Bulletin KRASEC. Phys. & Math. Sci. 2017. vol. 16, issue. 1. pp. 61-68. ISSN 2313-0156

DOI: 10.18454/2313-0156-2017-16-1-61-68

MSC 34A08

RICCATI EQUATION WITH VARIABLE HEREDITY

D. A. Tvyordyj

Vitus Bering Kamchatka State University, 683032, Petropavlovsk-Kamchatsky, Pogranichnaya st., 4, Russia
E-mail: diplomat95@mail.ru

The Riccati differential equation with fractional derivative of variable order is considered. Introduction of a derivative of fractional variable order into the initial equation determines the property of a medium, the memory effect or the heredity, which consists in the dependence of the dynamic system current state on its previous states.

Key words: Riccati equation, fractional derivative, heredity, numerical methods, differential equation.

Introduction

Differential equations of fractional orders are of great interest for the investigation as they are often applied in many areas of science such as: mathematics, physics and so on [1], [2]. Equations with fractional derivatives belong to the class of integro-differential equations and are called hereditary ones based on V. Volterra's terminology [3]. This notion implies the presence of memory effect or time nonlocality in the process under investigation. Time nonlocality, containing in the initial equation integral operator kernel, is called the memory function. If the memory function is a power one, we naturally turn to the equation with fractional derivatives which are studied within the framework of fractional calculus [4].

In the paper, the object of our investigation is the Riccati equation [5] taking into the account the heredity. The heredity in Riccati equation is characterized by a derivative of fractional variable order. We should not that in the paper [6], the authors studied the heredity Riccati equation, when the order of fractional derivative is a constant.

Problem statement and solution method

Consider the following hereditary equation [1]:

$$\int_0^t K(t-\tau)\dot{u}(\tau)d\tau + u^2(t) - 1 = 0, \tag{1}$$

where $K(t-\tau)$ is the memory function, $t \in [0,T], T > 0$ is the modeling time, $u(t)$ is the solution function. Equation (1) is the analogue of a classical Riccati equation [5] and it takes the memory effect into consideration.

Tverdiy Dmitry Alexandrovich – Master student of the second year of the direction of preparation «Applied Mathematics and Informatics Kamchatka State University named after Vitus Bering.

©Tvyordyj D. A., 2017.

If the memory function $K(t-\tau)$ is the Heaviside function, we can say that the process has the total memory. If it is a delta-function, the memory is absent. Thus, we shall consider the memory function in the form of a power function

$$K(t-\tau)=\frac{(t-\tau)^(-\alpha(t))}{\Gamma(1-\alpha(t))},0<\alpha(t)<1, \qquad (2)$$

where $\Gamma(1-\alpha(t))$ is the gamma-function (Euler function).

Processes with the memory function of the form (2) are called the processes with partial memory loss and they require special attention in investigation as long as many natural processes have power lows of distribution which in most cases will bring about the notion of fractality or a fractal [7].

Substituting the memory function (2) into the hereditary equation (1), we obtain the following integro-differential equation, called Riccati equation:

$$\frac{1}{\Gamma(1-\alpha(t))}\int_0^t\frac{\dot{u}(\tau)}{(t-\tau)^{\alpha(t)}}d\tau+u^2(t)-1=0. \qquad (3)$$

We introduce the following notation into equation (3):

$$\partial_{0t}^{\alpha(t)}u(\tau)=\frac{1}{\Gamma(1-\alpha(t))}\int_0^t\frac{\dot{u}(\tau)}{(t-\tau)^{\alpha(t)}}d\tau, \qquad (4)$$

which is a generalization of Caputo or Gerasimov-Caputo fractional operator.

We should note that there are other definitions for the derivative of fractional variable orders [8], but we decide upon the definition from (4). Now equation (3) can be written in a more compact form

$$\partial_{0t}^{\alpha(t)}u(\tau)+u^2(t)-1=0, \qquad (5)$$

for which the following initial condition is true:

$$u(0)=\rho, \qquad (6)$$

where ρ is a const.

From the abovesaid, the problem statement for hereditary Riccati equation has reduced to Cauchy problem (5) and (6) in this case.

We should note that for $\alpha(t)$ is a const, we come to the Cauchy problem which was considered in the paper [6]. And if $\alpha(t)=1$, the problem reduces to the classical Cauchy problem for Riccati equation (5).

As long as the Cauchy problem (5) and (6) in the general case does not have exact solution, we apply numerical methods for its solution. In order to do that, we divide the time segment $t\in[0,T]$ into N equal parts where $k=\frac{T}{N}$ is the sampling interval, and obtain that $t_i=ik, i=0,...,N-1$, and the solution function $u(t_i)=u_i$. We approximate the fractional derivative (4) according to the paper [9] as follows:

$$\partial_{0t}^{\alpha(t)}u(\tau)=\sigma_{\alpha_i,k}\sum_{j=1}^{i}\omega_{j,\alpha_i}(u_{i-j+1}-u_{i-j}), i=1,...,N-1, \qquad (7)$$

where $\sigma_{\alpha_i,k}=\frac{k^{-\alpha_i}}{\Gamma(2-\alpha_i)}$, $\omega_{j,\alpha_i}=j^{1-\alpha_i}-(j-1)^{1-\alpha_i}$. We can also show that approximation (7) has the first order.

