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ON SOME NEW ESTIMATES RELATED WITH DISTANCE FUNCTION FOR DIFFERENTIAL FORMS

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In this short note we consider a new extremal problem for some general spaces of differential forms and provide some estimates for related distance function.

Key words: extremal problems, differential forms, Hodge norms

УДК 517.53+517.55

О НЕКОТОРЫХ НОВЫХ ОЦЕНКАХ СВЯЗАННЫХ С ДИСТАНЦИЯМИ В ПРОСТРАНСТВАХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ

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В этой короткой заметке мы рассматриваем новую экстремальную задачу в некоторых пространствах дифференциальных форм и приводим некоторые новые оценки, связанные с функцией дистанции в этих пространствах. Эти результаты обобщают ранее полученные нами оценки для функциональных классов.

Ключевые слова: экстремальные проблемы, дистанции, дифференциальные формы, норма Ходжа.

We consider in this short note some new extremal problems in various spaces of differential forms related with distance function. Using known integral representations for differential forms from such type spaces some general new (not sharp) estimates will be provided.

We consider in this section only bounded D domains in C^n with $\partial D \in C^\infty$, so that $C^n \setminus \overline{D}$ is connected. We put as usual $D = \{z \in C^n : \rho(z) < \infty\}$, where ρ is a real valued function of $C^\infty(C^n)$ class and $d\rho \neq 0$ on (∂D), and we choose ρ, so that $|d\rho| = 1$ on (∂D).

In our previous papers we solved many extremal problems related with distances in various spaces of functions on various type of domains. It is very natural to try to solve similar type problems in general spaces consisting of differential forms. Note various similar type problems related with approximation of various forms were provided by various authors (see [1,2] and various references there).

We denote positive constants in this paper by $c_1, c_2, c_3, \ldots$. To formulate our results we need various standard definitions from [1-3].

The root of our approach is to try to replace Bergman kernel and Bergman representation formula which was heavily used in our previous papers on distances in various functional classes by similar reproducing formula, but for differential forms.

For that reason we apply general so-called Bochner-Martinelly-Koppelman formulas to be more precise their versions in some functional classes consisting of differential forms taken from [1,2]. Our paper is based on some results on differential forms taken from [1,2].

We first remind the reader some basic facts on mentioned reproducing formula. Then we define spaces,and formulate our problem and then give some new results on extremal problems. Approaches we used here are very similar to those we used previously in various functional classes via Bergman reproducing formula (see [4,5,6-7] and references there also).

We provide first Koppelman formula.

Let $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$; be growing multiindexes

$$1 \le i_1 < \cdots < i_p \le n; 1 \le j_1 < \cdots < j_q \le n, 0 \le p \le n, 0 \le q \le n.$$

For $q \le n-1$ we consider Koppelman kernels

$$U_{p,q}(\xi, z) = (-1)^{p(n-q-1)} \frac{(n-1)!}{(2\pi i)^n} \times$$

$$\times \sum_{I,J}{}' \sum_{k \notin J} \sigma(J,k)\sigma(I) \frac{\overline{\xi_k} - \overline{z_k}}{|\xi - z|^{2n}} d\overline{\xi}[J,k] \wedge d\xi[I] d\overline{z_j} \wedge dz_I,$$

where $d\overline{z_j} = d\overline{z_1} \wedge \cdots \wedge d\overline{z_{j_q}}, dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, where to get $d\xi[I]$ we have to remove $d\xi_{i_1} \ldots d\xi_{i_p}$ and

$$\sigma(J,k)dz = dz_k \wedge dz_j \wedge dz[J,k]$$

$$\sigma(I)dz = dz_I \wedge dz_{[}I].$$

The $U_{p,q}(\xi, z)$ kernel is a double differential form of $(n-p, n-q-1)$ type by ξ and of (p,q) type by z variable.

