spaces this problem is well-known (see, for example, [4, 13]). Note some other interesting problems related with Taylor coefficients exists in Nevanlinna spaces (growth rate, etc.) (see below).

Let $\Lambda$ be a bounded domain with $\mathbb{C}^2$ boundary. Let $\mu$ be the positive Borel measure on $\Lambda$. The embedding problem for area Nevanlinna spaces is to find characte-rizations of such positive Borel $\mu$ measures so that

$$\int_{\Lambda} (log^+|f(z)|)^p d\mu(z) \leq c||f||_X,$$

(or similar type embeddings) where $0 < p \leq \infty$, and where $X \subset H(\Lambda)$ is a fixed quazinormed subspace of $H(\Lambda)$ (see [17] for $n = 1$).

Even in particular cases of unit ball and polydisk a lot of open problems in this area related with both these problems in area Nevanlinna type spaces. Note if $p \geq 1$, then $(log^{+}|f(z)|)^{p}$ is subharmonic and this fact is some cases is crucial for proofs of such type embeddings (see, for example, [1, 17, 18, 19]).

Let $(Df)(z) = \frac{(\partial f)(z_1 \dots z_n)}{\partial z_1 \dots z_n}, z_j \in U, j = 1, \dots, n$ and let $\omega$ be weight from a special fixed $S$ class in $U$. We wish to find sharp conditions on pair $(\omega_1, \omega)$, so that the differentiation $D$ operator maps $\tilde{N}^{p,q}_{\vec{\omega}}$ or $N^{p,q}_{\vec{\omega}_1}$ to $N^{p,q}_{\vec{\omega}_1}$ or $\tilde{N}^{p,q}_{\vec{\omega}}$, where $\vec{\omega}_1$ also belongs to $S$ class, where

$$\vec{\omega}_1 = (\omega_1^1, \dots, \omega_n^1); \vec{\omega} = (\omega_1, \dots, \omega_n); \omega_j \in S, \omega_j^1 \in S, j = 1, \dots, n.$$

Next we can consider the problem of so-called diagonal mapping and related issues in analytic area Nevanlinna type spaces in higher dimension. Let $\Lambda$ be bounded domain. Let $f \in H(\Lambda^m)$, where $\Lambda^m = \Lambda \times \dots \times \Lambda, m \in \mathbb{N}$. $H(\Lambda^m)$ is a space of all analytic functions on $\Lambda^m$. Let

$$||f||^{p}_{N^{p}_{\vec{\alpha}}(\Lambda^m)} = \int_{\Lambda} \dots \int_{\Lambda} (log^{+}|f(z_1, \dots, z_m)|)^{p} dV_{\alpha_1}(z_1) \dots dV_{\alpha_m}(z_m) < \infty, \alpha_j > -1,$$

$$j = 1, \dots, m, 0 < p \leq \infty.$$

The problem is to indicate precisely X functional class, so that we have $f(z, \dots z) \in X$, where $X \subset H(\Lambda)$. For various analytic spaces this problem was solved (see [4, 5, 21] and various references there).

Let $dm_{2n}$ be the Lebeques measure on the unit polydisk $U^n$.

Let $T(\tau, f)$ be Nevanlinna characteristic of $f, f \in H(U)$ (see [11, 12]). Let below always $\omega$ be a function from a set of all positive growing functions, $(\omega \in L^1(0,1))$ such that there are two numbers $m_\omega > 0, M_\omega > 0$ and number $q_\omega \in (0,1)$ such that $m_\omega \leq \frac{\omega(\lambda \tau)}{\omega(\tau)} < M_\omega, \tau \in (0,1), \lambda \in [q_\omega, 1]$ (see [11, 12]). Let $\omega \in S$, then there are measurable functions $w(x), q(x)$ so that

$$\varepsilon(x) = exp\{q(x) + \int \frac{\omega(u)}{u} du\}, x \in (0,1)$$

(see [11, 12]). This characterization gives various examples of function from $S$ class/ A typical example is $\omega(r) = r^{\alpha}, \alpha > -1, r \in (0,1)$ or $\omega(r) = r^{\alpha}(\log \frac{C}{r})^{\beta}, \alpha > -1, \beta > 0, r \in (0,1)$.

It is obvious that for $q = \infty, \omega = 1$ the $N^{p,q}_{\omega}$ coincides with well-known $N^p$ spaces of holomorphic functions with bounded characteristic (see [11, 12, 20]).

In the recent papers (see, for example, [11, 12, 20]), it was noted that the following assertions concerning the action of differentiation $D(f)z = f'(z)$ and integration $I(f)(z) = \int_0^z f(t)dt$ in the unit disk are valid in mentioned analytic classes. $N^{q,q}_{\alpha}$ is closed under differentiation operator $D(f)$ is closed under differentiation and integration operator. $N^{q,q}_{\omega}$ and $N^{1,q}_{\omega}$ are closed under differentiation operator $D(f)$ if and only if $\int_0^1 \omega(t)(\ln \frac{1}{t})^{p} dt < +\infty$. The study $I(f), D(f)$ in Smirnov $N^{+}$ class were studied also before (see [11, 12] and references there).

We note much earlier in [11, 12] Frostman then W.K. Hayman (see [3, 4]) established that the $N^p$ class is not invariant under differentiation operator, but $N^p, p > 1$ are closed for differentiation operator, but not $N^1$.

The natural question is to study differentiation operator in $N_\alpha^{q,q}(\tilde{N}_\alpha^{q,q})$. The goal of this paper is to give in particular several new sharp results in this direction.

Finally we would like to indicate that some assertions of this section were obtained by modification of approaches and argument provided recently in [20]. All our results in higher dimension were obtained for $n = 1$ in [18, 20, 21, 22].

Motivated by mentioned results in this section we provide new assertions concerning differentiation operator $D(f)$ in new Nevanlinna type spaces that were defined above. In the following assertion, we provide several sharp results on the action of the differentiation operator in Nevanlinna type analytic spaces in the unit polydisk complementing previously known propositions of this type obtained before by various (see, for example, [4, 11, 12, 20] and references there).

Let further $Df(\vec{z}) = \frac{f(z_1,\ldots,z_n)}{\partial z_1 \ldots \partial z_n}$.

Now we formulate some new sharp results in higher dimentions from [11, 12].

**Theorem 1.** *Let* $0 < p < \infty, \int\limits_0^1 \omega_j(t)dt < +\infty, j = 1,2,\ldots,n$. *Then*

$$\int\limits_{I^n}(\int\limits_{T^n} ln^+|Df(\tau_1\xi_1,\ldots,\tau_n\xi_n)|d\xi_1\ldots d\xi_n)^p \prod_{j=1}^n \omega_j(1-\tau_i)d\tau_1\ldots d\tau_n \leq$$

$$\leq C\int\limits_{I^n}(\int\limits_{T^n} ln^+|f(\tau_1\xi_1,\ldots,\tau_n\xi_n)|d\xi_1\ldots d\xi_n)^p \prod_{j=1}^n \omega_j(1-\tau_i)d\tau_1\ldots d\tau_n$$

*if and only if*

$$\int\limits_0^1 \omega_j(t)(ln\frac{1}{t})^p dt < +\infty, j = 1,2,\ldots,n.$$

**Theorem 2.** *Let* $s \geq 1, s \geq max(q,p), \omega = \prod\limits_{j=1}^n w_j$. *Let*

$$\frac{2}{s} - \frac{1}{p} > 0, \tilde{\omega}_j(1-|z_j|) = \omega_j(1-|z_j|)^{\frac{q}{s}}(1-|z_j|)^{\frac{2q}{s}-\frac{q}{p}-1}.$$

*Then* $Df$ *is acting from* $N_{\tilde{\omega}}^{p,q}, \tilde{N}_{\tilde{\omega}}^{p,q}$ *to* $N_\omega^{s,s}$ *if and only if*

$$\int\limits_0^1 \omega_j(1-\tau)(ln\frac{1}{1-\tau})^s d\tau < +\infty, j = 1,2,\ldots,n.$$

Similar results may be valid in the following function spaces in the ball

$$N^1_{p,q,\omega}(B^n) = \{f \in H(B^n) : \int\limits_{S^n}\left(\int\limits_I (ln^+|f(\tau_1\xi_1,\ldots,\tau_n\xi_n)|d\vec{\xi})^p \omega(1-\tau)d\tau\right)^{\frac{q}{p}} d\vec{\xi} < +\infty\},$$

$$N^2_{p,q,\omega}(B^n) = \{f \in H(B^n) : \int\limits_I \left(\int\limits_{S^n} (ln^+|f(\tau_1\xi_1,\ldots,\tau_n\xi_n)|d\vec{\xi})^p \omega(1-\tau)d\vec{\xi}\right)^{\frac{q}{p}} d\tau < +\infty\},$$

$0 < p,q < \infty$.

Let us mention some lemmas that are needed for the proofs (see [11, 12]).

These are new interesting estimates for Nevanlinna characteristics and spaces in $U^n$.

