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Non-commutative phase space Landau problem in the presence of a minimal length

***F. A. Dossa*[1], *J. T. Koumagnon*[2], *J. V. Hounguevou*[2], *G. Y. H. Avossevou*[2]**

[1] Faculté des Sciences et Techniques (FAST) Université Nationale des Sciences, Technologies, Ingénierie et Mathématiques (UNSTIM) Abomey BP: 2282 Goho Abomey, Republic of Benin

[2] Unité de Recherche en Physique Théorique (URPT), Institut de Mathématiques et de Sciences Physiques (IMSP), 01 B.P. 613 Porto-Novo, Rep. du Bénin

E-mail: finedofas@yahoo.fr, jeankoumagnon81@gmail.com, jeanvignonhoung@yahoo.fr, gabavossevou@gmail.com

The deformed Landau problem under a electromagnetic field is studied, where the Heisenberg algebra is constructed in detail in non-commutative phase space in the presence of a minimal length. We show that, in the presence of a minimal length, the momentum space is more practical to solve any problem of eigenvalues. From the Nikiforov-Uvarov method, the energy eigenvalues are obtained and the corresponding wave functions are expressed in terms of hypergeometric functions. The fortuitous degeneration observed in the spectrum shows that the formulation of the minimal length complements that of the non-commutative phase space.

Keywords: Landau problem, non-commutative phase space, minimal length, Nikiforov-Uvarov method, hypergeometric functions

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Introduction

The concept of non-commutative coordinates arose in the context of superstring theory. The intrinsic length of the strings being the parameter which induces a non-commutative structure of space-time on a very small scale.In physics, the idea that the coordinates of space-time could not commute have been put forward by Heisenberg with the view that this could solve the problems of ultraviolet divergences in quantum

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field theory. A classic example of how non-commutativity can emerge in a physical situation is given in the Landau problem [1] by projecting the whole system to the lowest Landau level. This idea has introduced a long time ago by Snyder [2] and has been extensively studied by Connes [3]. Then, the term non-commutative geometry provides a mathematical framework in which a number of physical concepts can be expressed and sometimes unified. To clarify the role that non-commutative phase space variables can play in physics, a better understanding of quantum mechanics in non-commutative phase spaces would be useful. In fact, there are realistic physical systems like electrons in a uniform external magnetic field that actually move in a two-dimensional non-commutative phase space that is perpendicular to the magnetic field. In accordance with the first works concerning the variables of non-commutative phase space [4]-[8], quantum mechanics in non-commutative space has been defined in a simple and direct way by following several approaches [9]-[11] and some models baseline have been studied [12]-[17]. In quantum mechanics, the physical observables must be described by the operators and that the position and momentum operators do not commute. This leads to the well-known Heisenberg uncertainty principle which states that it is not possible to measure both the position and the momentum of a particle with an absolute precision. There are physical situations in which an electron is perfectly localized and there are physical situations in which an electron has a perfectly defined momentum, however, one cannot have both at the same time. The more the electron is localized, the less its momentum is defined, and vice versa. In fact, this description is idealistic, a generalized uncertainty relation leading to a non-zero minimum uncertainty would be closer to physical reality. It is this last point that has driven physicists in recent years to take an interest in this notion of minimal length by trying to introduce it into the treatment of physical problems, in quantum mechanics, through corrections to canonical commutation relations. The general formalism of this modified Heisenberg algebra has been studied by Kempf and his collaborators [18]-[20]. A theory which follows from this principle could give a better description of composite particles such as nucleons in nuclear potentials or nuclei in molecular potentials [20]. In recent years, several problems have been studied in connection with this deformed version of quantum mechanics. The one-dimensional and multi-dimensional harmonic oscillator has been solved exactly [21]-[23]. The problem of a charged particle of spin $1/2$ moving in a constant magnetic field has also been dealt with in this formalism, and the thermodynamic properties of the high temperature system have been examined [24]. Recently, minimal and maximal lengths from position-dependent non-commutativity have been studied in work [25]. The aim of this work is to study the formulation of a system made up of an electrically charged particle moving in a Euclidean plane and subjected to a homogeneous electromagnetic field. Next, we solve the fundamental equations within the framework of non-relativistic quantum mechanics with a minimal length in the non-commutative phase space. For so doing, we first mapped the problem into a non-commutative phase space using appropriate transformations. Then we solve it in the presence of a minimal length. The deformed ladder operators can be realized as differential form in momentum space. This realization enables one to convert the time-independent Schrodinger equation of a dynamical system into a differential equation. We adopt the Nikiforov-Uvarov method [26] which has been used to solve several quantum mechanical problems in physics and applied mathematics. By means of this method we obtain the energy eigenvalues and the corresponding wave functions in terms of hypergeometric functions. Then, we study the degeneracy of Landau levels.