We consider (p,q) type differential form $\gamma = \sum\limits_{I,J}{}' \gamma_{I,J} dz_I \wedge d\overline{z_J}$ and we put as usual

$$\overline{\partial}\gamma = \sum_{k=1}^{n} \sum_{I,J}{}' \Big(\frac{\partial \gamma_{I,J}}{\partial \overline{z_k}} \Big) d\overline{z_k} \wedge dz_I \wedge d\overline{z_J}$$

Theorem $\widetilde{A}$. *(Koppelman) (see [1,2]). Let D be a bounded region in $\mathbb{C}^n$, with a boundary ∂D partially smooth. Let γ be a differential form of (p,q) type with coefficients of $C^1(\overline{D})$ class, then we have*

$$\int\limits_{\partial D} \gamma(\xi) \wedge U_{p,q}(\xi,z) - \int\limits_{D} \overline{\partial}\gamma(\xi) \wedge U_{p,q}(\xi,z) -$$

$$-\overline{\partial} \int\limits_{D} \gamma(\xi) \wedge U_{p,q-1}(\xi,z) = \begin{cases} \gamma(z), z \in D \\ 0, z \notin \overline{D} \end{cases}.$$

If γ is a differential form of (p,q) type

$$\gamma = \sum_{I,J}{}' (\gamma_{I,J}(z)) dz_I \wedge d\overline{z_J},$$

$$I = (i_1, \ldots, i_p); J = (j_1, \ldots, j_q), 0 < p,q \leq n$$

where summation moves via $1 \leq i_1 < \cdots < i_p \leq n, 1 \leq j_1 < \cdots < j_q \leq n.$

Then the Hodge operator on forms $*$ is defined as follows

$$(*\gamma) = \sum_{I,J} (\gamma_{I,J}(z)) * (dz_I \wedge d\overline{z_J}) * (dz_I \wedge d\overline{z_J}) =$$

$$= (2^{p+q-n})(-1)^{np}(i^n)(\sigma(I))(\sigma(I)) dz([J]) \wedge d\overline{z}[I]$$

where $d(z_I) \wedge dz[I] = (\sigma[J])dz$, so $(*\gamma)$ is a form of $(n-q, n-p)$ type. For properties of $*$ operator we refer the reader to [1,2].

If γ, φ are two forms of (p,q) type with coefficients of $L^2(D)$ class then scalar product of Hodge (γ,φ) can be defined as $(\gamma,\varphi) = \Big(\int\limits_D \gamma \wedge *\overline{\varphi} \Big)$, and $||\gamma|| = \sqrt{(\gamma,\gamma)}$ (norm of a differential form).

Note also that

$$\overline{\partial}\gamma = \sum_{k=1}^{n} \sum_{I,J} \frac{\partial \gamma_{I,J}}{\partial \overline{z_k}} d\overline{z_k} \wedge dz_I \wedge d\overline{z_J}$$

$$\partial\gamma = \sum_{k=1}^{n} \sum_{I,J}{}' \frac{\partial \gamma_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z_J}$$

Let $W^s(D) = W_2^s(D), s = 0,1,2,3,\ldots$ be a Sobolev space with f functions whose derivatives up to s order are in $L^2(D)$ with the usual scalar product. We define $W_q^s(D)$ as a spaces of forms of $(0,q)$ type with coefficients from $W^s(D), W_0^s = W^s(D)$.

We consider here bounded D domains in C^n, so that $\partial D \in C^\infty$, and so that $C^n \setminus \overline{D}$ is connected.

In this paper we put always q is less or equal than 2. And s is a natural always. This with natural inclusions of W_q^s spaces allows us to use representation formulas in our theorems. We need some more definitions and assertions now to formulate our results below.

Let X be a closed subspace of $W^s_q, X \subset (W^s_q)$. The natural question is to estimate in Hodge norms

$$(dist)(\gamma, X) = \inf_{g\in X}(||\gamma - g||), \gamma \in W^s_q,$$

.

Note in a recent series of papers of the first authors such type problems were solved in various functional spaces in various domains (see [4,5-7] and various references there also).

We first provide a short scheme for function spaces in the unit disk D. Let $H(D)$ is a space of all analytic functions in D.

