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Assembling classical and dynamic inequalities accumulated on calculus of time scales

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In this paper, we present an extension of dynamic Renyi's inequality on time scales by using the time scale Riemann–Liouville type fractional integral. Furthermore, we find generalizations of the well–known Lyapunov's inequality and Radon's inequality on time scales by using the time scale Riemann–Liouville type fractional integrals. Our investigations unify and extend some continuous inequalities and their corresponding discrete analogues.

Keywords: time scales, fractional Riemann–Liouville integral, Renyi's inequality, Lyapunov's inequality, Radon's inequality.

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1. Introduction

We introduce here some classical inequalities.

If $x_k \geq 0$, $y_k \geq 0$ for $k = 1,2,\ldots,n$ and $\lambda > 0$ with $\lambda \neq 1$, then

$$\frac{1}{\lambda - 1}\left(\sum_{k=1}^{n} x_k\right)^{\lambda}\left(\sum_{k=1}^{n} y_k\right)^{1-\lambda} \leq \frac{1}{\lambda - 1}\sum_{k=1}^{n} x_k^{\lambda} y_k^{1-\lambda}. \tag{1}$$

The inequality from (1) is called, in literature, Renyi's inequality as given in [10].

If $x_k > 0$, $y_k > 0$, $k = 1,2,\ldots,n$ and $0 < \lambda_1 < \lambda_2 < \lambda_3 < \infty$, then

$$\left(\sum_{k=1}^{n} x_k y_k^{\lambda_2}\right)^{\lambda_3-\lambda_1} \leq \left(\sum_{k=1}^{n} x_k y_k^{\lambda_1}\right)^{\lambda_3-\lambda_2}\left(\sum_{k=1}^{n} x_k y_k^{\lambda_3}\right)^{\lambda_2-\lambda_1}. \tag{2}$$

The inequality from (2) is called, in literature, Lyapunov's inequality as given in [12].

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If $x_k \geq 0$, $y_k > 0$ for $k = 1, 2, \ldots, n$ and $\lambda \geq 0$, then

$$\frac{\left(\sum\limits_{k=1}^{n} x_k\right)^{\lambda+1}}{\left(\sum\limits_{k=1}^{n} y_k\right)^{\lambda}} \leq \sum_{k=1}^{n} \frac{x_k^{\lambda+1}}{y_k^{\lambda}}. \tag{3}$$

The inequality from (3) is called, in literature, Radon's inequality as given in [13].

We will unify and extend these results on time scales. The calculus of time scales was initiated by Stefan Hilger as given in [9]. A time scale is an arbitrary nonempty closed subset of the real numbers. The theory of time scales is applied to combine results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The time scales calculus is studied as delta calculus, nabla calculus and diamond$-\alpha$ calculus. This hybrid theory is also widely applied on dynamic inequalities. The basic work on dynamic inequalities is done by Ravi Agarwal, George Anastassiou, Martin Bohner, Allan Peterson, Donal O'Regan, Samir Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adopted from monographs [6, 7].

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right–scattered, while if $\rho(t) < t$, we say that t is left–scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right–dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left–dense. If $\mathbb{T}$ has a left–scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^{Δ} is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\varepsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t, and $f^{\Delta}(t)$ is called the delta derivative of f at t.

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd–continuous), if it is continuous at each right–dense point and there exists a finite left–sided limit at every left–dense point. The set of all rd–continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

The next definition is given in [6, 7].

Definition 2.1. A function $F:\mathbb{T}\to\mathbb{R}$ is called a delta antiderivative of $f:\mathbb{T}\to\mathbb{R}$, provided that $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b)-F(a).$$

The following results of nabla calculus are taken from [5, 6, 7].

If $\mathbb{T}$ has a right–scattered minimum m, then $\mathbb{T}_k=\mathbb{T}-\{m\}$, otherwise $\mathbb{T}_k=\mathbb{T}$. A function $f:\mathbb{T}_k\to\mathbb{R}$ is called nabla differentiable at $t\in\mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t)\in\mathbb{R}$ such that given any $\varepsilon>0$, there is a neighborhood V of t, such that

$$|f(\rho(t))-f(s)-f^{\nabla}(t)(\rho(t)-s)|\le\varepsilon|\rho(t)-s|,$$

for all $s\in V$.