Non-commutative phase space and the minimal length

It is well known that in the two-dimensional commutative space, the coordinates x_i and momenta p_i satisfy the usual canonical commutation relations

$$[x_i, p_j] = i\hbar\delta_{ij}\mathbb{1}, \quad [x_i, x_j] = 0 = [p_i, p_j]. \tag{1}$$

At very tiny scales, say string scale, the space may not commute anymore. Let us denote the operators of coordinates and momenta in non-commutative phase space as $\hat{x}_i$ and $\hat{p}_i$ respectively, then in the two-dimensional non-commutative phase space [27],[28], the operators $\hat{x}_i$ and $\hat{p}_i$ satisfy the following commutation relations

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\mathbb{1}, \quad [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\bar{\theta}_{ij}, \tag{2}$$

where θ_{ij} and $\bar{\theta}_{ij}$ are totally antisymmetric matrices which are proportional to the antisymmetric tensor ε_{ij},

$$\theta_{ij} = \varepsilon_{ij}\theta, \quad \bar{\theta}_{ij} = \varepsilon_{ij}\bar{\theta}, \quad \varepsilon_{12} = 1, \ \varepsilon_{21} = -1. \tag{3}$$

Using the commutation relations (1) and (2), we can study the representations of the non-commutative operators $\hat{x}_i$ and $\hat{p}_i$ in terms of x_i and p_i. Once these representations are obtained, the non-commutative problems can be changed into problems in the usual commutative space that we know. So, we have the following representations for the operators $\hat{x}_i$ and $\hat{p}_i$,

$$\hat{x}_i = \alpha x_i - \frac{1}{2\alpha\hbar}\varepsilon_{ij}\theta p_j, \quad \hat{p}_i = \bar{\alpha} p_i + \frac{1}{2\bar{\alpha}\hbar}\varepsilon_{ij}\bar{\theta} x_j, \tag{4}$$

where the scaling constants α and $\bar{\alpha}$ should not be equal to zero, and the parameters θ and $\bar{\theta}$ are related by the relation,

$$\theta\bar{\theta} = 4\alpha\bar{\alpha}\hbar^2(1 - \alpha\bar{\alpha}). \tag{5}$$

In two-dimensional quantum mechanics, in the presence of a minimal length, the canonical commutation relations (1) become

$$\begin{gathered}[\tilde{x}_i, \tilde{p}_j] = i\hbar(1 + \beta p^2)\delta_{ij}\mathbb{1}, \quad [\tilde{p}_i, \tilde{p}_j] = 0, \\ [\tilde{x}_i, \tilde{x}_j] = -2i\hbar\beta(1 + \beta p^2)\varepsilon_{ijk}L_k, \quad \beta = \frac{\bar{\beta}}{\hbar\mu\omega}, \quad 0 \leq \bar{\beta} \leq 1,\end{gathered} \tag{6}$$

where β has the dimension of an inverse squared momentum and $\bar{\beta}$ is dimensionless. The limit $\bar{\beta} \longrightarrow 0$ corresponds to the normal quantum mechanics while the limit $\bar{\beta} \longrightarrow 1$ corresponds to the extreme quantum gravity.