The ingegro-differential Cauchy problem (5) and (6) can be written in a differential setting

$$\sigma_{\alpha_i,k}\sum_{j=1}^{i}\omega_{j,\alpha_i}(u_{i-j+1}-u_{i-j})=1-u_i^2, u_0=\rho. \qquad (8)$$

In the result we obtain a system of nonlinear algebraic equations the numerical solution of which, depending on the type of the function $\alpha(t)$, was implemented in C++ software. The graphs presented in the paper were constructed in Maple computer mathematics system.

Modeling results and discussion

We consider the following examples of numerical solutions of the Cauchy problem (5) and (6) depending on different presentations of the function $\alpha(t)$ and the control parameter values.

FOREXAMPLE 1. We consider a case which was studied in the paper [6], when $\alpha(t) = \text{const}$. We choose the following values of control parameters: $t \in [0,T], T = 3, N = 1000, k = 0.003, \rho = 0.2$. On a graph we construct several calculated curves which correspond to different values of $\alpha(t)$.

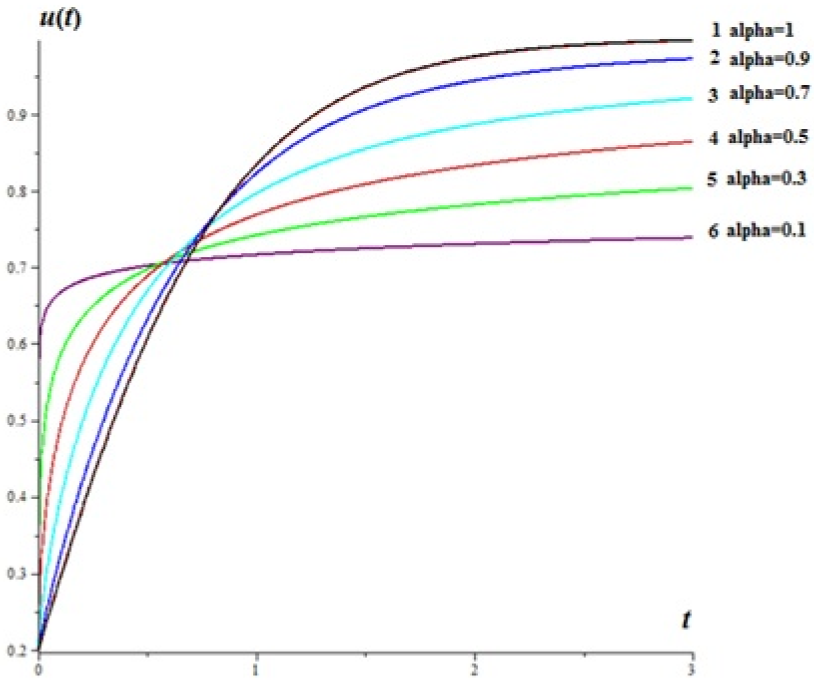

Fig. 1. A family of calculated curves corresponding to system (8) solution for different values of the parameter $\alpha(t)$.

Fig. 1 shows the family of the curves corresponding to the solution of Cauchy problem (5) and (6) depending of fractional parameter values: $\alpha(t) = 1$ (curve 1 is a classical solution of Riccati equation), $\alpha(t) = 0.9$(curve 2), $\alpha(t) = 0.7$ (curve 3), $\alpha(t) = 0.5$ (curve 4), $\alpha(t) = 0.3$ (curve 5), $\alpha(t) = 0.1$ (curve 6).

It was noted that the presence of a fractional parameter $\alpha(t)$, when its value is decreasing, causes rearrangement of numerical solution calculated curves for Cauchy problem (5) and (6). It is due to the fact that memory presence in the investigated process causes "heavy long tails"in distribution curves of the obtained solutions, for example, curve 5 from (**??**).

If a medium has the memory effects, sometimes such a medium is called fractal and the fractional parameter $\alpha(t)$ is associated with its characteristics, the medium fractal dimension. Thus, investigation of this parameter is important for different applications where medium or material properties are under study.

Now we consider the cases when $\alpha(t)$ is a function including a random one.

FOREXAMPLE 2. Let $\alpha(t) \in [0,1]$ based on the uniform law. The control parameter values are as follows: $t \in [0,T], T = 3, N = 1000, k = 0.01, \rho = 0.2$.