**Proposition A.** Let $f \in H(U^n), s \geq max(p,q), s > 1$. Then

$$\int\limits_{U^n} (ln^+|f(\vec{z})|)^s \prod_{k=1}^{n} \omega(1-|z_k|)dm_{2n}(\vec{z}) \leq$$

$$\leq C_1 \int\limits_{I^n} \prod_{k=1}^{n} (\omega(1-|z_k|))^{\frac{q}{s}} (1-|z_k|)^{\frac{2q}{s}-\frac{q}{p}-1} \left( \int\limits_{T^n} log^+|f(z)|^q dm_n(\xi) \right)^{\frac{p}{q}} d|z_1|\dots d|z_n|$$

and

$$\int\limits_{U^n} (ln^+|f(\vec{z})|)^s \prod_{k=1}^{n} \omega(1-|z_k|)dm_{2n}(\vec{z}) \leq$$

$$\leq C_2 \int\limits_{T^n} \left( \int\limits_{I^n} \prod_{k=1}^{n} (\omega(1-|z_k|))^{\frac{q}{s}} (1-|z_k|)^{\frac{2q}{s}-\frac{q}{p}-1} log^+|f(\vec{z})|d|z| \right)^{\frac{p}{q}} dm_n(\vec{\xi}).$$

**Lemma 1.**

*1) The following estimations are true*

$$\int\limits_{T^n} ln^+|Df(\tau_1\varphi_1,\dots,\tau_n\varphi_n)|d\varphi_1\dots\varphi_n \leq C_3 \left( \sum_{j=1}^{n} ln\frac{1}{1-\tau_j} + \int\limits_{T^n} ln^+|f(\vec{\tau}\xi)|dm_n(\vec{\xi}) \right),$$

*where*

$$\vec{\tau} = \left( \frac{1+\tau_1}{2},\dots,\frac{1+\tau_n}{2} \right), \tau \in (0,1), i = 1,2,\dots,n;$$

*2)*

$$\ln^+ T\left(\frac{1+\tau}{2}, f\right) \leq C_4 T\left(\frac{1+\tau}{2}, f\right), \tau \in (0,1),$$

$$T(R,f) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} ln^+|f(R\xi)|d\xi, R \in (0,1).$$

**Remark** $1^0$. The proof of 1) is based on 2) in Lemma 1.

**Lemma 2.**

*1) Let* $R_{m_j} = exp\left(-\frac{1}{2^{\lambda m_j}}\right) \in (0,1], t \in (0,+\infty), \lambda > 1, j = 1,2,\dots,n.$

*Then there exists a function* $f, f \in H(U^n)$,

$(ln^+|Df(R_{m_1}e^{i\varphi_1},\dots,R_{m_n}e^{i\varphi_n})|)^t \geq C \sum\limits_{j=1}^{n} \left( ln\frac{1}{1-R_{m_j}} \right)^t, \varphi \in (2\pi].$

*2)* $\int\limits_{T^n} (ln^+|Df(\tau_1 e^{i\varphi_1},\dots,\tau_n e^{i\varphi_n})|)^s d\varphi_1\dots\varphi_n$

*is growing as a function of* $\tau_1,\dots,\tau_n$ *for every* $s \geq 1, f \in H(U^n)$.

**Remark 1.** The statements in the Theorem 2 for $q = p = s, n = 1$, where established in [20].

We at the end of this section discuss some related problems and introduce some new analytic spaces of Nevanlinna type.

In the unit ball $B_n$ consider $f$ function , $f(z_1,\dots,z_m), \vec{z} \in B_n, z_j \in \mathbb{C}$, $j = 1,\dots,m$.

The interesting problem is to find the image of the following operator in various Nevanlinna spaces in the unit ball

$$T_f : f(z_1,\dots,z_n) \to f(z_1,\dots,z_{k-1},0,\dots,0), z_j \in \mathbb{C}, j = 1,\dots,n, k \in [2,n)$$

or

$\tilde{T}_f : f(z_1,\dots,z_n) \to f(z_1,\dots,z_{k-1},z_k^0,\dots,z_n^0), \vec{z}, \vec{z^0} \in \mathbb{C}, k \in [2,n)$, $z = (z_1,\dots,z_n), z_j \in \mathbb{C}, j = 1,\dots,n$ (see [4, 5, 6] for such operators).

If $f$ is from a certain analytic function space in $\mathbb{C}$. Consider slice function $f_\xi(u)$, $u \in U = \{|z| < 1\} : (f_\xi)(u) = f(\xi u), \xi \in \partial B_n = S_n$ (slice function in $B_n$). This slice function was used by many authors to provide at least partially some info on $f$ function in $B_n$ by properties of it is one dimensional version in $U$ (see [4, 5, 6]).

In connection with first problem in various analytic spaces we refer the reader to [4, 5, 6], with second problem we refer the reader to [4, 5, 6], with third problem in various analytic spaces we refer the reader to [4, 5, 6]. The problem is to study such type problems in various Nevanlinna spaces in $\mathbb{C}^n$.

We mention finally also some new extremal problems in area Nevanlinna type spaces.

Let $\Lambda$ be a bounded domain with $\mathbb{C}^2$ boundary or an unbounded domain. Let $X, Y \subset H(\Lambda)$ be normed or quazinormed subspaces of $H(\Lambda)$. Let $X \subset Y, f \in Y$.

We wish to estimate $dist_Y(f,X)$ assuming that $X$ or $Y$ is area Nevanlinna type space of analytic functions in $\Lambda$. The same problem can be posed in various other domains and spaces (see [14] for some results Nevanlinna spaces in this direction and also in another analytic function spaces).

All problems can be posed also in cases of more general product domains and analytic spaces on them. Next problems we posed for bounded domains on $\mathbb{C}^n$ can be posed similarly for Nevanlinna type spaces in unbounded domains in $\mathbb{C}^n$. We leave this simple procedure to interested readers. We mention for example tube domains over symmetric cones (see [23, 24]) as typical example of such unbounded domains.

We now introduce also some other area Nevanlinna type spaces in $\mathbb{C}^n$ (Smirnov spaces, Nevanlinna-Lumer spaces, Nevanlinna spaces in $\mathbb{C}^n$ ).

Note all problems we formulated can be also considered similarly in these spaces.

Let further $\Sigma$ be $S_n$ unit sphere or product of unit circles torus that is $T^n = T \times \cdots \times T, T = \{|z| = 1\}$ .

Let $d\sigma$ be normalized Lebeques measure on $\Sigma$. Then we define Privalov-Nevanlinna $N^p(\Sigma)$ space

$$N^p(\Sigma) = \{f \in H(\Sigma) : (\sup_{0<r<1}) \int_\Sigma (log^+|f(r\xi)|)^p d\sigma(\xi) < \infty\}, 0 < p \leqslant \infty,$$

where $d\sigma$ is appropriate measure if $log^+(|f(r\xi)|)^p, r \in (0,1), 0 < p \leqslant \infty$ is uniformly can be integrated then we define Smirnov space $N_p^+(\Sigma)$ ( or $N_*^p(\Sigma)$).

Note $H^p \subset N_1^+ \subset N^1$, if $H^p, 0 < p \leq \infty$ is a classical Hardy class in the unit ball and polydisk.

These $N_1^+, N^1$ classes are functional algebras in $B_n$ and $U^n$. $N^1$ in $\Sigma$ is called usually Nevanlinna space.

The topology can be given in $N_1^+, N^1$ by invariant metric

$$||f-g||_{N^1} = (\sup_{r<1})||log(1+|f(r\xi)-g(r\xi)|)||_{L^1(\Sigma)}$$

$N_1^+$ is a complete linear metric space by this metric ($F$ space) (see [4]). $N^p(\Sigma)$ is a complete linear metric space (see [4]).

Putting $\alpha = -1$ formally in $N_\alpha^p$ (see definition above) we get $N^p$ space.

Having at hand in $\mathbb{C}^n$ various analytic area Nevanlinna type spaces in this expository paper we formulate below a list of new results concerning these classes and we indicate various references concerning these new sharp (or not sharp) results.

Note an interesting idea to use the fact that $(log^+|f|)^p, p \geqslant 1$ is subharmonic and introduce and study various properties of more general spaces of subharmonic functions of area Nevanlinna type in higher dimension. In particular similar problems can be posed in such general spaces.

We finally add a word on new Lumer-Nevanlinna interesting classes of analytic functions ([5, 6, 25]). First note that $N^1(B_n)$ or $N^1(U^n)$ spaces can be equivalently defined as spaces of analytic $f$ functions with the following condition $log^+|f(z)| \leq u(z)$ for some $n-$harmonic $u$ function, $u$ - is $n-$harmonic for $\Sigma = U_n$ (or $M$ harmonic in $B^n$).

So called Lumer-Nevanlinna spaces $(LN)(\Lambda)$ can be defined if we change $u$ function to pluriharmonic function (majorants). We refer the reader to [5, 6, 25], where some equivalent definitions of $(LN)(\Lambda)$ spaces were also provided.

Let $\tau > -1, T_\Lambda$ be tubular domain, $\Delta$ be determinant function, $H(T_\Lambda)$ be usual space of analytic functions on $T_\Lambda$ (see [23] for more details).

We can also define and study Nevanlinna type spaces in unbounded tubular $T_\Lambda$ domains over cones as follows

$$N^p(T_\Lambda) = \{f \in H(T_\Lambda) : \int_{T_\Lambda} (log^+|f(z)|)^p \times \Delta^\tau(Imz)dV(z) < \infty\}, 0 < p < \infty,$$

$dV$ is a Lebeques measure on $T_\Lambda$, and consider same problems here. Similarly same type general analytic spaces can be defined also in bounded psevdoconvex domains. It is a new nice problem to study such spaces.