A representation of $\tilde{x}_i$ and $\tilde{p}_i$ which realizes (6) is given by

$$\tilde{x}_i = i\hbar(1 + \beta p^2)\frac{\partial}{\partial p_i}, \quad \tilde{p}_i = p_i, \quad i = 1, 2, \quad p^2 = p_1^2 + p_2^2. \tag{7}$$

The commutation relations (2) become

$$\begin{gathered}[X_i, P_j] = i\hbar(1 + \beta p^2)\left[\delta_{ij} - i\beta\tfrac{\alpha\bar{\theta}}{\bar{\alpha}}\varepsilon_{i\ell}\varepsilon_{j\ell k}L_k\right], \\ [X_i, X_j] = i\hbar(1 + \beta p^2)\left[\tfrac{\theta}{\hbar}\varepsilon_{i\ell}\delta_{j\ell} - 2\beta\alpha^2\varepsilon_{ijk}L_k\right], \\ [P_i, P_j] = i\bar{\theta}(1 + \beta p^2)\delta_{ij},\end{gathered} \tag{8}$$

where

$$X_i = \alpha\tilde{x}_i - \frac{1}{2\alpha\hbar}\varepsilon_{ij}\theta\tilde{p}_j, \quad P_i = \bar{\alpha}\tilde{p}_i + \frac{1}{2\bar{\alpha}\hbar}\varepsilon_{ij}\bar{\theta}\tilde{x}_j. \tag{9}$$

Deformed Landau problem

The Landau problem has been considered in physics as an example which allows to characterize non-commutative geometry. This problem is a system consisting of a charged particle with electric charge e and mass μ moving in a two-dimensional plane and subjected to a homogeneous electromagnetic field. We would like to consider background field configurations that are homogeneous in that plane, static namely time independent as well, such that the magnetic field be perpendicular to that plane and the electric one lying inside that plane. The dynamics of the system is described by the Hamiltonian of the form,

$$H = \frac{1}{2\mu}(p_i + eA_i(x_i))^2 + V(x_i) + e\phi(x_i), \tag{10}$$

where p_i is the canonical momentum, $A_i(x_i)$ is the vector potential, $V(x_i)$ is a harmonic potential and $\phi(x_i)$ is a scalar potential.

Hence we have the following form for the gauge fields,

$$A_i(x_i) = -\frac{1}{2}B^0\varepsilon_{ij}x_j, \quad \phi(x_i) = -x_iE_i^0, \tag{11}$$

E_i^0 and B^0 are indeed the electric and magnetic fields of the background. In addition we shall also use for the potential energy,

$$V(x_i) = \frac{1}{2}\mu\omega_0^2x_i^2, \tag{12}$$

hoping to see how that confining potential, would allow for an exact solution.

The Hamiltonian (10) can be extended to more general algebras by assuming that the operators X_i and P_i satisfy the relations (8), with a harmonic potential which is a function of the non-commutative coordinates X_i in the presence of a minimal length. Now let us consider the system (10) on non-commutative phase space in the presence of a minimal length. According to equation (9) the corresponding Hamiltonian can be written as

$$\mathscr{H} = \frac{1}{2\mu}\left(P_i - \frac{1}{2}B\varepsilon_{ij}X_j\right)^2 + \frac{1}{2}\mu\omega_0^2X_i^2 - X_iE_i, \quad B = eB^0, \ E_i = eE_i^0. \tag{13}$$

When wanting to complete with the electric field coupling the square defined by the harmonic potential, one is led to the following change of variables, which is a canonical transformation in phase space,

$$\mathscr{X}_i = X_i - \frac{E_i}{\mu\omega_0^2}, \quad \mathscr{P}_i = P_i - \frac{1}{2}B\varepsilon_{ij}\frac{E_j}{\mu\omega_0^2}. \tag{14}$$

We note that these changes of variable are ill-defined if one wants to set $\omega_0 = 0$. The reason for this is the following. In the presence of a magnetic and an electric field but no other confining force, then the magnetic center moves at a constant velocity, and one needs to apply a Galilei boost; quantum states are no longer all normalisable. In order to avoid that singularity, when wanting to remove the harmonic confining potential, first one needs to turn off the electric field E_i lying in the plane, and only then set $\omega_0 = 0$ [29].