Let $\widetilde{B}^{-t} = \{f \in H(D) : (\sup_{z\in D})|f(z)|(1-|z|)^{-t} < \infty\}, t < 0$. Then we have for $f \in \widetilde{B}^{-t}$

$$(dist_{\widetilde{B}^{-t}})(f, B^q_s) \le c\inf\{\varepsilon > 0 : \int_D \Big(\int_{\Lambda_{\varepsilon,-t}(f)} \frac{(1-|w|)^{\beta+t}}{|1-\bar{z}w|^{2+\beta}} dA(w)\Big)^q \times$$

$$\times (1-|z|)^{-sq-1} dA(z) < \infty\},$$

where dA is a Lebegues measure in D, $0 < q < \infty; s < 0, \beta > \beta_0$, β_0 is large enough, and

$$(B^q_s) = \{f \in H(D) : \int_D |f(z)|^q (1-|z|)^{-sq-1} dA(z) < \infty\},$$

$$\Lambda_{\varepsilon,-t} = \{z \in D : |f(z)|(1-|z|)^{-t} \ge \varepsilon\}, t < 0, \varepsilon > 0.$$

This is a known result (see [4,6,7]). Here is the short proof of this result which we wish to extend to much larger spaces of differential forms in this note, using Koppelman's formulas repeating same type arguments.

For $\beta > \beta_0$ by Bergman representation formula we have.

$$f(z) = c(\beta)\Big(\int_{D\setminus\Lambda_{\varepsilon,-t}} \frac{f(w)(1-|w|)^\beta}{(1-\overline{w}z)^{\beta+2}} dA(w) + \int_{\Lambda_{\varepsilon,-t}} \frac{f(w)(1-|w|)^\beta}{(1-\overline{w}z)^{\beta+2}} dA(w)\Big) =$$

$$= f_1(z) + f_2(z), z \in D$$

For $t < 0$

$$|f_1(z)| \le c \int_{D\setminus\Lambda_{\varepsilon,-t}} \frac{|f(w)|(1-|w|)^\beta}{|1-wz|^{\beta+2}} dA(w) \le c\varepsilon \frac{1}{(1-|z|)^{-t}}; z \in D.$$

So we have $(\sup_{z\in D})|f_1(z)|(1-|z|)^{-t} \le c\varepsilon$. For $s < 0, t < 0$ we also have $||f_2||_{B^q_s} \le c$, and hence finally we have

$$dist_{B^{-t}}(f, B^q_s) \le c||f - f_2||_{\widetilde{B}^{-t}} = c||f_1||_{\widetilde{B}^{-t}} \le c\varepsilon$$

We turn here to the same problem, but in more general spaces of differential forms. The main idea here in this paper is to use Koppelmans formulas (reproducing formulas) instead of Bergman reproducing formulas as we did in our previous papers on this topic previously.

We refer for all these to [1,2]. We consider only kernels for forms of type $(0,q), q \in [0, n-1]$ in reproducing formula of Bocher-Martinell-Koppelman that is

$u_{0,q}(\xi,z)=\sum_J{}' \sum_{K\notin J} \sigma(J,k)\frac{\partial g}{\partial \xi_k} d\overline{\xi}[J\bigcup k]\wedge d\xi d\overline{z_J}$, where $J\bigcup k$ means $J(J_1,\ldots,J_q)$ with k within and $\sigma(J,k)$ we define as $d\xi_k \wedge d\xi_J \wedge d\xi[J\cup k]=\sigma(J,k)d\xi$ where $g(\xi,z)$ is a standard fundamental solution of Laplace equation (see [1,2]).

We consider now the following bounded operators acting on spaces of differential forms (see [1,2])

$$M : W_q^s(D) \to W_q^s(D),$$

where

$$M\gamma(z) = \int_{\partial D} \gamma(\xi)\wedge U_{0,q}(\xi,z);$$

$$P : W_q^s(D) \to W_{q-1}^{s+1}(D),$$

where $P\gamma(z) = -\int_{\partial D} \gamma(\xi)\wedge U_{0,q-1}(\xi,z)$; (see [1,2]). Here is version of Bochner-Koppelman which formula we need for W_q^s spaces.

Theorem *A. If $\gamma\in W_q^s(D), s\geq 1$, then $\gamma = M\gamma+\overline{\partial}P\gamma+P\overline{\partial}\gamma$ in D.*

This formula is exactly the one we need for our purposes as substitution of Bergman integral reproductiong formula discussed above, we will need also it is versions for various subspaces of W_q^s (see [1,2]).