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be left–dense continuous (ld–continuous), provided it is continuous at all left–dense points in $\mathbb{T}$ and its right–sided limits exist (finite) at all right–dense points in $\mathbb{T}$. The set of all ld–continuous functions is denoted by $C_{ld}(\mathbb{T},\mathbb{R})$.

The next definition is given in [5, 6, 7].

Definition 2.2. A function $G:\mathbb{T}\to\mathbb{R}$ is called a nabla antiderivative of $g:\mathbb{T}\to\mathbb{R}$, provided that $G^{\nabla}(t)=g(t)$ holds for all $t\in\mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b)-G(a).$$

We need the following results.

Theorem 2.1 ([1]). *Let $w,f,g\in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$. If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, then*

$$\int_a^b |w(x)||f(x)g(x)|\Delta x\le\left(\int_a^b |w(x)||f(x)|^p\Delta x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q\Delta x\right)^{\frac{1}{q}}. \quad (4)$$

If $\frac{1}{p}+\frac{1}{q}=1$ with $p<0$ or $q<0$, then inequality (4) is reversed.

Inequality (4) is called dynamic Rogers–Hölder's inequality on time scales.

We consider the Kantorovich's ratio defined by

$$K(h):=\frac{(h+1)^2}{4h},\quad \textit{where}\quad h>0.$$

The function K is decreasing on $(0,1)$ and increasing on $[1,+\infty)$, $K(h)\ge1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following reverse of Young's inequality [11] in terms of Kantorovich's ratio holds

$$\frac{a}{p}+\frac{b}{q}\le K^{\delta}\left(\frac{a}{b}\right)a^{\frac{1}{p}}b^{\frac{1}{q}}, \quad (5)$$

where $a,b>0$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$ and $\delta=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

The following definition is taken from [2, 4].

Definition 2.3. For $\alpha \geq 1$, the time scale Δ–Riemann–Liouville type fractional integral for a function $f \in C_{rd}$ is defined by

$$\mathscr{I}_a^{\alpha} f(t) = \int_a^t h_{\alpha-1}(t,\sigma(\tau)) f(\tau)\Delta\tau, \tag{6}$$

which is an integral on $[a,t)_{\mathbb{T}}$, as given in [8] and $h_\alpha : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise rd–continuous functions, such that $h_0(t,s) = 1$,

$$h_{\alpha+1}(t,s) = \int_s^t h_\alpha(\tau,s)\Delta\tau, \ \forall s,t \in \mathbb{T}. \tag{7}$$

Notice that

$$\mathscr{I}_a^1 f(t) = \int_a^t f(\tau)\Delta\tau,$$

which is absolutely continuous in $t \in [a,b]_{\mathbb{T}}$, see [8].

The following definition is taken from [3, 4].

Definition 2.4. For $\alpha \geq 1$, the time scale ∇–Riemann–Liouville type fractional integral for a function $f \in C_{ld}$ is defined by

$$\mathscr{J}_a^{\alpha} f(t) = \int_a^t \hat{h}_{\alpha-1}(t,\rho(\tau)) f(\tau)\nabla\tau, \tag{8}$$

which is an integral on $(a,t]_{\mathbb{T}}$, as given in [8] and $\hat{h}_\alpha : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise ld–continuous functions, such that $\hat{h}_0(t,s) = 1$,

$$\hat{h}_{\alpha+1}(t,s) = \int_s^t \hat{h}_\alpha(\tau,s)\nabla\tau, \ \forall s,t \in \mathbb{T}. \tag{9}$$

Notice that

$$\mathscr{J}_a^1 f(t) = \int_a^t f(\tau)\nabla\tau,$$

which is absolutely continuous in $t \in [a,b]_{\mathbb{T}}$, see [8].

3. Renyi's inequality

In order to present our main results, first we give a simple proof for an extension of dynamic Renyi's inequality on time scales by using the time scale Δ–Riemann–Liouville type fractional integral.