From here on the solution of the system follows a standard path. One introduces the quantities

$$\omega = \sqrt{\omega_0^2 + \frac{1}{4}\omega_c^2}, \quad \omega_c = \frac{B}{\mu}, \quad \Omega_c = \omega_c - \frac{1}{\hbar}\mu\omega^2\theta - \frac{1}{2\mu\hbar}\bar{\theta}, \tag{15}$$

$$M = \left(\frac{\bar{\alpha}^2}{\mu} - \frac{\bar{\alpha}\theta\omega_c}{2\alpha\hbar} + \frac{1}{4\alpha^2\hbar^2}\mu\omega^2\theta^2\right)^{-1}, \tag{16}$$

$$\Omega = \left[\left(\alpha^2\mu\omega^2 + \frac{\bar{\theta}^2}{4\mu\bar{\alpha}^2\hbar^2} - \frac{\alpha\bar{\theta}\omega_c}{2\bar{\alpha}\hbar}\right)\left(\frac{\bar{\alpha}^2}{\mu} - \frac{\bar{\alpha}\theta\omega_c}{2\alpha\hbar} + \frac{1}{4\alpha^2\hbar^2}\mu\omega^2\theta^2\right)\right]^{\frac{1}{2}}, \tag{17}$$

and

$$\mathscr{A}_i = \sqrt{\frac{M\Omega}{2\hbar}}\left(\tilde{x}_i + \frac{i}{M\Omega}\tilde{p}_i\right), \quad \mathscr{A}_i^\dagger = \sqrt{\frac{M\Omega}{2\hbar}}\left(\tilde{x}_i - \frac{i}{M\Omega}\tilde{p}_i\right), \tag{18}$$

with

$$\begin{aligned}
\left[\mathscr{A}_i, \mathscr{A}_j^\dagger\right] &= (1+\beta p^2)\left(\delta_{ij} - i\beta M\Omega\varepsilon_{ijk}L_k\right), \\
\left[\mathscr{A}_i, \mathscr{A}_j\right] &= -i\beta M\Omega(1+\beta p^2)\varepsilon_{ijk}L_k, \\
\left[\mathscr{A}_i^\dagger, \mathscr{A}_j^\dagger\right] &= -i\beta M\Omega(1+\beta p^2)\varepsilon_{ijk}L_k.
\end{aligned} \tag{19}$$

Next,

$$\mathscr{A}_\pm = \frac{1}{\sqrt{2}}\left(\mathscr{A}_1 \mp i\mathscr{A}_2\right), \quad \mathscr{A}_\pm^\dagger = \frac{1}{\sqrt{2}}\left(\mathscr{A}_1^\dagger \pm i\mathscr{A}_2^\dagger\right), \tag{20}$$

with

$$[\mathscr{A}_\pm, \mathscr{A}_\pm^\dagger] = (1+\beta p^2)\left(1 \pm \beta M\Omega L_z\right), \tag{21}$$

$$[\mathscr{A}_+, \mathscr{A}_-] = \beta M\Omega(1+\beta p^2)L_z, \; [\mathscr{A}_+^\dagger, \mathscr{A}_-^\dagger] = -\beta M\Omega(1+\beta p^2)L_z, \tag{22}$$

$$[\mathscr{A}_\pm, \mathscr{A}_\mp^\dagger] = 0, \; [\mathscr{A}_\pm, \mathscr{A}_\pm] = 0, \; [\mathscr{A}_\pm^\dagger, \mathscr{A}_\pm^\dagger] = 0. \tag{23}$$

In terms of these operators the Hamiltonian (13) takes the following form

$$\begin{aligned}
\mathscr{H} &= \frac{\hbar\Omega}{2}\left(\mathscr{A}_+^\dagger\mathscr{A}_+ + \mathscr{A}_+\mathscr{A}_+^\dagger + \mathscr{A}_-^\dagger\mathscr{A}_- + \mathscr{A}_-\mathscr{A}_-^\dagger\right) \\
&\quad + \frac{\hbar\Omega_c}{2}\left(\mathscr{A}_+^\dagger\mathscr{A}_+ + \mathscr{A}_+\mathscr{A}_+^\dagger - \mathscr{A}_-^\dagger\mathscr{A}_- - \mathscr{A}_-\mathscr{A}_-^\dagger\right) - \frac{E_i^2}{2\mu\omega_0^2}.
\end{aligned} \tag{24}$$