Let $D$ be a star shaped bounded circular domains with the Bergman-Silov boundary $\tilde{B}$, and $0 \in D$.

In [26, 27] the authors showed the equivalence of two new Nevanlinna type spaces in $D$:

$$N(D) = \{f \in H(D) : \sup_{0<r<1} \int_{\tilde{B}} log^+|f(rb)|d\lambda(b) < \infty\}$$

and $N_*(D) = \{f \in H(D) : (log^+|f(rb)|, 0 < r < 1$ *is uniformly integrable on* $\tilde{B}\}$, where $H(D)$ is a space of analytic functions on $D$ and $d\lambda$ is a Lebeques measure on $D$. In the unit ball we have

$\sup_{0<r<1} \int_S log^+|f_r|d\tau < \infty$ implies $\sup_{0<r<1} \int_S |log|f_r||d\tau < \infty$.

Also in the ball we have $H^{p_1} \subset H^{p_2} \subset N^1, p_1 > p_2$.

Let $H_\varphi(B) = \{f \in H(B) : \sup_{0<r<1} \int_S \varphi((log|f_r|)d\tau < \infty\}$.

($\varphi$ is positive nondecreasing convex function $\varphi : (-\infty,\infty) \to [0,\infty)$, see [5, 6]). This space includes $H^p$ and $N^1$ obviously.

We will discuss in next section coefficient multiplier problem and discuss Taylor coefficients listing some recent results. Then consider recent theorems on embeddings and some related operators in area Nevanlinna type spaces of several variables.

## 3. On some new embeddings theorems, Taylor coefficients of area Nevanlinna type spaces in higher dimention and related problems.

The plan of this section is the following. We review some new results on Taylor coefficients then we provide results on embeddings, then consider new theorems on some operators in area Nevanlinna type spaces $\mathbb{C}^n$.

A typical result on Taylor coefficients is the following Theorem 3.

**Theorem 3.** *(see [4, 28])*

*Let $f \in N^1_\alpha(U^n), \alpha > -1$. Then we have $log^+|a_{k_1,\dots,k_n}| = 0\left((k_1,\dots,k_n)^{\frac{2+\alpha}{3+\alpha}}\right)$ for each $f, f(z) = \sum\limits_{k_1\geq 0}\dots\sum\limits_{k_n\geq 0} a_{k_1,\dots,k_n} z_1^{k_1}\dots z_n^{k_n}, f \in N^1_\alpha, \alpha > -1$ and this estimate also is the best possible.*

It will be nice to extend this result to all $N^{p,q}_\alpha$ and $\widetilde{N}^{p,q}_\alpha$ spaces in polydisk.

We refer to [29] for a description of coefficient multipliers of Smirnov spaces obtained by Nawrocki. He provided complete generalization of well-known theorem of Yanagihara (see [30]) in the unit disk. Various one dimensional results on functionals and multipliers on Nevanlinna analytic spaces and related spaces are known now (see [4, 8, 9, 10, 16]). It will be nice to extend such theorems similarly to get similar to these nice results of Nawrocki in $U^n$.

The Smirnov class $N_*(U^n)$ is the subspace of $N^1(U^n)$ consisting of those $f$, so that $\{\log^+|f_r|, r \in I\}$ is uniformly integrable on $T^n, f_\tau(\omega) = f(\tau\omega)$. The functional $\sup\limits_{0<r<1}\int_{T^n}\log(1+|f_r|)dm_n$ is a complete $F-$norm on $N_*(U^n)$, the radial limit $\lim\limits_{r\to 1^-} f(r\omega)$ exists almost everywhere (see [5]). Here $dm_n$ is the Lebeques measure on $T^n$.

Here is a known typical result in the unit disk on coefficient multipliers (see [4]).

**Theorem 4.** *Let $\{\Lambda_k\}$ be multiplier from $N^p(U)$ to $H^q(U), 0 < q \leq \infty$. Then $\bigwedge$ maps bounded subsets of $N^p$ into bounded subsets of $H^p$. Let $0 < q \leq \infty, 1 < p < \infty, \{\Lambda_k\}$ is a multiplier from $N^p$ into $H^q$ if and only if for some positive constant $\Lambda_k = O(exp(-ck^{\frac{1}{p+1}})$.*

Nawrocki also provided full description of the countinuous linear functionals of $N_*(U^n)$. The Frechet envelope of $N_*(U^n)$ also was provided in his vital work [29]-[32]. He proved that for $n > 1$ $N_*(U^n)$ is not isomorphic to $N_*(U^1)$. We refer to [29]-[32] for similar results in the unit ball (see [4] for other such new results).

It is a problem to find a solid hull of Nevanlinna and Smirnov spaces in the unit ball in $\mathbb{C}^n$ and other domains. For $n = 1$ see [4, 8, 16] and references there.

We now discuss some problems related with embeddings in area Nevanlinna type spaces. We refer to [4, 17, 19, 33] for unit ball, disk cases. We provide recent results from [34], then some results based on technique from [33, 35], then finally some related results embeddings from [36, 37]. Note these results also many have Applications (see [4, 14, 38]).

Let $q > p$, then we have the following embeddings in the unit disk

$$N^q(U) \subset (N^p)(U), \bigcup_{p>0} H^p \subset \bigcap_{p>1} (N^p(U)), \bigcup_{p>1} (N^p) \subset (N^+(U))$$

.

We first give an embeddings theorem obtained by Cho-Kwon for $N^p_\alpha$ spaces in $\mathbb{C}^n$ .

Let $D \subset C^n$ be any a bounded domain in $C^n$. Let $H(D) = \{f, f \textit{ holomorphic in } D\}, \delta_D(z)$ is a distance from $z$ to boundary of $D$.

Let

$$||f||^p_{A^p_\alpha} = \int_D |f(z)|^p \delta_D(z)^{\alpha-1} dV(z); \alpha > 0, 0 < p < \infty,$$

$dv$ is a Lebeques measure on $D$ and

$$A^{-\sigma}(D) = \{f \in H(D) : ||f||_{-\sigma} = \sup\{\delta_D(\omega)^\sigma |f(\omega)|; \omega \in D\}\} < \infty$$

be classical Bergman spaces in $D$.

Let also further

$$||f||^p_{A^p_\alpha} = \int_D (\log^+ |f(\omega)|)^p \delta_D^{\alpha-1} dV(\omega) < \infty.$$

If $D$ has $C^2$ boundary, then we have

$$A^p_\alpha(D) \subset A^{-\frac{n+\alpha}{p}}(D) \subset N^1_\beta(D), \sigma \in (0, \frac{\alpha}{p}), \beta > 0, 0 < p < \infty.$$

The complete analogue of the following embedding theorems is valid also in the unit ball (see [36, 37]).

We need some definitions. Let further

$$||f||^p_{A^p(\varphi)} = \int_{U^n} |f(z)|^p \exp\left(-\varphi\left(\frac{1}{1-|z|}\right)\right) dm_{2n}(z).$$

Let also $g \in H(U^n)$, denote by $M^p_p(g, r)$ the following growing function

$$\int_{T^n} |g(r_1\xi_1, \ldots, r_n\xi_n)|^p dm_n(\xi), 0 < p < \infty, r \in I^n.$$

**Theorem 5.** *(see [36, 37])*

*Let $\varphi(\vec{r}) = (\varphi_j(r_j))_{j=1}^n, r_j \in [0, +\infty), j = 1, \ldots, m$, let $\varphi_j$ be an increasing positive on $R_+$ and*

$$\lim_{r \to +\infty} \frac{\ln r}{\varphi_j(r)} = 0, j = 1, \ldots, n.$$

$$\varphi_j \nearrow +\infty, (x \to +\infty), \lim_{x \to +\infty} \frac{\ln x}{\varphi_j(x)} = 0, \varphi_j \in C^{(2)}(R_+), j = 1, \ldots, n.$$

$$\frac{\varphi_j''(x)}{\varphi_j^2(x)} \searrow 0, x \to +\infty.$$

*Let $f(z) \neq 0, z \in U^n$, then for $0 < p < \infty$*

*1)* $\sup_{z\in U^n} ln^+|f(z)|\Big(\sum\limits_{j=1}^{n}\varphi_j(\frac{1}{1-|z_j|})\Big)\leq c_1||f||^p_{A^p(\varphi)}$ *and*

*2)* $[M_1(ln^+|f|,r)]\Big(\sum\limits_{j=1}^{n}\varphi_j(\frac{1}{1-r_j})\Big)\leq c_2||f||^p_{A^p(\varphi)}, r\in I^n;$

*where* $\exp(\widetilde{\varphi}(z))=\prod\limits_{j=1}^{n}\exp(\varphi_j(z_j))$.

We refer the reader to [36, 37] for complete analogue of Theorem 5 in the unit ball.