Theorem 3.1. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ be Δ–integrable functions and $h_{\alpha-1}(.,.) > 0$. If $\lambda \in \mathbb{R}_0^+ - \{0,1\}$, then for $\alpha \geq 1$*

$$\frac{1}{\lambda-1}\left(\mathscr{I}_a^{\alpha}(|w(x)||f(x)|)\right)^{\lambda}\left(\mathscr{I}_a^{\alpha}(|w(x)||g(x)|)\right)^{1-\lambda} \leq \frac{1}{\lambda-1}\mathscr{I}_a^{\alpha}\left(|w(x)||f(x)|^{\lambda}|g(x)|^{1-\lambda}\right). \tag{10}$$

Proof. Case (1). If $\lambda > 1$, then dynamic Rogers–Hölder's inequality (4) becomes

$$\int_a^x |w(y)||f(y)g(y)|\Delta y \leq \left(\int_a^x |w(y)||f(y)|^{\lambda}\Delta y\right)^{\frac{1}{\lambda}} \left(\int_a^x |w(y)||g(y)|^{\frac{\lambda}{\lambda-1}}\Delta y\right)^{\frac{\lambda-1}{\lambda}}. \tag{11}$$

Replacing $|g(y)|$ by $|g(y)|^{\frac{\lambda-1}{\lambda}}$ in inequality (11), we obtain

$$\int_a^x |w(y)||f(y)||g(y)|^{\frac{\lambda-1}{\lambda}}\Delta y \leq \left(\int_a^x |w(y)||f(y)|^{\lambda}\Delta y\right)^{\frac{1}{\lambda}} \left(\int_a^x |w(y)||g(y)|\Delta y\right)^{\frac{\lambda-1}{\lambda}}. \tag{12}$$

Taking power $\lambda > 1$ on both sides of inequality (12) and replacing $|w(y)|$ by $h_{\alpha-1}(x,\sigma(y))$ $|w(y)|$ with $h_{\alpha-1}(x,\sigma(y)) > 0$, $\forall x \in [a,b]_{\mathbb{T}}$, we get

$$\left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)||g(y)|^{\frac{\lambda-1}{\lambda}}\Delta y\right)^{\lambda} \left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||g(y)|\Delta y\right)^{1-\lambda}$$
$$\leq \int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^{\lambda}\Delta y. \tag{13}$$

Replacing $|f(y)|$ by $|f(y)||g(y)|^{\frac{1-\lambda}{\lambda}}$ in inequality (13) and multiplying both sides by $\frac{1}{\lambda-1}$, we get the desired claim.

Case (2). If $\lambda \in (0,1)$, then by using dynamic Rogers–Hölder's inequality (4) for $p = \frac{1}{\lambda} > 1$ and $q = \frac{1}{1-\lambda} > 1$, we obtain

$$\int_a^x |w(y)||f(y)|^{\lambda}|g(y)|^{1-\lambda}\Delta y \leq \left(\int_a^x |w(y)||f(y)|\Delta y\right)^{\lambda} \left(\int_a^x |w(y)||g(y)|\Delta y\right)^{1-\lambda}. \tag{14}$$

Replacing $|w(y)|$ by $h_{\alpha-1}(x,\sigma(y))|w(y)|$ with $h_{\alpha-1}(x,\sigma(y)) > 0$, $\forall x \in [a,b]_{\mathbb{T}}$ and multiplying both sides of inequality (14) by $\frac{1}{\lambda-1}$, we get the desired claim. Thus, the proof of Theorem 3.1 is now complete. □

Next, we give an extension of dynamic Renyi's inequality on time scales by using the time scale ∇–Riemann–Liouville type fractional integral.