Now, we can write the differential form of the ladder operators $\mathscr{A}_\pm$ and $\mathscr{A}_\pm^\dagger$,

$$\mathscr{A}_\pm = i\sqrt{\frac{\hbar M\Omega}{4}}e^{\mp i\varphi}\left[\frac{p}{\hbar M\Omega} + (1+\beta p^2)\frac{\partial}{\partial p} \mp i\frac{(1+\beta p^2)}{p}\frac{\partial}{\partial\varphi}\right], \tag{25}$$

$$\mathscr{A}_{\pm}^{\dagger}=i\sqrt{\frac{\hbar M\Omega}{4}}e^{\pm i\varphi}\left[-\frac{p}{\hbar M\Omega}+(1+\beta p^2)\frac{\partial}{\partial p}\pm i\frac{(1+\beta p^2)}{p}\frac{\partial}{\partial\varphi}\right]. \tag{26}$$

The above Hamiltonian takes the following form

$$\begin{aligned}\mathscr{H} &= -\frac{1}{2}\hbar^2M\Omega^2(1+\beta p^2)^2\left(\frac{\partial^2}{\partial p^2}+\frac{1}{p}\frac{\partial}{\partial p}+\frac{1}{p^2}\frac{\partial^2}{\partial\varphi^2}\right)\\ &\quad -\hbar^2M\Omega^2\beta(1+\beta p^2)p\frac{\partial}{\partial p}+\frac{1}{2M}p^2-i\frac{\hbar}{2}\Omega_c(1+\beta p^2)\frac{\partial}{\partial\varphi}-\frac{E_i^2}{2\mu\omega_0^2}.\end{aligned} \tag{27}$$

Then the time-independent Schrödinger equation gives the differential equation for wave function $\psi(p)$,

$$\begin{aligned}&\left[(1+\beta p^2)^2\left(\frac{\partial^2}{\partial p^2}+\frac{1}{p}\frac{\partial}{\partial p}+\frac{1}{p^2}\frac{\partial^2}{\partial\varphi^2}\right)+2\beta(1+\beta p^2)p\frac{\partial}{\partial p}+\frac{1}{\hbar^2M^2\Omega^2}p^2\right.\\ &\left.+i\frac{\Omega_c}{\hbar M\Omega^2}(1+\beta p^2)\frac{\partial}{\partial\varphi}+\frac{2}{\hbar M\Omega^2}\left(\frac{E_i^2}{2\mu\omega_0^2}+\mathscr{E}\right)\right]\psi(p)=0.\end{aligned} \tag{28}$$

By use the analytical Nikiforov-Uvarov technique[26],[30], we obtain the eigenvalues

$$\begin{aligned}\mathscr{E}_{m,n}^{\beta} &= \frac{1}{2}\hbar\Omega_c m+\frac{1}{2}\beta\hbar\Omega\bar{\Omega}\left[(2n+|m|+1)^2+m^2+1\right]\\ &+ \hbar\Omega(2n+|m|+1)\sqrt{1+\beta^2\bar{\Omega}^2\left(1+m^2+\frac{\Omega_c m}{\beta\Omega\bar{\Omega}}\right)}-\frac{E_i^2}{2\mu\omega_0^2},\end{aligned} \tag{29}$$

where m is the angular momentum quantum number and n is the principal quantum number. $\bar{\Omega}$ is given by $\bar{\Omega}=\hbar M\Omega$.