**Theorem 6.** *(see [13, 19])*

*Let*

$$l_j\in[0,1),\theta_j\in[-\pi,\pi],\Delta_{l_j}(\theta_j)=\left\{z\in D: 1-l_j\leq|z_j|<1, |argz-\theta_j|\leq\frac{l_j}{2}\right\},$$

$$j=1,\dots,m, \Delta_{\vec{l_j}}(\vec{\theta}_j)=\prod_{j=1}^{m}(\Delta_{l_j}(\theta_j)).$$

*Let* $\{\mu_j\}$ *be finite nonnegative Borel measure in the unit disk,* $j=1,\dots,m$, *let* $\mu=\prod\limits_{j=1}^{m}\mu_j$. *Let* $\mu_j(\Delta_{l_j}(\theta_j))\leq c_1\prod\limits_{j=1}^{m}\omega(l_j)l_j^{p+1}$*; for all* $l_j\in(0,1),\theta_j\in[-\pi,\pi]$. *Then*

$$\int\limits_{U^n}(ln^+|f(\xi_1,\dots,\xi_n)|)^p\prod_{j=1}^{m}d\mu_j(\xi_j)\leq$$

$$\leq c\int\limits_0^1\cdots\int\limits_0^1\prod_{j=1}^{m}\omega(1-r_j)\Big(\int\limits_{T^n}(log^+|f(r_1\xi_1,\dots,r_n\xi_n)|)^q dm_n(\xi)\Big)^{\frac{p}{q}}dr_1\dots dr_n,$$

*for* $q=1$, *where* $\omega\in S$.

Similar result is valid for $N^{p,q}_{\omega}(U^n)$ instead of $\widetilde{N}^{p,q}_{\omega}$ and all $q,p\geq q$.

**Remark 2.** Various such type other results are also valid for analytic spaces with the same type norms or for spaces with the following norms

$$\int\limits_{T^n}\Big(\int\limits_{I^n}(\log^+|f(r_1\xi_1,\dots,r_n\xi_n)|)^q\omega(\vec{r})d\vec{r}\Big)^{\frac{p}{q}}d\xi_1\dots d\xi_n, d\vec{r}=dr_1\dots dr_n, \vec{r}=(r_1,\dots,r_n),$$

where $p\geq 1, q\geq 1$ (see [13, 19]), where $\omega(\vec{r})$ is weight of certain $S$ class, $\omega(\vec{r})=\prod\limits_{j=1}^{m}\omega_j(r_j)$.

We now consider some other direct extensions of $N^p_\alpha$ area Nevanlinna type classes in $C^n$. Let $\Gamma_\alpha(\xi)=\{z\in U: |1-\vec{\xi}z|<\alpha(1-|z|)\}, \alpha>1$ be Lusin cone in the unit disk. Let $D(z,r)$ be the Bergman ball in the unit disk $z\in U, r>0$ (see [4, 19]). We may also consider a problem of finding precise conditions on positive Borel $\mu$ measure in $U^n$, so that

$$\int\limits_{T^n}\Big(\int\limits_{\Gamma_{\alpha_1}(\xi_1)}\cdots\int\limits_{\Gamma_{\alpha_n}(\xi_n)}\log^+|f(z_1,\dots,z_n)|\prod_{j=1}^{n}(1-|z_j|)^{\beta_j}d\mu(\vec{z})\Big)^p dm_n(\xi)\leq$$

$$\leq c\int\limits_0^1\cdots\int\limits_0^1\prod_{j=1}^{n}(1-r_j)^{\gamma_j}\Big(\int\limits_{T^n}\log^+|f(r\vec{\xi})|dm_n(\vec{\xi})\Big)^q\prod_{j=1}^{n}dr_j,$$

$$0 < p,q < \infty, \gamma_j > -1, \beta_j > -1, j = 1,\dots,n.$$

or

$$\int\limits_{U^m} \Big( \int\limits_{D(z_1,r)} \cdots \int\limits_{D(z_m,r)} (\log^+ |f(\omega_1,\dots,\omega_m)|)^q \prod_{j=1}^{m} (1-|\omega_j|)^{\alpha_j} d\mu(\vec{\omega}) \Big)^{\frac{p}{q}} \times dm_{2m}(\vec{z}) \leq$$

$$c_1 \int\limits_{U^m} (\log^+ |f(\omega_1,\dots,\omega_m)|)^s \prod_{j=1}^{m} (1-|\omega_j|)^{\beta_j} dm_{2m}(\vec{\omega})$$

for $0 < q,\ p,s < \infty, \alpha_j > -1, \beta_j > -1, j = 1,\dots,m, \vec{z} = (z_1,\dots,z_m)$,

$\vec{\xi} = (\xi_1,\dots,\xi_m)$ estimates are valid.

Note in the first case for $p \geq q,\ p,q \geq 1$ and in the second case for $\min(p,q) \geq s \geq 1$, it can be shown that the following condition (Carleson type condition).

$\mu(\Delta_{\vec{l}}(\vec{\theta})) \leq c\big(\prod\limits_{j=1}^{n} l_j\big)^{\tau}$ for some $\tau = \tau(p,q,s)$ is sufficient for these embeddings (based in particular on known methods and results of first author, see also [4, 19]).

Very similar problems similarly can be posed in the unit ball (see [33]).

Concerning polydisk we mention the following interesting vital fact. Various embedding theorems for various analytic function spaces are known in literature ( for Bergman, Hardy, mixed norm classes). In many of those results the main fact used in proofs is the $n$-subharmonicity of the function $|f(z_1,\dots,z_n)|^p, p \geq 1$. This gives also embedding for various area Nevanlinna spaces in the polydisk regarding the fact that $(\log^+ |f(z_1,\dots,z_n|)^p, p \geq 1$ is also a $n$-subharmonic function and indeed this in unit disk and polydisk was used by Mihic-Shamoyan (see [8]).

Let further $H^p(B_n)$ be Hardy class in the unit ball, then we have $f \in H^p(B_n) \Leftrightarrow f(z) \in (A^p_{n-k})(B_{k-1})$ and also $f \in N^1(B_n) \Leftrightarrow f \in (N^1_{n-k})(B_{k-1})$ (see [5, 6]) where $A^p_\alpha$ is a Bergman space in the ball (see [5, 6]). Then also the operator of restriction $f(z_1,\dots,z_n) \to f(z_1,\dots,z_{k-1},o,\dots,o)$ maps as a bounded operator from $(N^1)(B^n)$ into $(N^1_{n-k})(B_{k-1})$ (see [5, 6]). In polydisk this operator maps $N^1(U^n)$ into $(N^1)(U^{k-1})$, and $(N_*)(U^n)$ maps into $(N_*)(U^{k-1})$ (see [5, 6]).

The natural and interesting problem to solve this problem in other classes of area Nevanlinna type and not only in the unit ball.

We finally mention some other problems and provided hints for solutions of other problems in area Nevanlinna type spaces in $C^n$ and pose some questions also.

Concerning slice function we have the following results in the unit ball and in the unit polydisk (see for example [5, 6] and references there). If, for example, $f \in (N(B_n))$ then $(f_\xi)(u) \in (N_*)(U)$. It will be interesting to solve this problem in other spaces of analytic area Nevanlinna type spaces in the unit ball and in the unit polydisk and also in such spaces in other domains.

Concerning the problem of diagonal map $(Df) = (f)(z,\dots,z), z \in U$, the following results are valid (see [4, 21, 39] and various references there).

Note first the fact that $log^+|f(z_1,\dots,z_m)|^p, p \geq 1, |z_j| < 1, j = 1,\dots,m$, is a $m$-subharmonic function gives various results on diagonal map directly from related results on diagonal map of Bergman type spaces, since the key ingredient in those proofs is the fact that $(|f(z_1,\dots.z_n)|^p), p \geq 1$ is also $n$-subharmonic function in the unit polydisk.

We have based on this idea for Nevanlinna and area Nevanlinna spaces (see [4, 21, 39]).

$$Diag(N^p)(U^n) = (N^p_{n-2})(U); p \geq 1, n > 1.$$

And also $Diag(N^p_\alpha)(U^n) = (N^p_{\alpha n+2n-2})(U)$, for $p \geq 1, n \geq 1, \alpha > -1$. But for mixed norm such type spaces results are not known. We formulate one more related result in this direction. Let $u$ be a nonnegative $n$-subharmonic function in $U^n$ for which $(\sup_{r\in(0,1)} \int_{T^n} (|u(rw)|^p dm_n(w)) < \infty$, $1 < p < \infty$, where $dm_n$ is a Lebegues measure on $T^n$.

If there exists a constant $C > 0$, such that $\mu(\Delta_l(\xi)) \leq Cl^n$, where $\xi \in T^n$, $l \in (0,1); \Delta_l(\xi) = \{z \in U^n : 1 - l_j \leq |z_j| < 1, |argz_j - arg\xi_j| \leq \frac{l_j}{2}; 1 \leq j \leq n\}$ then $\int_U Diag(u(z))d\mu(z) \leq C||u||^p_{L^p(T^n)}$ (see [4, 21]).

We finally mention another interesting problem related with isometries in Nevanlinna type spaces in higher dimension. Any linear map from a Banach space $X$ to $X$, so that $||T_y||_X = ||y||_X, y \in X$ we call isometry (see also next section for discussion).

The isometries of $(N_*)(B_n)$ are $(Tf)(z) = (g(z))(f(\varphi(z))), z \in B_n, f \in N_*, g(z) = (T1)(z), z \in B_n$, where $\varphi$ - is so-called inner map of $B_n$ (the same result is valid in the polydisk, see [4, 40]).

For various other results related with this issue we refer also to [4, 5, 6, 26, 38, 40, 41]. It will be nice to find such results for many other area Nevanlinna type spaces and not only in $B_n$ and $U^n$.