Theorem 3.2. *Let $w,f,g \in C_{ld}([a,b]_{\mathbb{T}},\mathbb{R})$ be ∇–integrable functions and $\hat{h}_{\alpha-1}(.,.) > 0$. If $\lambda \in \mathbb{R}_0^+ - \{0,1\}$, then for $\alpha \geq 1$*

$$\frac{1}{\lambda-1}\left(\mathscr{J}_a^{\alpha}(|w(x)||f(x)|)\right)^{\lambda}\left(\mathscr{J}_a^{\alpha}(|w(x)||g(x)|)\right)^{1-\lambda}$$
$$\leq \frac{1}{\lambda-1}\mathscr{J}_a^{\alpha}\left(|w(x)||f(x)|^{\lambda}|g(x)|^{1-\lambda}\right). \tag{15}$$

Proof. Similar to the proof of Theorem 3.1. □

Remark 3.1. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = n+1$, $w \equiv 1$, $f(k) = x_k \in [0,+\infty)$ and $g(k) = y_k \in [0,+\infty)$ for $k = 1,2,\ldots,n$. Then inequality (10) reduces to (1).

4. Lyapunov's inequality

In this section, we give a simple proof for an extension of dynamic Lyapunov's inequality and its reverse version via time scales by using the time scale Δ–Riemann–Liouville type fractional integral.

Theorem 4.1. *Let $w,f,g,h \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}-\{0\})$ be Δ–integrable functions and $h_{\alpha-1}(.,.) > 0$. Further assume that $|f|^{\lambda_1(\lambda_3-\lambda_2)}|g|^{\lambda_2(\lambda_1-\lambda_3)}|h|^{\lambda_3(\lambda_2-\lambda_1)} = M$ for $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$, where M is a positive real number.*

(1) *If* $\lambda_1 < \lambda_2 < \lambda_3$*, then for* $\alpha \geq 1$

$$\left(\mathscr{I}_a^\alpha\left(|w(x)||f(x)|^{\lambda_1}\right)\right)^{\lambda_3-\lambda_2}\left(\mathscr{I}_a^\alpha\left(|w(x)||g(x)|^{\lambda_2}\right)\right)^{\lambda_1-\lambda_3} \times\left(\mathscr{I}_a^\alpha\left(|w(x)||h(x)|^{\lambda_3}\right)\right)^{\lambda_2-\lambda_1} \geq M. \quad (16)$$

(2) *If* $\lambda_2 < \lambda_1 < \lambda_3$*, then for* $\alpha \geq 1$

$$\left(\mathscr{I}_a^\alpha\left(|w(x)||f(x)|^{\lambda_1}\right)\right)^{\lambda_3-\lambda_2}\left(\mathscr{I}_a^\alpha\left(|w(x)||g(x)|^{\lambda_2}\right)\right)^{\lambda_1-\lambda_3} \times\left(\mathscr{I}_a^\alpha\left(|w(x)||h(x)|^{\lambda_3}\right)\right)^{\lambda_2-\lambda_1} \leq M. \quad (17)$$

Proof. Case (1). We note that $\lambda_1-\lambda_2+\lambda_3-\lambda_1+\lambda_2-\lambda_3=0$. Set $P=\frac{\lambda_3-\lambda_1}{\lambda_3-\lambda_2}>1$, $Q=\frac{\lambda_3-\lambda_1}{\lambda_2-\lambda_1}>1$. Then $\frac{1}{P}+\frac{1}{Q}=1$.

Applying Rogers–Hölder's inequality (4), we get

$$\int_a^x |w(y)||f(y)h(y)|\Delta y \leq \left(\int_a^x |w(y)||f(y)|^P \Delta y\right)^{\frac{1}{P}}\left(\int_a^x |w(y)||h(y)|^Q \Delta y\right)^{\frac{1}{Q}}. \quad (18)$$

Replacing $|f(y)|$ by $|f(y)|^{\frac{\lambda_1}{P}}$ and $|h(y)|$ by $|h(y)|^{\frac{\lambda_3}{Q}}$ in inequality (18), we obtain

$$\int_a^x |w(y)||f(y)|^{\lambda_1\left(\frac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}\right)}|h(y)|^{\lambda_3\left(\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right)}\Delta y \leq \left(\int_a^x |w(y)||f(y)|^{\lambda_1}\Delta y\right)^{\frac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}}\left(\int_a^x |w(y)||h(y)|^{\lambda_3}\Delta y\right)^{\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}}. \quad (19)$$