The corresponding wave functions are given by

$$\begin{aligned}\psi_n^m(p) &= \mathscr{N}\,(-\beta p^2)^{\frac{|m|}{2}}(1+\beta p^2)^{-\sqrt{\lambda}}\\ &\quad\times{}_2F_1(-n,n+|m|-2\sqrt{\lambda},|m|+1,-\beta p^2)e^{im\varphi},\end{aligned} \tag{30}$$

where

$$\lambda=\frac{1}{2\beta\hbar\Omega\bar{\Omega}}\left(\frac{E_i^2}{2\mu\omega_0^2}+\mathscr{E}_{m,n}^{\beta}\right)+\frac{1}{4\beta^2\bar{\Omega}^2}. \tag{31}$$

From (29) it follows that the energy is real provided

$$\left[1+\beta^2\bar{\Omega}^2\left(1+m^2+\frac{\Omega_c m}{\beta\Omega\bar{\Omega}}\right)\right]>0. \tag{32}$$

In the limit, $\beta\longrightarrow 0$, we obtain the Landau problem in non-commutative phase space. We note that, for a given value n, m can take the values $n, n-2, n-4, \cdots, -n$, i.e. $m=n-2k$, with $0\le k\le n$. Thus, the energy eigenvalues and the corresponding wave functions are given by

$$\mathscr{E}_{m,n}^{\beta=0}=\frac{1}{2}\hbar\Omega_c m+2\hbar\Omega\left(\frac{|m|}{2}+n+\frac{1}{2}\right)-\frac{E_i^2}{2\mu\omega_0^2}, \tag{33}$$

$$\psi(p) = p^{-|m|}e^{-\frac{1}{2}\hbar M\Omega p^2}\,{}_1F_1(-n,|m|+1,\hbar M\Omega p^2)e^{im\varphi}. \tag{34}$$

From (32), it also follow that if $0<\Lambda<\sqrt{\frac{m^2}{4(1+m^2)}}$ there is a critical value τ_c such that for $\tau>\tau_c$ the energy is real. This value is given by

$$\tau_c = \frac{-m+\sqrt{m^2-4\Lambda^2(1+m^2)}}{2\Lambda(1+m^2)}, \tag{35}$$

where

$$\Lambda = \frac{\Omega}{\Omega_c},\ \ \tau = \beta\bar{\Omega},\ \ m=-1,-2,-3,\cdots. \tag{36}$$

The case $\Lambda=\sqrt{\frac{m^2}{4(1+m^2)}}$ corresponds to

$$\tau_c = \frac{-m}{2\Lambda(1+m^2)},\ \ m=-1,-2,-3,\cdots. \tag{37}$$

Concerning the case $\Lambda>\sqrt{\frac{m^2}{4(1+m^2)}}$, there is no relation between the parameters m, τ and Λ. The angular momentum quantum number becomes $m=0,\pm1,\pm2,\cdots$.

For $\tau=0$, when $\alpha\longrightarrow1$, we have $\bar{\theta}\longrightarrow0$. If we further set $\theta\longrightarrow0$, then $M\longrightarrow\mu$ as well as $\Omega_c\longrightarrow\omega_c$ and $\Omega\longrightarrow\omega$. The results will smoothly transform to the case of general quantum mechanics. In this case, by turning off the electric field and only then set $\omega_0=0$, the energy of the system is reduced to that of a particle subjected to a harmononic potential whose angular frequency is related to the cyclotron frequency of the charged particle moving in a magnetic field. But in addition, a term proportional to the orbital angular momentum of the particle is added to this energy, with a normalization such that infinite degeneration appears in the spectrum for one of the helicity modes. If $\omega_0\neq0$, there is a lifting of this degeneration. But this degeneration has not been completely lifted, there are fortuitous degenerations that appear in the spectrum in the specific value of the ratio $\Lambda=\frac{\Omega}{\Omega_c}$ (see Fig. 1).

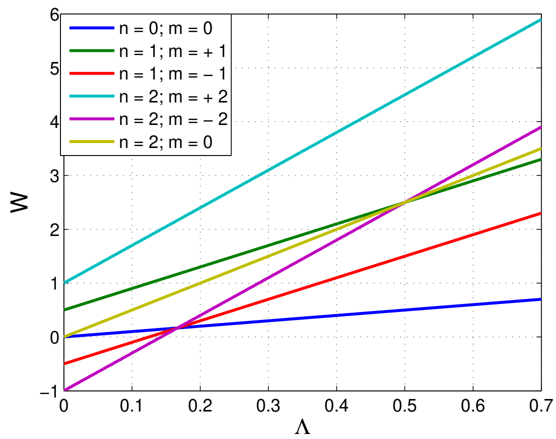

Fig. 1. The first three energy levels for $\beta=0$ as a function of the ratio $\Lambda=\frac{\Omega}{\Omega_c}$. The lines intersect for $\Lambda=\frac{1}{6}$ and $\Lambda=\frac{1}{2}$, and represent fortuitous degeneracies.