We refer to last section for more results. We now provide shortly a uniqueness theorem in $\mathbb{C}^n$ of $F$. Beatrous for analytic area Nevanlinna type spaces in $\mathbb{C}^n$ (see [42]). This result probably may have many applications in Nevanlinna space function theory of several complex variables.

Let $D$ be bounded domain, a characterizing function of $D$ is a real valued $\mathbb{C}^1$ function $\rho$ on $\mathbb{C}^n$, such that $D = \{\rho < 0, d\rho \neq 0\}$ on $\partial D$.

There is $\varepsilon_0 > 0$, so that $D_\varepsilon = \{\rho < -\varepsilon\} \subset D$, $\varepsilon < \varepsilon_0$. Let $D$ be with $\mathbb{C}^2$ boundary. We denote by $\sigma_\varepsilon$ surface measure on $\partial D_\varepsilon$. Privalov-Nevanlinna class on $D, N^p$ in $\mathbb{C}^n$ is a subspace of $H(D)$ , so that

$$||f||_p = \sup_{0<\varepsilon<\varepsilon_0} \Big( \int_{\partial D_\varepsilon} (log^+|f|)^p d\sigma_\varepsilon \Big)^{\frac{1}{p}} < +\infty, 0 < p < \infty$$

(if $log^+|f|$ is $(|f|)^p$ then we have $H^p$).

Note $H^s(D) \subset N^1, 0 < s \leq \infty$ and also any function of $(N_1)$ has non-tangential limits at almost every point of $\partial D$. We denote them by $f^*(\xi)$ (see [42]).

**Theorem 7.** *(see [42]) Let $D$ be bounded domain with $\mathbb{C}^2$ boundary. Let $E$ be a Borel set in $\partial D$ with positive Lebeques measure. Let $f \in N_1(D)$ then if $|f^*|_E = 0$ then $f \equiv 0$.*

To extend this nice result to various $N^p_\alpha$ type analytic spaces in the ball and in the unit polydisk is an open problem.

When $D$ is bounded symmetric domain in $\mathbb{C}^n$ with Bergman-Silov boundary $B$ and $0 \in D$, $N_*$ is a spaces of all holomorphic functions $D$ on for which the family $\{log^+(|f_r|) : 0 < r < 1\}$ is uniformly integrable on $B$ and if $f, g \in N_*$ we put $\rho(fg) = \int_B (log)(1 + |f^* - g^*|)d\mu$, where by $(f^*)$ and $(g^*)$ we denote the boundary values of $f, g$ respectively.

Next we mention M. Stoll's results on these area Nevanlinna type spaces in $C^n$ (see [43]-[45]). M. Stoll proved (1976) that $(N_*(D), \rho)$ is an $F$ algebra that is an $F$ space with. We mention M. Stoll results $(N_*), \rho$ is an $F$ algebra that is an $F$ space with a continuous multiplication, where $N_*$ is this Smirnov type class on bounded symmetric domain $D$.

He also provided other extensions of some known one dimensional results on $N_*$ class.

For example, that if $\gamma$ is a continuous linear functional on $N_*(D)$, then there exists a holomorphic function $g$ on such that the following equality is valid

$$\gamma(f) = \lim_{\rho\to 1} \int_D f(\rho^{-1}t)g(\rho t)d\mu(t)$$

for all $0 < r < \rho < 1$.

Some nice properties of invertible functions and outer function in these classes were also provided by M. Stoll (see [43]-[45]).

If $D = D_1 \times \cdots \times D_k$, where each $D_i$ is irreducible domain of dimension $(n_i)$ then let

$$M_\infty(f,r) = \sup\{|f(r_1t_1,\dots,r_kt_k)|; (\bar{t}) \in D_1 \times \cdots \times D_k\}.$$

Let also $(F_*)(D)$ denote the space of holomorphic functions $f$ on $D$ for which

$$||f||_c = \int_{I^k} (\exp)[-c\prod_{i=1}^{m}(1-r_i)^{-n_i}](M_\infty(f,(r)))dr_1\dots dr_k < \infty;$$

for all $c > 0$, $I^k = (0,1]^k$. M. Stoll showed (1983) (see [43]-[45]) that $(F_*)(D)$ is a countable normed Frechet space containing $(N_*)(D)$ as a dence subspace. Furthermore if $D$ is irreducible of dimension $n$ and if for $\{\varphi_{k_i}\}$ orthonormal system of functions $f \sim \{a_{k\nu}\}; f = \sum a_{k_j}\varphi_{k_j}$ then $f \in F_*(D)$ and also $|a_{k,\nu}(f)| \le (\exp)\{\lambda_k k^{\frac{n}{(n+1)}}\}$ for some sequence $\{\lambda_k\}$ decreasing to zero. All linear bounded functionals of $F_*$ were also characterized by Stoll in same paper.

Note all these results were known previously in onedimensional case (see [43]-[45]).

His papers also contain interesting results concerning the rate of growth of the means $M_\infty(\cdot,r)$ of certain Poisson integrals of measures and of functions from the $(N(D))$ space and $(N_*)(D)$, where $N(D)$ is a Nevanlinna class that is a space of functions for which we have that

$$\sup_{0<r<1} \int_D log^+|f(rt)|d\lambda(t) < \infty.$$

These results also extend some known classical one dimensional theorems.

It will be nice to get similar theorems for general weighted area Nevanlinna spaces of several complex variable in such general domains.

Let further $\mu$ be positive Borel measure on $B_n$. For $\alpha > -1$ define weighted Lebeques measure by $dv_\alpha(z) = (1-|z|)^\alpha dv(z), z \in B_n, \alpha > -1$. Let $\xi \in S$, $\delta > 0, S(\xi,\delta) = \{z \in \bar{B}_n : |1- < z,\xi > | < \delta\}$. For $p \ge 1$ define the Bergman-Privalov space in the unit ball $(AN)^p(v_\alpha)$ by

$(AN)^p(v_\alpha) = \{f \in H(B_n) : \int_{B_n} \{\log(1+|f|)\}^p dv_\alpha < \infty\}$.

In recent paper [46] it was shown that $f \in (AN)^p(v_\alpha)$ if and only if $(1+|f|)^{-2}\{\log(1+|f|)\}^{p-2}|\tilde{\nabla}f|^2 \in L^1(v_\alpha), 1 < p < \infty$, or $(1+|f|)^{-2}|f|^{-1}|\tilde{\nabla}f^2| \in L^1(v_\alpha)$ in the case of $p = 1$, where $\tilde{\nabla}$ is the gradient of $f$ with respect to the Bergman metric on $B_n$ (see [35, 46] for definition of gradient).

Let $\varphi$ be holomorphic self map on $B$, let $C_\varphi f = f(\varphi(z))$ (linear operator). In [35, 41, 47] the following space of Privalov type was studied

$$\tilde{N}^p(B_n) = \{f \in H(B_n) : \sup_{0<r<1} \int_S \{\log(1+|f(r\xi)|)\}^p d\sigma(\xi)\}, 1 < p < \infty.$$

Composition $C_\varphi$ operators on this space and this type space was studied in [35, 46]. It was shown that $C_\varphi$ is metrically bounded if and only if $\mu(S(\xi,\delta)) \le c\delta^n$, similarly it was shown that $C_\varphi$ is metrically compact if and only if $\mu(S(\xi,\delta)) = 0(\delta^n), \delta \to 0$ uniformly by $\xi \in S$, and metrically bounded if $\mu(S(\xi,\delta)) \le c\delta^n$.

Let $B(\xi,\delta) = S(\xi,\delta) \cap B$. Let $\mu(B(\xi,\delta)) \le C(\sigma(S(\xi,\delta))), \xi \in S, \delta > 0$. Then $\int_B \{\log(1+|f|)\}^p d\mu \le \tilde{C}||f||^p_{N^p(B_n)}, 1 < p < \infty$ (see [35, 46]).

Let $\Omega$ be bounded strictly pseudoconvex open set $\Omega = \{\xi \in C^n : \rho(\xi) < \infty\}$, $\Omega_\varepsilon = \{\xi : \rho(\xi) = \varepsilon\}$ where $\rho$ is a strictly plurisubharmonic function near $\bar{\Omega}; d\rho(\xi) \neq 0, \xi \in \partial\Omega$.

Let $N_1(\Omega) = \{f \in H(\Omega), \overline{\lim}_{\varepsilon>0,\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} \log^+|f| dS_\varepsilon < \infty\}$, and also let

$N^p(\Omega) = \{f \in H(\Omega), \overline{\lim}_{\varepsilon>0,\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} |f|^p dS_\varepsilon < \infty\}$,

where $\Omega = \{\xi \in C^n : \rho(\xi) < -\varepsilon\}$, $dS_\varepsilon$ is a Euclidean area measure of $\partial\Omega$, where $H(\Lambda)$ is a set of all analytic function on $\Omega$, then note $H^\infty(\Omega) \subset H^p(\Omega) \subset N_1(\Omega)$, $0 < p < \infty$ (see [42]).

Let $\bar{N}^{p,q}_\alpha = \{f \in H(\Omega) : \int_0^\rho (\int_{\partial\Omega} (\log^+|f|)^p dS_\varepsilon)^{\frac{q}{p}})\varepsilon^\alpha d\varepsilon < \infty\}$, where $0 < p,q < \infty; \alpha > -1$. We have many open problems for this new interesting mixed norm space.