Replacing $|w(y)|$ by $h_{\alpha-1}(x,\sigma(y))|w(y)|$ with $h_{\alpha-1}(x,\sigma(y))>0$, $\forall x \in [a,b]_{\mathbb{T}}$ and taking power $(\lambda_3-\lambda_1)>0$ on both sides of inequality (19), we get

$$\left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^{\lambda_1\left(\frac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}\right)}|h(y)|^{\lambda_3\left(\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right)}\Delta y\right)^{\lambda_3-\lambda_1} \leq \left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^{\lambda_1}\Delta y\right)^{\lambda_3-\lambda_2} \times\left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||h(y)|^{\lambda_3}\Delta y\right)^{\lambda_2-\lambda_1}. \quad (20)$$

Using the condition that

$$|f|^{\lambda_1(\lambda_3-\lambda_2)}|g|^{\lambda_2(\lambda_1-\lambda_3)}|h|^{\lambda_3(\lambda_2-\lambda_1)}=M$$

for $\lambda_1<\lambda_2<\lambda_3$, where M is a positive real number, the inequality (20) takes the form

$$\left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)|M^{\frac{1}{\lambda_3-\lambda_1}}|g(y)|^{\lambda_2}\Delta y\right)^{\lambda_3-\lambda_1} \leq \left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^{\lambda_1}\Delta y\right)^{\lambda_3-\lambda_2} \times\left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||h(y)|^{\lambda_3}\Delta y\right)^{\lambda_2-\lambda_1}. \quad (21)$$

This directly yields (16).

Similarly, we can prove the Case (2) by applying reverse Rogers–Hölder's inequality. Thus, the proof of Theorem 4.1 is now complete. □

Next, we present an extension of dynamic Lyapunov's inequality and its reverse version via time scales by using the time scale ∇–Riemann–Liouville type fractional integral.

Theorem 4.2. *Let $w, f, g, h \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be ∇–integrable functions and $\hat{h}_{\alpha-1}(.,.) > 0$. Further assume that $|f|^{\lambda_1(\lambda_3-\lambda_2)}|g|^{\lambda_2(\lambda_1-\lambda_3)}|h|^{\lambda_3(\lambda_2-\lambda_1)} = M$ for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, where M is a positive real number.*

(1) *If $\lambda_1 < \lambda_2 < \lambda_3$, then for $\alpha \geq 1$*

$$\left(\mathscr{I}_a^{\alpha}\left(|w(x)||f(x)|^{\lambda_1}\right)\right)^{\lambda_3-\lambda_2}\left(\mathscr{I}_a^{\alpha}\left(|w(x)||g(x)|^{\lambda_2}\right)\right)^{\lambda_1-\lambda_3} \times\left(\mathscr{I}_a^{\alpha}\left(|w(x)||h(x)|^{\lambda_3}\right)\right)^{\lambda_2-\lambda_1} \geq M. \quad (22)$$

(2) *If $\lambda_2 < \lambda_1 < \lambda_3$, then for $\alpha \geq 1$*

$$\left(\mathscr{I}_a^{\alpha}\left(|w(x)||f(x)|^{\lambda_1}\right)\right)^{\lambda_3-\lambda_2}\left(\mathscr{I}_a^{\alpha}\left(|w(x)||g(x)|^{\lambda_2}\right)\right)^{\lambda_1-\lambda_3} \times\left(\mathscr{I}_a^{\alpha}\left(|w(x)||h(x)|^{\lambda_3}\right)\right)^{\lambda_2-\lambda_1} \leq M. \quad (23)$$

Proof. Similar to the proof of Theorem 4.1. □

Remark 4.1. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = n+1$, $w(k) = x_k \in (0, +\infty)$ and $f(k) = g(k) = h(k) = y_k \in (0, +\infty)$ for $k = 1, 2, \ldots, n$. Then inequality (16) reduces to (2).