It is the presence of a minimal length that totally removes the degeneration of the energies of the system. This minimal length fixes a value at the ratio $\Lambda = \sqrt{\frac{m^2}{4(1+m^2)}}$ and reduces the orbital quantum number to $m < 0$ (see Fig. 2).

Now, we plot the quantity $W = \frac{1}{\hbar\Omega_c}\left(\mathscr{E}^{\beta}_{m,n} + \frac{E_i^2}{2\mu\omega_0^2}\right)$ for various values of the ratio $\Lambda = \frac{\Omega}{\Omega_c}$ in Fig. 1 and for various values of the parameter $\tau = \beta\bar{\Omega}$ in Fig. 2.

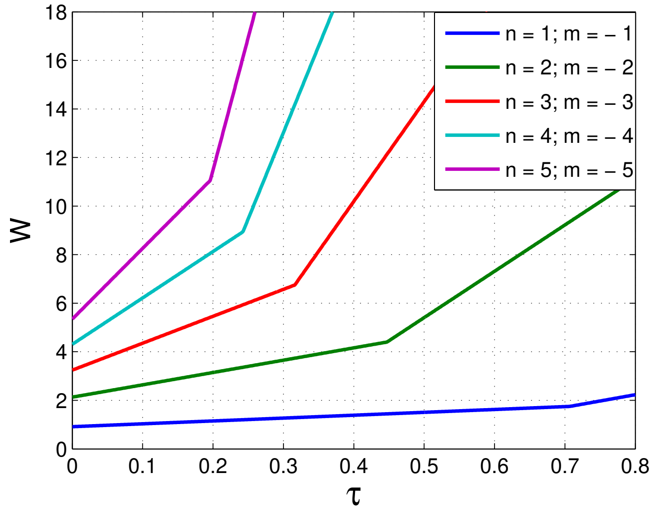

Fig. 2. W as a function of $\tau = \beta\bar{\Omega}$ for $\Lambda = \frac{\Omega}{\Omega_c} = \sqrt{\frac{m^2}{4(1+m^2)}}$. The energy levels are not degenerated in the presence of a minimal length. The change in direction of the energy levels corresponds to $\tau = \frac{1}{\sqrt{1+m^2}}$.

Results and Discussions

We have been able to solve analytically for the first time in the presence of a minimal length a system consisting of a charged particle moving in a plane and subjected to an electromagnetic field such that the magnetic field is perpendicular in the plane and the electric field lying inside that plane. Fock algebras have been well established for the non-commutative phase space in the presence of the minimal length. The momentum space is more practical to solve any problem of eigenvalues. We made use of the Nikiforov-Uvarov method to solve the eigenvalues equations. Wave functions are expressed in terms of hypergeometric functions. The energy levels have been obtained. It has been found that its energy levels are expressed in terms of two quantum numbers namely the principal quantum number $n = 0, 1, 2, \cdots$, and the angular momentum quantum number m. It has been found that the presence of minimal length reduces the latter to the negative non-zero integer values i.e. $m = -1, -2, -3, \cdots$. This completely removes the degeneration of the system. In the absence of the minimal length, the quantum number associated with the angular momentum can take the values $m = 0, \pm 1, \pm 2, \cdots$. The parameters of non-commutativity do not completely remove degeneration. There is then the presence of fortuitous degeneration for some value of the ratio $\frac{\Omega}{\Omega_c}$.