Note also if we replace in $\bar{N}^{p,q}_\alpha$ $\log^+|f|$ by $|f|^p$ then this space is well-known and studied (see [4]).

Note finally in the ball the following embeddings are valid for Nevanlinna type space in $B_n$ (see [36, 37, 48]).

Let $A^p(\varphi) = \Bigg\{ f \in H(B_n)$:

$$||f||_{A^p(\varphi)} = \left( \int_{B_n} |f(\xi)|^p \exp\left(-\varphi\left(\frac{1}{1-|\xi|}\right)\right) dv(\xi) \right)^{\frac{1}{p}} < +\infty \Bigg\}, 0 < p < \infty.$$

**Theorem 8.** *A) Let $f \in A^p(\varphi), f(z) \neq 0, z \in B_n, 0 < p < \infty, \frac{\varphi''(x)}{\varphi'^2(x)} \searrow 0, x \to +\infty$, let*

$$\lim_{x\to+\infty} \left( \frac{\varphi'(x)x}{\varphi(x)} \right) = \alpha_\varphi; \alpha_\varphi \in (0,\infty].$$

*Let $\varphi \in C^{(2)}(R_+)$; let also $\lim_{x\to+\infty} \left(\frac{\varphi(x)}{ln(x)}\right) = +\infty$.*

*Then we have that*

*1) $ln|f(z)| \le \frac{1}{p}\varphi\left(\frac{1}{1-|z|}\right) + \left(\frac{2n}{p}\right) ln \left( \frac{\varphi'\left(\frac{1}{1-|z|}\right)}{1-|z|^2} \right) + c(f,n)$;*

*for some constant $c(f,n); z \in B_n$;*

*2) $ln|f(z)| \le \tilde{c}\varphi\left(\frac{1}{1-|z|}\right) \left( \frac{\varphi'\left(\frac{1}{1-|z|}\right)}{1-|z|^2} \right)^{m+1}, m \in \mathbb{Z}_+$.*

*B) Let* $F(z)=lnf(z)$*. Then we have* $|R^mF(z)|\le\varphi\left(\frac{1}{1-|z|}\right)\left(\frac{\varphi'\left(\frac{1}{1-|z|}\right)}{1-|z|^2}\right)^{m+1}, z\in B_n,$

$m\in\mathbb{Z}_+$*; where* $R^m$ *is a differential operator on* $H(B_n),(R^mf)=(D^mf)_{z'}(\xi)\Big|_{\xi=|z|},$

$z\in B_n; D\psi(z)=z\psi'(z); z\in B_1, \psi\in H(B_1); f\in H(B_n), f_z(\xi)=f(\xi z);$
$z=\left(\frac{z_1}{|z_1|},\dots,\frac{z_n}{|z_n|}\right); z\neq 0, |\xi|=1.$

**Remark 3.** Similar results are valid in the unit polydisk (see [36, 37, 48]).

## 4. Isometries and boundary behaviour of $N_\alpha$ Nevanlinna type spaces in $\mathbb{C}^n$ and related problems

In recent papers A.V. Subbotin and V. Gavrilov investigated some new interesting general area Nevanlinna type Privalov type spaces in $\mathbb{C}^n$. We provide some short rewiew of results of his and their papers below mostly referring sometimes the reader for details to [38]. Note these results were published mostly only in Russian previously.

We also note that practically all Subbotin's and their results in one dimension are known, for further details we also refer the reader to his Subbotin's paper [38, 49] or to their papers [16, 50].

Let $G$ be the unit polydisk or the unit ball, $\Gamma$ is $\partial G$,

$$N^q(G)=\{f\in H(G):\int_\Gamma ln_+^q|f(r\gamma)|d\sigma(\gamma)\le K<\infty\};$$

($d\sigma$ has a usual sense of Lebeques measure), where $q>1$ and where $ln_+a=(\max)(0,lna)$; for $a>0$ and $ln_+0=0$ (it is a known Privalov class, but in higher dimension).

Let $\varphi(t), t\ge 0$ be an arbitrary nonnegative nonincreasing, concave (from below) function. We say (see [38]) $f\in\varphi(N)$ if the following condition holds

$$\int_\Gamma \varphi(ln_+|f(r\gamma)|)d\sigma(\gamma)\le N<\infty.$$

This $\varphi(N)$ in particular case is known as $H^p(G)$ Hardy space and is known as $N$-Nevanlinna class (for special $\varphi$).

Let $P(z,w)$ be a usual Poisson kernel in $G$. We say further that $f\in\varphi(M)$, if the following condition holds

$$\int_\Gamma \varphi(ln_+M_{rad}f(\gamma)d\sigma(\gamma)<+\infty,$$

where we denote $M_{rad}f(\gamma)=\sup\limits_{0\le r<1}|f(r\gamma)|, \gamma\in\Gamma$. And finally $\varphi(N_*)$ is a space of all analytic functions, for which $\psi_r(\gamma)=\varphi(ln_+|f(r\gamma)|), 0<r\le 1$, function is absolutely continuous (see [38]).

We have the following inclusion $\varphi(M)\subseteq\varphi(N_*)\subseteq\varphi(N)$. If $\varphi$ grows (by the order of growth at $\infty$), then these spaces also are nonincreasing (see [38]).

The following vital results can be found in recent paper of A. Subbotin [38]. We put below $N^1=N$, and for $\varphi=t^q, \varphi(M)=M^q, q>0$.

The following results are valid for the unit ball and the unit polydisk.

**Theorem 9.** *Let $f \in N, \tilde{f}(\gamma) = \lim_{r\to 1} f(r\gamma)$ be their radial boundary values almost every where on $\Gamma$. For $q > 1$ the following properties of $f$ are equivalent*

*1) $f \in N^q$;*

*2)$f \in N_*, ln_+^q|\tilde{f}| \in L^1(\Gamma,\sigma)$;*

*3)$(ln_+^q)|\tilde{f}| \in L^1(\Gamma,\sigma)$ and $(ln_+^q)|f(z)| \leq \int_\Gamma (P(z,\gamma))(ln_+^q)|\tilde{f}(\gamma)|d\sigma(\gamma)$;*

*4) $f \in M^q, q \geq 1$;*

*5)for $r \in (0,1)$*

$$\int_\Gamma (ln_+^q)|f(r\gamma)|d\sigma(\gamma) \leq \int_\Gamma (ln_+^q)|\tilde{f}(\gamma)|d\sigma(\gamma) < +\infty$$

*and*

$$\lim_{r\to 1}\int_\Gamma (ln_+^q)|f(r\gamma)|d\sigma(\gamma) \geq \int_\Gamma (ln_+^q)|\tilde{f}(\gamma)|d\sigma(\gamma) < +\infty;$$

*6)$f \in N_*^q = (N_*)^q$.*

Various applications for ball or polydisk for $q = 1$ of this theorem were known and were obtaned earlier by M.Stoll and B.R.Choe and H.Kim (see [38]). This Theorem 9 is valid also if we change $ln_+^q|\tilde{f}(z)|$ to $ln^q(1+|f(z)|)$ (see [38]).

Let further

$$|f|_{N^q} = \left(\int_\Gamma ln^q(1+|\tilde{f}(\gamma)|)d\sigma(\gamma)\right)^{\frac{1}{q}},$$

where $\tilde{f}(\gamma) = \lim_{r\to 1} f(r\gamma)$ is radial limit of $f(z)$.

**Theorem 10.** *Let $q > 1; f \in N^q$. Then*

$$ln_+|f(z)| = o(1-|z|)^{-\frac{n}{q}}, |z| \uparrow 1$$

.

*Let $\rho_{N_q}(f-g) = |f-g|_{N^q}, f,g \in N^q$.*

*This is an invariant metric for $N^q, q \geq 1$ and $N^q, q \geq 1$, is closed $(N^q,|f|)$, so $N^q$ is a communicative algebra, $|f|_{N^q}$ is a quazinorm.*

**Theorem 11.** *For $q > 1$ and for $\rho_{N_q}$ metric $N^q$ is a standard $F$ algebra with respect to pointwise multiplicative and addition operations and*

$$|f+g|_{N^q} \leq |f|_{N^q} + |g|_{N^q}.$$

**Remark 4.** For $q = 1$ Theorem 10 not true anymore (see, for example, [16, 50, 51]).

**Theorem 12.** *1) If $q > 1, f \in N^q, f_r(z) = f(rz), z \in G, 0 \leq r \leq 1$, then*

*$f_r \to f$ for $r \to 1 \in N^q$ metric;*

*2) Polynomials are dense in $N^q, q > 1$, $N^q$ is a separable $F-$algebra, $q > 1$.*

Concerning bounded sets and completely bounded sets of $N^q$ we have the following results (see [16, 38, 50, 51]).

**Theorem 13.** *The boundedness of $L$ set in $N^q, q > 1$ space is equivalent to the following two conditions*

*1)There is $K < \infty$, so that $|f|_{N^q} \leq K$ for all $f \in L$;*

*2)$\{ln^q_+|\tilde{f}|\}_{f \in L}$ family has absolutely continuous integral on $\Gamma$.*

**Theorem 14.** *Let $q > 1$. Then $L \subset N^q$ is completely bounded if:*

*1) $L$ is bounded in $N^q$;*

*2) radial limits of functions from $L$ are forming relatively compact set by $\sigma$ measure as set of functions.*

We give a standard example of bounded set in $N^q$. Let $f \in N^q, f_r(z) = f(rz)$, $r \in [0,1)$. Then $f_r, 0 \leq r < 1$, is a bounded subset in $N^q$ space.