5. Radon's inequality

In order to conclude our main results, now we present an extension of dynamic Radon's inequality by using Kantorovich's ratio and the time scale Δ–Riemann–Liouville type fractional integral. Extensions of dynamic Radon's inequality are also proved in [14, 15, 16].

Theorem 5.1. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be Δ–integrable functions and $h_{\alpha-1}(.,.) > 0$. If $\lambda > 0$ and $\gamma \geq 1$, then for $\alpha \geq 1$*

$$\mathscr{I}_a^{\alpha}\left(\frac{|w(x)||f(x)|^{\lambda+\gamma}}{|g(x)|^{\lambda}}\right) \leq \frac{\left(\mathscr{I}_a^{\alpha}\left(K^{\delta}\left(\frac{\Omega(x)|f(x)|^{\lambda+\gamma}}{\Lambda(x)|g(x)|^{\lambda+\gamma}}\right)|w(x)||f(x)||g(x)|^{\gamma-1}\right)\right)^{\lambda+\gamma}}{\left(\mathscr{I}_a^{\alpha}\left(|w(x)||g(x)|^{\gamma}\right)\right)^{\lambda+\gamma-1}}, \quad (24)$$

where
$\Lambda(x) = \mathscr{I}_a^{\alpha}\left(\frac{|w(x)||f(x)|^{\lambda+\gamma}}{|g(x)|^{\lambda}}\right)$, $\Omega(x) = \mathscr{I}_a^{\alpha}\left(|w(x)||g(x)|^{\gamma}\right)$, $\forall x \in [a,b]_{\mathbb{T}}$, $\delta = \max\left\{\frac{1}{\lambda+\gamma}, \frac{\lambda+\gamma-1}{\lambda+\gamma}\right\}$, *and $K(.)$ is the Kantorovich's ratio.*

Proof. Let $p = \lambda+\gamma > 1$ and $q = \frac{\lambda+\gamma}{\lambda+\gamma-1} > 1$. Setting

$$\Phi(y) = \frac{|w(y)||f(y)|^p}{\int_a^x |w(y)||f(y)|^p \Delta y} \text{ and } \Psi(y) = \frac{|w(y)||g(y)|^q}{\int_a^x |w(y)||g(y)|^q \Delta y}$$

on $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. From Young's inequality (5), we get

$$\frac{|w(y)||f(y)|^p}{p\left(\int_a^x |w(y)||f(y)|^p \Delta y\right)} + \frac{|w(y)||g(y)|^q}{q\left(\int_a^x |w(y)||g(y)|^q \Delta y\right)} \leq \frac{K^{\delta}\left(\frac{\Omega_1(x)|f(y)|^p}{\Lambda_1(x)|g(y)|^q}\right)|w(y)||f(y)g(y)|}{(\Lambda_1(x))^{\frac{1}{p}}(\Omega_1(x))^{\frac{1}{q}}}, \quad (25)$$

where $\Lambda_1(x) = \int_a^x |w(y)||f(y)|^p \Delta y$ and $\Omega_1(x) = \int_a^x |w(y)||g(y)|^q \Delta y$. By integrating both sides of inequality (25) over y from a to x, we obtain

$$1 \leq \frac{\int_a^x K^{\delta}\left(\frac{\Omega_1(x)|f(y)|^p}{\Lambda_1(x)|g(y)|^q}\right)|w(y)||f(y)g(y)|\Delta y}{\left(\int_a^x |w(y)||f(y)|^p \Delta y\right)^{\frac{1}{p}}\left(\int_a^x |w(y)||g(y)|^q \Delta y\right)^{\frac{1}{q}}}. \quad (26)$$

Thus

$$\left(\int_a^x |w(y)||f(y)|^p \Delta y\right)^{\frac{1}{p}}\left(\int_a^x |w(y)||g(y)|^q \Delta y\right)^{\frac{1}{q}} \leq \int_a^x K^{\delta}\left(\frac{\Omega_1(x)|f(y)|^p}{\Lambda_1(x)|g(y)|^q}\right)|w(y)||f(y)g(y)|\Delta y. \quad (27)$$