We define $V_q^s(D)$ spaces and pose a problem. Let $V_q^s(D)=\{\gamma\in W_q^s(D): M\gamma=0, MP\gamma=0,\ldots,MP^q\gamma=0\} inD$.

Then V_q^s is closed subspace of $W_q^s(D)$. (see [1,2])

Our idea is to generalize a distance problem for function spaces to more general spaces of differential forms and to find concrete estimates for distances $dist_X(\gamma,Y)$, in Hodge norms in spaces of forms, where $X\subset Y$,and X,Y are various spaces of differential forms, and $\gamma\in X$.

1) We pose a new problem to find estimates for $dist(\gamma,V_q^s), \gamma\in W_q^s(D), V_q^s\subset W_q^s$ in Hodge norm using theorem A above.

2) Let next $\varphi\in W_0^s(D)$ then $\varphi = M\varphi+(P\overline{\partial}\varphi)$ ([1,2]) by theorem A and we can consider a problem of estimates of $(dist)(\varphi,G)$, for closed subspaces G of $W_0^s, G\subset W_0^s$.

3) For all γ-functions (see [1,2]) $\gamma\in V_0^s : \varphi=(P\overline{\partial}\varphi)(*)$. We consider problem of estimates of $(dist)(\gamma,X), X\subset V_0^s, \gamma\in(V_0^s), X\subset V_0^s$ using equality $(*)$.

4) Also we can look at closed subspaces X, $X\subset V_q^s$, since theorem A with $M\gamma=0$ for V_q^s is valid also (see [1,2]).

5) Let X be a closed subspace of $Q_0^s, X\subset Q_0^s(D)=W_0^s\setminus V_0^s(D)$ then we look at $(dist)(f,X), f\in Q_0^s(D)$, note $f=Mf$ for all $f\in Q_0^s(D)$ (see [1,2]).

All these questions are interesting enough and precise estimates are from our point of view an important issue, which may have various applications also.

We give some partial answers following our previous papers on extremal problems in various functional spaces in various domains.

We note that our results probably can be sharpened. The answer for us is unknown at this moment.

We fix indexes I_0,J_0. Then we put

$$X_{\gamma,\varepsilon}=X_{\gamma,\varepsilon}^{I_0,J_0}=\{\xi\in\partial D: |\gamma_{I_0,J_0}(\xi)||\phi(\xi)|\geq\varepsilon\}, Y_{\gamma,\varepsilon}^{I_0,J_0}=\{z\in D: |\gamma_{I_0,J_0}(z)||\phi(z)|\geq\varepsilon\},$$

where we choose $\phi\in C^\infty(\overline{D})$, we omit I_0,J_0 below.

We have the following results following argument of our previous papers on distances.

We have the following theorem for the fifth problem.

Theorem 1. *Let $\gamma \in (Q_0^s)(D)$, let X be a closed subspace of $(Q_0^s)(D)$. Then*

$$(dist_{Q_0^s})(\gamma, X) \leq c_1 \inf\{\varepsilon > 0 : \Big|\Big| \int_{X_{\varepsilon,\gamma}} \gamma(\xi) \wedge U_{0,q}(\xi, z) \Big|\Big|_X < \infty\};$$

if

$$\Big|\Big| \int_{D \setminus X_{\varepsilon,\gamma}} \gamma(\xi) \wedge U_{0,q}(\xi, z) \Big|\Big|_{Q_0^s} < \varepsilon$$

For the second problem we have finally the following.

Theorem 2. *Let $\gamma \in (W_0^s)$. Let X be a closed subspace of $(W_0^s)(D)$. Then*

$$(dist)_{W_0^s}(\gamma, X) \leq c_3\{\varepsilon > 0 : \Big|\Big| \int_{X_{\varepsilon,\gamma}} (\gamma(\xi)) \wedge (U_{0,q}(\xi, z)) - \int_{Y_{\gamma,\varepsilon}} \overline{\partial}\gamma(\xi) \wedge U_{0,q+1}(\xi, z) \Big|\Big| < \infty\}$$

if

$$\Big|\Big| \int_{\partial D \setminus X_{\varepsilon,\gamma}} \gamma(\xi) \wedge (U_{0,q}(\xi, z)) - \int_{\partial \setminus Y_{\gamma,\varepsilon}} \overline{\partial}\gamma(\xi) \wedge U_{0,q+1}(\xi) \Big|\Big|_{W_0^s} < \varepsilon$$

for some positive constants c_3.