We can see that when $\alpha = \bar{\alpha} = 1$, corresponding to $\bar{\theta} = 0$, then the energies levels (29) correspond to the case where the space is non-commutative while the momenta are commuting. If we set further more $\theta = 0$ then the result corresponds to the usual two dimensions Landau probem in commutative space in the presence of a minimal length i.e.

$$\begin{aligned} \mathscr{E}^{\beta}_{m,n} &= \frac{1}{2}\hbar\omega_c m + 2\beta\hbar\omega\bar{\omega}\left[\frac{1}{2} + \frac{m^2}{4} + \left(\frac{|m|}{2} + n\right)\left(\frac{|m|}{2} + n + 1\right)\right] \\ &+ 2\hbar\omega\left(\frac{|m|}{2} + n + \frac{1}{2}\right)\sqrt{1 + \beta^2\bar{\omega}^2\left(1 + m^2 + \frac{\omega_c m}{\beta\omega\bar{\omega}}\right)} - \frac{E_i^2}{2\mu\omega_0^2}, \end{aligned} \tag{38}$$

where

$$\omega_c = \frac{B}{\mu},\ \ \omega = \sqrt{\omega_0 + \frac{1}{4}\omega_c},\ \ \bar{\omega} = \hbar\mu\omega. \tag{39}$$

Conclusion

Exact solution of the Landau problem in the non-commutative phase space in the presence of a minimal length has been obtained. The energy levels of the system are given. The wave functions are obtained in terms of hypergeometric functions. We have also analysed the degeneracy of the levels. For $\beta = 0$, the degeneracy is possible if $\Lambda = \frac{\sigma}{\delta}$ where σ and δ are integers. The case $\Lambda = 0.5$ can correspond to $\omega_0 = 0$ in the commutative space. Thus by controlling the frequency ω_0 or the mass μ of the system, we may observe the degeneracy. The have shown that the presence of a minimal length totally removes the degeneracy of the spectrum. We have found that the deformation parameter is well defined only in the case where the orbital quantum number can take negative non-zero integer values. This allowed us to define a relation between the ratio $\frac{\Omega}{\Omega_c}$ and the orbital quantum number. The formulation of minimal length in non-commutative phase space then corresponds to the study of a system depending on the position and momentum operators which verify the generalized Heisenberg uncertainty relation and satisfy a non-canonical commutation algebra whose phenomenological consequences are widely used in the context of quantum theory.

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УДК 537.8 Научная статья

Некоммутативная задача Ландау о фазовом пространстве при наличии минимальной длины

***Ф. А. Доса*[1], *Дж. Т. Куманьон*[2], *Ж. В. Унгву*[2], *Г. И. Ю. Авосву*[2]**

[1] Факультет наук и технологий (FAST), Национальный университет наук, технологий, инженерии и математики (UNSTIM), Абомей, BP: 2282 Гохо Абомей, Республика Бенин
[2] Лаборатория исследований в области теоретической физики (URPT), Институт математики и физических наук (IMSP), 01 BP 613 Порто-Ново, Республика Бенин
E-mail: finedofas@yahoo.fr, jeankoumagnon81@gmail.com, jeanvignonhoung@yahoo.fr, gabavossevou@gmail.com

Изучается деформированная задача Ландау в электромагнитном поле, в которой алгебра Гейзенберга подробно строится в некоммутативном фазовом пространстве при наличии минимальной длины. Мы показываем, что при наличии минимальной длины импульсное пространство более практично для решения любой проблемы собственных значений. С помощью метода Никифорова-Уварова получаются собственные значения энергии, а соответствующие волновые функции выражаются через гипергеометрические функции. Случайное вырождение, наблюдаемое в спектре, показывает, что формулировка минимальной длины дополняет формулировку некоммутативного фазового пространства.

Ключевые слова: задача Ландау, некоммутативное фазовое пространство, минимальная длина, метод Никифорова-Уварова, гипергеометрические функции

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Конкурирующие интересы. Авторы заявляют, что конфликтов интересов в отношении авторства и публикации нет.

Авторский вклад и ответсвенность. Все авторы участвовали в написании статьи и полностью несут ответственность за предоставление окончательной версии статьи в печать. Окончательная версия рукописи была одобрена всеми авторами.

Финансирование. Исследование выполнялось без финансирования