Replacing $|f(y)|$ and $|g(y)|$ by $|F(y)|$ and $|G(y)|$ in (27), respectively, we have

$$\left(\int_a^x |w(y)||F(y)|^{\lambda+\gamma} \Delta y\right)^{\frac{1}{\lambda+\gamma}}\left(\int_a^x |w(y)||G(y)|^{\frac{\lambda+\gamma}{\lambda+\gamma-1}} \Delta y\right)^{\frac{\lambda+\gamma-1}{\lambda+\gamma}} \leq \int_a^x K^{\delta}\left(\frac{\Omega_2(x)|F(y)|^{\lambda+\gamma}}{\Lambda_2(x)|G(y)|^{\frac{\lambda+\gamma}{\lambda+\gamma-1}}}\right)|w(y)||F(y)G(y)|\Delta y, \quad (28)$$

where $\Lambda_2(x) = \int_a^x |w(y)||F(y)|^{\lambda+\gamma} \Delta y$ and $\Omega_2(x) = \int_a^x |w(y)||G(y)|^{\frac{\lambda+\gamma}{\lambda+\gamma-1}} \Delta y$.
Replacing $|F(y)|$ by $\left|\frac{f(y)}{g(y)}\right|^{\frac{1}{\lambda+\gamma}}$ and $|G(y)|$ by $|f(y)|^{\frac{\lambda+\gamma-1}{\lambda+\gamma}}|g(y)|^{\frac{1}{\lambda+\gamma}}$ in (28), we obtain

$$\left(\int_a^x \frac{|w(y)||f(y)|}{|g(y)|} \Delta y\right)^{\frac{1}{\lambda+\gamma}}\left(\int_a^x |w(y)||f(y)||g(y)|^{\frac{1}{\lambda+\gamma-1}} \Delta y\right)^{\frac{\lambda+\gamma-1}{\lambda+\gamma}} \leq \int_a^x K^{\delta}\left(\frac{\Omega_3(x)}{\Lambda_3(x)|g(y)|^{\frac{\lambda+\gamma}{\lambda+\gamma-1}}}\right)|w(y)||f(y)|\Delta y, \quad (29)$$

where $\Lambda_3(x) = \int_a^x \frac{|w(y)||f(y)|}{|g(y)|} \Delta y$ and $\Omega_3(x) = \int_a^x |w(y)||f(y)||g(y)|^{\frac{1}{\lambda+\gamma-1}} \Delta y$.
Taking power $\lambda+\gamma$ on both sides of inequality (29), we get

$$\int_a^x \frac{|w(y)||f(y)|}{|g(y)|} \Delta y \leq \frac{\left(\int_a^x K^{\delta}\left(\frac{\Omega_3(x)}{\Lambda_3(x)|g(y)|^{\frac{\lambda+\gamma}{\lambda+\gamma-1}}}\right)|w(y)||f(y)|\Delta y\right)^{\lambda+\gamma}}{\left(\int_a^x |w(y)||f(y)||g(y)|^{\frac{1}{\lambda+\gamma-1}} \Delta y\right)^{\lambda+\gamma-1}}. \quad (30)$$

Replacing $|g(y)|$ by $\left|\frac{g(y)}{f(y)}\right|^{\lambda+\gamma-1}$ and $|w(y)|$ by $h_{\alpha-1}(x,\sigma(y))|w(y)||g(y)|^{\gamma-1}$ with $h_{\alpha-1}(x,\sigma(y))>0$, $\forall x\in[a,b]_{\mathbb{T}}$ in (30), we get

$$\int_a^x \frac{h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^{\lambda+\gamma}}{|g(y)|^{\lambda}}\Delta y \leq \frac{\left(\int_a^x h_{\alpha-1}(x,\sigma(y))K^{\delta}\left(\frac{\Omega(x)|f(y)|^{\lambda+\gamma}}{\Lambda(x)|g(y)|^{\lambda+\gamma}}\right)|w(y)||f(y)||g(y)|^{\gamma-1}\Delta y\right)^{\lambda+\gamma}}{\left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||g(y)|^{\gamma}\Delta y\right)^{\lambda+\gamma-1}}, \quad (31)$$

where $\Lambda(x)=\int_a^x \frac{h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^{\lambda+\gamma}}{|g(y)|^{\lambda}}\Delta y$, $\Omega(x)=\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||g(y)|^{\gamma}\Delta y$. Thus (24) follows from (31). □