We leave the formulation of completely similar theorems for the first and fourth problems to readers though it is more complicated technically.

Similar results are valid for other embeddings and problems similar to those we put as for as appropriate integral representation of a differential form exists. We add some lines of short proofs of these assertions below.

Indeed the simple argument for the proof of the first theorem is the following.

We have $f = f_1 + f_2$, where

$$f_1(z) = \int_{X_{\varepsilon,\gamma}} \gamma(\xi) \wedge U_{0,q}(\xi, z), z \in D$$

$$f_2(z) = \int_{\partial D \setminus X_{\varepsilon,\gamma}} (\gamma(\xi)) \wedge U_{0,q}(\xi, z), z \in D$$

by theorem A.

It remains to note that under conditions of our theorem, we have obviously the following

$$(dist)_{Q_0^s}(f, X) \leq c||f - f_1||_{Q_0^s} = c||f_2||_{Q_0^s} < c\varepsilon$$

Other our theorems have similar proofs. It will be nice to show if these estimates are sharp or not.

Similar results are valid for so -called $\varepsilon^q(\overline{D})$ type spaces of differential forms. To formulate related version of Koppelman-Martinelly-Bochner formula, we will need some definitions in bounded strongly convex domains in C^n with C^∞ boundary.

Here we consider domains which are given as follows $D = \{Z \in C^n : \rho(Z) < 0\}$ and we denote by $\varepsilon^q(\overline{D})$ the space of all differential forms of type $(0, q)$ in $\overline{D}$ with coefficients in C^∞ in $\overline{D}$.

By theorem from [1,2] each differential form $u \in \varepsilon^q(\overline{D}), q \geq 0$ can be represented as

$$u = F_q u + P_{q+1}(\overline{\partial} u) + \overline{\partial}(P_q u).$$

(see [1,2])

where $P_q = T_q + L_q$, for $1 \leq q \leq n, P_q = 0$, for $q = 0, q = n+1$

$$(F_q u)(z) = (-1)^q \frac{(n-1)!}{(2\pi i)^n} \int_{\partial D} (u(\xi)) \wedge \omega_q' \Big(\frac{P(\xi)}{\phi(z,\xi)} \Big) \wedge d\xi$$

$$(T_q u)(z) = (-1)^q \frac{(n-1)!}{(2\pi i)^n} \int_{D} (u(\xi)) \wedge \omega_{q-1}' \Big(\frac{\overline{\xi} - \overline{z}}{|\xi - z|^2} \Big) \wedge d\xi$$

$$(L_q u)(z) = (-1)^q \frac{(n-1)!}{(2\pi i)^n} \int_{\partial D \times [0,1]} (u(\xi)) \wedge \omega_{q-1}' \Big((-1-\lambda) \Big(\frac{\overline{\xi} - \overline{z}}{|\xi - z|^2} \Big) + \frac{\lambda P(\xi)}{\phi(z,\xi)} \Big) \wedge d\xi$$

where we put

$$\phi(\xi, z) = < P(\xi), \xi - z >= \sum_{j=1}^{n} \frac{\partial \rho}{\partial \xi_j} (\xi_j - z_j)$$

for $(z,\xi) \in C^n \times C^n$, and $\partial \rho \neq 0$ on ∂D, and $P(\xi) = \Big(\frac{\partial \rho}{\partial \xi_1}, \dots, \frac{\partial \rho}{\partial \xi_n} \Big)$, ρ real valued and is of C^∞ class and $\omega'(\eta) = \sum\limits_{j=1}^{n} (-1)^j \eta_j d\eta[j]$ is Lere form (see [1,2]).

Using this representation we can following arguments of proof of our previous theorems find upper estimates for $dist_{(\varepsilon^q)}(f,X); f \in \varepsilon^q$, for X closed subspaces of $\varepsilon^q(\overline{D})$. We leave this to readers.

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