Let further (it is a particular case of $\varphi(M)$ space)

$$M^q(G) = \{f \in H(G) : \int\limits_\Gamma (ln^+)|Mf(\zeta)|)^q d\sigma(\zeta) < +\infty\}, q \in (0,\infty),$$

where

$$(Mf)(\zeta) = \sup_{0<r\leq 1} |f(r\zeta)|,$$

if $q \geq 1$ then this is Privalov class. For $q < 1$ all functions of $M^q$ have radial finite limits (see [38]).

We can define a metric in $M^q, q > 0$ as follows $(M^1 = M)$

$$\rho_{M^q}(f,g) = \left( \int\limits_\Gamma ln(1 + M(f-g)(\zeta)) d\sigma(\zeta) \right)^{\frac{\alpha_q}{q}}$$

$f, g \in M^q$, where $\alpha_q = \min(1,q)$. This is a $F-$ space (see [38]).

Note various new results on $M^q$ in $C^n$ can be seen in [38]. Various authors studied isometries of various analytic spaces in one and also is several complex variables (see, for example, [16, 38, 49, 50, 51]). We add such results also for $N^q$ and $M^q$ classes.

Note the following embedding is valid $M^q(B_n) \in N_{n(\frac{1}{q}-1)}(B_n)$ for $q < 1$, where $N_\alpha, \alpha > -1$, are spaces with the following quazinorms

$$||f||_{N_\alpha} = \int\limits_{B_n} (ln_+|f(z)|)(1-|z|)^\alpha dv(z) < \infty$$

(so-called Nevanlinna-Djzrbashian spaces in the unit ball (see, for example, [38])).

Note for $(N_\alpha)(B_n)$ spaces and Nevanlinna spaces the complete description of all zero sets is well-known (via Hausdorff measures) (see [38]).

Note very similarly similar area Nevanlinna spaces can be defined also in the unit polydisk. Note to study such spaces is a new vital open reserach area.

**Theorem 15.** *Let $q > 0$. The $A$ mapping is a surjective linear isometry of $M^q$ (or $N^q, q > 1$) if and only if for each $f, f \in M^q, (Af)(z) = \alpha f(\phi(z)), z \in G$,where $\alpha \in \mathbb{C}, |\alpha| = 1$ and $\phi(z)$ is a biholomorphic avtomorfism of $G$ region, which keeps $\sigma$ on $\Gamma$ zero untouched.*

**Theorem 16.** *An $A$ mapping from $N^q, q > 1$ into $N^q$ is a linear isometry if $Af(z) = \psi(z) f(\phi(z)), z \in G, f \in N^q, q > 1$, where $\psi$ is inner function and $\phi$ is an inner mapping on $G$, whose radial boundary values keep $\sigma$ measure on $\Gamma$.*

For natural values of $q, q \in N$ this result can be seen in another paper of A.Subbotin. For such type spaces such results in one dimension were also discusses by A.Subbotin in [38] and also by various authors. The same results (with the same description of surjective linear isometries) are valid for Smirnov and Privalov spaces $N_*$ and $N^q$, spaces for $q > 1$.

As it was shown in [38] the set of all linear isometries of $M^q$ spaces and Privalov $N^q$ spaces are different for each $q > 1$. It is an open question to obtain complete description of all linear isometries of $M^q$ spaces at last for one value of $q > 0$. Isometries of $N^*$ are were known (see [38]).

**Theorem 17.** *If $(Af)(z) = (\psi(z))f(z), z \in G, f \in M^q$ is an isometry for $M^q$, $q > 0$ for some inner $\psi$ function, then we have $\psi(z) = const$.*

We also refer the reader for some short review to an important paper of V.Gavrilov and A.Subbotin on area Nevanlinna type spaces in higher dimensions (in $C^n$) (on topics related to so-called maximal theorems and Hintschin-Ostrovski property). We give only some recent results from these papers on this interesting topic below.

**Theorem 18.** *1) If boundary values of $f, f \in N(G)$ are equal to zero on a set with positive measure on $\Gamma$ then $f \equiv 0$ on $G$.*

*2) For $0 < q < 1, \alpha > 1$ we have that*

$$\int_\Gamma (ln_+^q) M_\alpha f(\zeta) d\sigma(\zeta) \le C_{q,\alpha} \sup_{0<r<1} \left( \int_\Gamma (ln_+)|f(r\zeta)| d\sigma(\zeta) \right)^q,$$

where $(M_\alpha)f(\zeta) = \sup_{z \in D_\alpha(\zeta)} |f(z)|, \zeta \in \Gamma$;

$$D_\alpha(\zeta) = \{z \in B_n : |1- < z, \zeta > | < \alpha(1-|z|)\};$$

and for the unit polydisk

$$D_\alpha(\zeta_k) = \{z \in C : |1 - z\bar{\zeta}_k| < \alpha(1-|z|)\}; 1 \le k \le n,$$

where $\alpha > 1$;

$$D_\alpha(\zeta) = \{z \in D_\alpha(\zeta_1) \times \ldots \times D_\alpha(\zeta_n) : \frac{1}{\alpha} \le \frac{1-|z_k|}{1-|z_l|} \le \alpha, 1 \le k \le n\}.$$

We refer to mentioned paper for important applications of these results.

Here is the multidimensional analogue of Hintschin-Ostrovski theorem (see [49].)

**Theorem 19.** *(see [38]). Let $f_k \in H(G)$. Let*

$$\int_\Gamma \ln_+ |f_k(r\xi)| d\sigma(\xi) \le C < +\infty; k \in N; r \in [0,1)$$

*and $(f_k^*) \to (f_k)$ by measure on $E; m(E) > 0$ where $f_k^*$ is a boundary value of $f_k$.*

*Then $f_k$ functions on any $G$ compact set are tending to $f, f \in N(G)$, and $f_k^* \to f^*$, where $f^*$ is a boundary value of $f$ on a set $E$ (by measure).*

Note for $n = 1$ these results were known (see [4, 38]).

$$\widetilde{M^q}(G) = \left\{ f \in H(G) \int_\Gamma (\ln_+^q \sup_{D_\alpha(\xi)} |f(r\xi)|) d\sigma(\xi) < +\infty \right\}, 0 < q < \infty.$$

A natural metric for this class is $p(f,g) = \int\limits_{\Gamma}(\ln(1+M(f-g))(\xi)d\sigma(\xi)$,

$f,g \in \widetilde{M}^q(G)$. Returning to $M(G)$ we note $M(G)$ is an $F$ algebra, $M(G) \subset N(G)$, obviously, so for each $f, f \in M(G)$ the boundary values $f^*(\xi), \xi \in \Gamma$ exists.

**Theorem 20.** *(see [38]). $L$ is absolutely bounded in $M(G)$ if*

*1) $\{ln_+Mf(\xi), \xi \in \Gamma\}_{f\in L}$ family has uniformly absolutely continuous integrals on $\Gamma$;*

*2) $\{ln_+(\xi), \xi \in \Gamma\}_{f\in L}$ family of functions is relatively compact on $\Gamma$ in topology of convergence by measure.*

Can we say the same for $\widetilde{M}(G)$. It is an open problem (see [38]).

A well-known Smirnov theorem says if $f \in H^p, p > 0$, and $|f^*|^{p\prime}, p\prime > p$ is integrable on $\Gamma$, then $f \in H^p$. We have similar type results for Nevanlinna type spaces (see [38]).

**Theorem 21.** *1) Let $f \in N^q, q > 1, ln_+^{q'}|f^*|, q' > q$ is integrable on $\Gamma$,*

*then $f \in N^{q'}$;*

*2) Let $q > 1, p > 0$. If $f \in N^q, |f^*|^p$ is integrable on $\Gamma$, then $f \in H^p$;*

*3) If $f \in N^q, q > 1, |f^*| \leq k < +\infty$ almost everywhere on $\Gamma$, then $f$ is bounded on $G$ and $|f(z)| \leq k$, for some constant $k$.*

Similar results are valid for so-called Zigmund $NlnN$ space, we refer the reader to [49] for these results and new results on isometries of $NlnN$ space in higher dimension. (See also some results below taken from [38] in this direction).

Choosing in our main definition $\varphi(t) = t(ln_+^{\alpha}t), \alpha > 0$, we get $(Nln^{\alpha}N)$spaces so-called Zigmund spaces. We have the following inclusions $Nln^{\alpha}N \subset N^* \subset N$; further $NlnN$ is an $F$ algebra, it is separable metric space and polynomials are dense in $NlnN$ (see [38]).

Moreover $NlnN$ is a functional algebra and this operation of multiplication is continuous in $p$ metric, so we have

$$f_r \to f, 0 < r < 1,$$

if $f \in NlnN$.

Put further $w(t) = tln(e+t), t \geq 0$. We provide two another equivalent definitions of $NlnN$ functional class below. We have

$$|f|_{NlnN} = \sup_{r<1} \int\limits_{\Gamma} w(ln(1+|f(r\gamma)|))d\sigma(\gamma) < \infty$$

and

$$|f|^1_{NlnN} = \sup_{0<r<1} \int\limits_{\Gamma} \varphi(ln(1+|f(r\gamma)|))d\sigma(\gamma) < \infty.$$

Then also we have the following estimates and inclusions.