Theorem 5.2. *Let $w,f,g\in C_{ld}([a,b]_{\mathbb{T}},\mathbb{R}-\{0\})$ be ∇–integrable functions and $\hat{h}_{\alpha-1}(.,.)>0$. If $\lambda>0$ and $\gamma\geq 1$, then for $\alpha\geq 1$*

$$\mathscr{J}_a^{\alpha}\left(\frac{|w(x)||f(x)|^{\lambda+\gamma}}{|g(x)|^{\lambda}}\right)\leq \frac{\left(\mathscr{J}_a^{\alpha}\left(K^{\delta}\left(\frac{\Omega(x)|f(x)|^{\lambda+\gamma}}{\Lambda(x)|g(x)|^{\lambda+\gamma}}\right)|w(x)||f(x)||g(x)|^{\gamma-1}\right)\right)^{\lambda+\gamma}}{\left(\mathscr{J}_a^{\alpha}(|w(x)||g(x)|^{\gamma})\right)^{\lambda+\gamma-1}}, \quad (32)$$

where $\Lambda(x)=\mathscr{J}_a^{\alpha}\left(\frac{|w(x)||f(x)|^{\lambda+\gamma}}{|g(x)|^{\lambda}}\right)$, $\Omega(x)=\mathscr{J}_a^{\alpha}(|w(x)||g(x)|^{\gamma})$, $\forall x\in[a,b]_{\mathbb{T}}$, $\delta=\max\left\{\frac{1}{\lambda+\gamma},\frac{\lambda+\gamma-1}{\lambda+\gamma}\right\}$, and $K(.)$ is the Kantorovich's ratio.

Proof. Similar to the proof of Theorem 5.1. □

Remark 5.1. Let $\alpha=1$, $\mathbb{T}=\mathbb{Z}$, $a=1$, $x=b=n+1$, $w\equiv 1$, $f(k)=x_k\in(0,+\infty)$ and $g(k)=y_k\in(0,+\infty)$ for $k=1,2,\dots,n$. Then discrete version of inequality (24) reduces to

$$\sum_{k=1}^{n}\frac{x_k^{\lambda+\gamma}}{y_k^{\lambda}}\leq \frac{\left(\sum\limits_{k=1}^{n}K^{\delta}\left(\frac{\Omega x_k^{\lambda+\gamma}}{\Lambda y_k^{\lambda+\gamma}}\right)x_k y_k^{\gamma-1}\right)^{\lambda+\gamma}}{\left(\sum\limits_{k=1}^{n}y_k^{\gamma}\right)^{\lambda+\gamma-1}}, \quad (33)$$

where $\Lambda=\sum\limits_{k=1}^{n}\frac{x_k^{\lambda+\gamma}}{y_k^{\lambda}}$ and $\Omega=\sum\limits_{k=1}^{n}y_k^{\gamma}$.

The inequality (33) is obtained by using Kantorovich's ratio and is a reverse version of inequality (3) for $\gamma=1$.

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УДК 519.644 Научная статья

Объединение классических и динамических неравенств, возникающих при анализе временных масштабов

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В этой статье мы представляем расширение динамического неравенства Реньи на шкалы времени с помощью дробного интеграла типа Римана-Лиувилля. Кроме того, мы находим обобщения хорошо известного неравенства Ляпунова и неравенства Радона на шкалах времени с помощью дробных интегралов типа Римана-Лиувилля на шкале. Наши исследования объединяют и расширяют некоторые непрерывные неравенства и соответствующие им дискретные аналоги.

Ключевые слова: шкалы времени, дробный интеграл Римана-Лиувилля, неравенство Реньи, неравенство Ляпунова, неравенство Радона.

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Конкурирующие интересы. Конфликтов интересов в отношении авторства и публикации нет.

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