Let $f \in NlnN$, then

$$|f|_{NlnN} = \int\limits_{\Gamma} w(ln(1+|f^*(\gamma)|))d\sigma(\gamma) < \infty$$

and we have the following inclusions

$$NlnN \subset M \subset N^* \subset N.$$

If further $f \in NlnN$, then we have

$$ln(1+|f(z)|) \leq w^{-1}\left(\frac{(1+|z|)^n}{(1-|z|)^n}\right), z \in G,$$

and this estimate can not be improved (see [38]).
Also we have that

$$w(ln(1+|f(z)|)) = o\left(\frac{1+|z|}{1-|z|}\right)^n, |z| \to 1_-, f \in NlnN$$

and $|f|_{NlnN} = 0$ for $f \equiv 0$; $|-f|_{NlnN} = |f|_{NlnN}, f \in NlnN$, and
$|f \pm g|_{NlnN} \leq |f|_{NlnN} + |g|_{NlnN}$, $f, g \in NlnN$.

**Theorem 22.** *(see [38]). Let $f \in N$, then we have the following equivalent properties*
*1) $f \in NlnN$;*
*2)$f \in N^*, ln(1+|f^*|) \in NlnN$;*
*3)$f \in M, ln(1+|f^*|) \in NlnN$;*
*4) $ln(1+|f^*|) \in NlnN, w(ln(1+|f(z)|)) \leq \int_\Gamma P(z,\gamma)w(ln(1+|f^*(\gamma)|))d\sigma(\gamma)$;*
*and $w(ln(1+|f(r\gamma)|))$ family has for $\gamma \in \Gamma$ absolutely continuous integrals, where $P$ is a a usual Poisson kernel in $G$.*

Note $M^q = N^q$ for $q > 1$, but for all $q > 0$ this not valid and we have following results. For each $f$, $f \in N$ there are boundary limits $\lim_{r \to 1^{-0}} f(r\zeta) = f^*(\zeta), \zeta \in S_n$ almost everywhere and even more $\lim_{z \to \zeta} f(z)$ ($K$ limit) also exists, here $z \in D_\alpha(\zeta)$, $\alpha > 1$, almost everywhere.

Let $\alpha > 1$. Denote $\widetilde{M}^q(B_n) = \{f \in H(B_n) : \int_{S_n} ln_+^q M_\alpha(f(\zeta))d\sigma(\zeta) < \infty\}$. For this class we have next theorem.

**Theorem 23.** *1) For each $f \in \widetilde{M}^q, q > 0$, limit $\lim_{z \to \xi} f(\xi) \in S_n$ exists almost everywhere on $S_n$.*

*2) For each $f, f \in \widetilde{M}^q$ radial limit $f^*(\zeta), \zeta \in S_n$ exists and is finite almost everywhere on $S_n$, $ln_+^q|f^*|$ is integrable on $S_n$, $|f^*| = M_{rad}f$, almost everywhere on $S_n$.*

*3) $N \subset \widetilde{M}^q$ for $0 < q < 1$ in $B_n$.*

*4) The $\widetilde{M}^q$ space, $q > 0$, is $F$ algebra $(\widetilde{M}^q, p_q)$ and $\widetilde{M}^q$ is separable and all polynomials are dense in $\widetilde{M}^q, q > 0$, where*

$$||f||_{q,\alpha} = \left(\int_{S_n} ln^q(1+M_\alpha f(\zeta))d\sigma(\zeta)\right)^{\frac{1}{q}},$$

$$M_\alpha f(\zeta)) = \sup_{D_\alpha(\zeta)} |f(z)|, z \in D_\alpha(\zeta); f \in \widetilde{M}^q, q > 0, \alpha > 1;$$

*moreover we have*

$$ln(1+|f(z)|) \leq \frac{c||f||_{q,\alpha}}{(1-|z|)^{\frac{n}{q}}}, z \in B_n, f \in \widetilde{M}^q, q > 0;$$

*(the same is valid also for $||f||_q^*$ with $M_{rad}$ instead of $M_\alpha$).*

*5) Let $f_r(z) = f(rz), r \in (0,1), z \in B_n$. Then we also have $f_r \to f$ by $p_q$ metric for $r \to 1$, if $f \in M^q$.*

It is an open problem can we assert these results of Theorem 23 for the unit polydisk namely for $M^q(U^n)$ spaces.

**Remark 5.** All these assertions are valid also for the $\rho_{q,\alpha}$ metric, where

$$\rho_{q,\alpha}(f,g) = ||f-g||_{q,\alpha}^{\alpha_q}, \alpha_q = \min(1,q), q > 1, \alpha > 1.$$

We finally add some results on boundary behaviour of these area Nevanlinna type spaces in $C^n$ from [38].

**Theorem 24.** *Let $E \subset \Gamma$, let $\varphi$ be a function defined on $E$, then $\varphi = f$, where $f$ is a boundary function of a certain function $f, f \in N (f \in N^q, q > 1)$, if and only if*

*1) $P_v(\zeta) \to \varphi(\zeta), \zeta \in E$ almost everywhere on $E$*

*and*

*2) $\overline{\lim\limits_{v\to\infty}} \int\limits_{\Gamma} ln_+|P_v(\zeta)|d\sigma(\zeta) < \infty$ (for $N^q, q > 1$ we have to replace only $ln_+$ by $ln_+^q$ in this theorem), where $P_v$ is a sequence of algebraic polynomials.*

The only change for the same results for $M$ space is that we have to change the last condition to $\overline{\lim\limits_{v\to+\infty}} \int\limits_{\Gamma} ln_+ M_{rad}(P_v(\zeta))d\sigma(\zeta) < +\infty$ . The only change for $N_*$ class for the same to Theorem 24 result is that we have to change the last condition in Theorem 24 to

$$\int\limits_{\Gamma} ln_+|P_v(\zeta)|d\sigma(\zeta);$$

and here integrals are absolutely continuous uniformly on $\Gamma$.

Note also in addition this last condition, but for $f \in X$ is making any $X$ subset; $X \subset N_*$ a bounded set of linear topological space $N_*(G)$ (see [38]).

**Remark 6.** We do not discuss important spaces of bounded continues functionals and coefficient multipliers of various area Nevanlinna type spaces of several complex variables in details. M.Nawrocki provided in [29, 31, 32] for $N^*(U^n)$ complete description of multipliers (coefficient) from $N^*$ to $H^p$ in the polydisk $U^n$ for all $p > 0$. (Smirnov class multipliers in polydisk). In one dimension many results in this direction are well-known (see [4, 7, 8, 9, 10, 29, 31, 32]).

Many new interesting problems in this nice research area are still open.

**Remark 7.** Note embeddings we provided from papers of F.Shamoyan were used by him in solutions of some problems related with weak invertibility in spaces of analytic functions of several variable in the unit ball and unit polydisk (see [29, 31]).

Many new interesting problems in this nice research area are still open.

**Competing interests.** The author declares that there are no conflicts of interest regarding authorship and publication.

**Contribution and Responsibility.** The author contributed to this article. The author is solely responsible for providing the final version of the article in print. The final version of the manuscript was approved by the author.

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Вестник КРАУНЦ. Физ.-Мат. Науки. 2021. Т. 36. №. 3. С. 40–64. ISSN 2079-6641

УДК 517.55+517.33 Научная статья

# Интегральные операторы, теоремы вложения, изометрии, граничное поведение и коэффициенты Тэйлора многомерных пространств типа Неванлинны и связанные с ними проблемы

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В обзорной работе собраны воедино различные утверждения, полученные различными авторами в последнее время по аналитическим многомерным пространствам типа Неванлинны в различных многомерных областях. В статье также сформулированы и кратко обсуждаются различные новые актуальные интересные проблемы, возникающие естественным образом в указанных многомерных классах аналитических функций в различных областях в $C^n$. Особое внимание в работе уделяется изометриям, действию различных интегральных операторов, различным теоремам вложения, и оценкам коэффициентов Тейлора в упомянутых аналитических пространствах типа Неванлинны в различных многомерных областях. Вдобавок в данной статье вместе с ранее изученными многомерными классами функций подобного типа вводятся также новые различные шкалы многомерных пространств типа Неванлинны в различных областях в $C^n$.

*Ключевые слова: полидиск, шар, классы типа Неванлинны, аналитические функции, теоремы вложения, коэффициенты Тэйлора, изометрии, интегральные операторы, характеристика типа Неванлинны, трубчатые и псевдовыпуклые области.*

DOI: 10.26117/2079-6641-2021-36-3-40-64

Поступила в редакцию: 01.04.2021 В окончательном варианте: 15.07.2021

**Для цитирования.** Shamoyan R. F. Integral operators, embedding theorems, Taylor coefficients, isometries, boundary behaviour of Area-Nevanlinna type spaces in higher dimension and related problems // *Вестник КРАУНЦ. Физ.-мат. науки*. 2021. Т. 36. № 3. С. 40-64. DOI: 10.26117/2079-6641-2021-36-3-40-64

**Конкурирующие интересы.** Конфликтов интересов в отношении авторства и публикации нет.

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