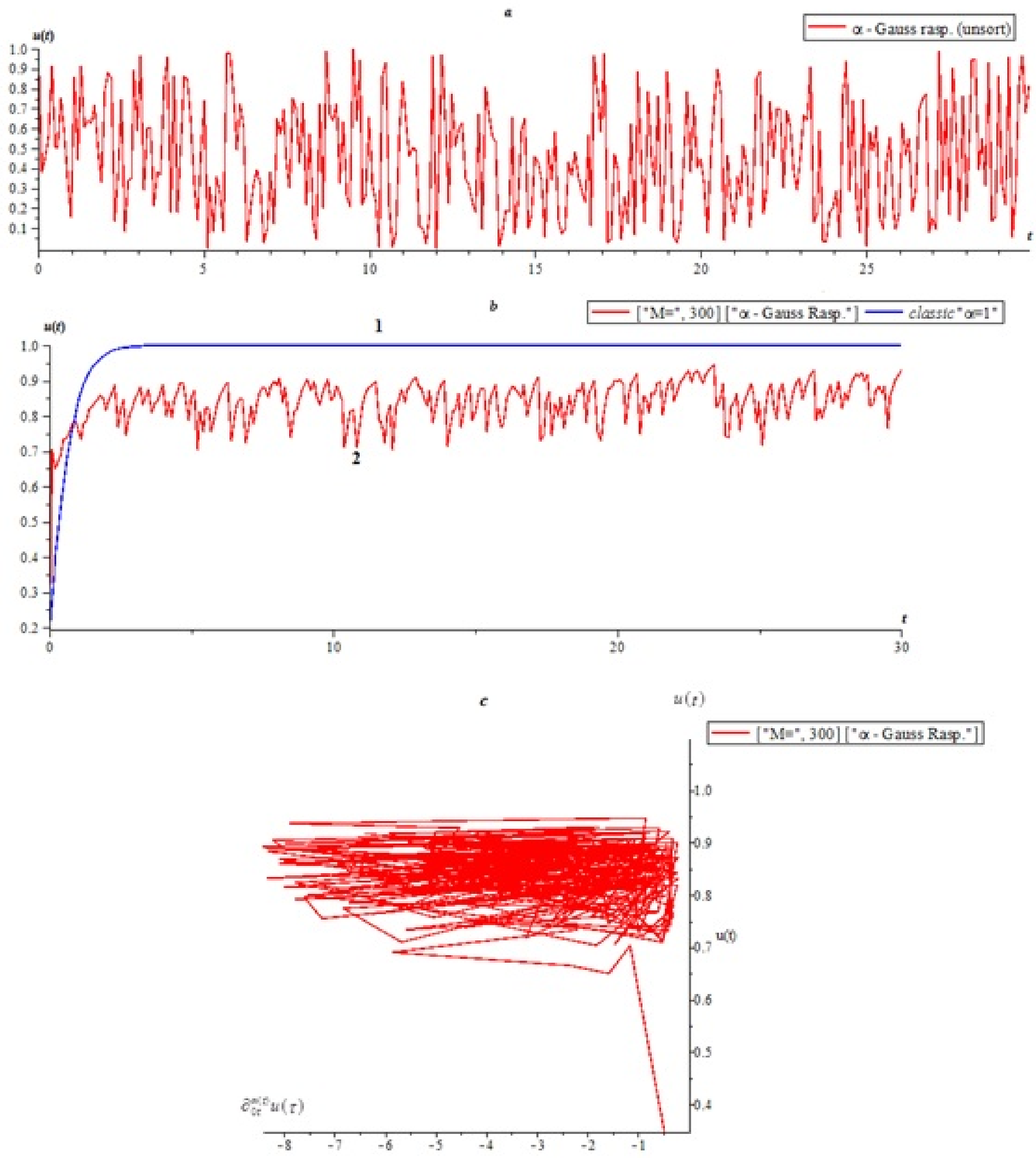

Fig. 2. Modeling results for example 2: a) behavior of $\alpha(t)$; b) calculated curves: 1 is a classical solution $\alpha(t) = 1$; numerical solution curve for system (8); c) phase trajectory.

Fig.2 shows the modeling results. Introduction of $\alpha(t)$ as a random function, the parameter change is presented (Fig. 2a), results in a solution random function (Fig. 2b) that is also reflected in the chaotic regime on the phase plane (Fig. 3c) plotted in the coordinates $(\partial_{0t}^{\alpha(t)} u(\tau), u(t))$.

FOREXAMPLE 3. Consider an example when $\alpha(t) = \dfrac{(1-\delta-\theta)\cos(\mu t)+(\theta-\delta+\phi)}{2}$, and the control parameter values are as follows: $\delta = 0, \theta = 0.05, \mu = 9, \phi = 1, t \in [0,T], T = 30, N = 1000, k = 0.03, \rho = 0$.

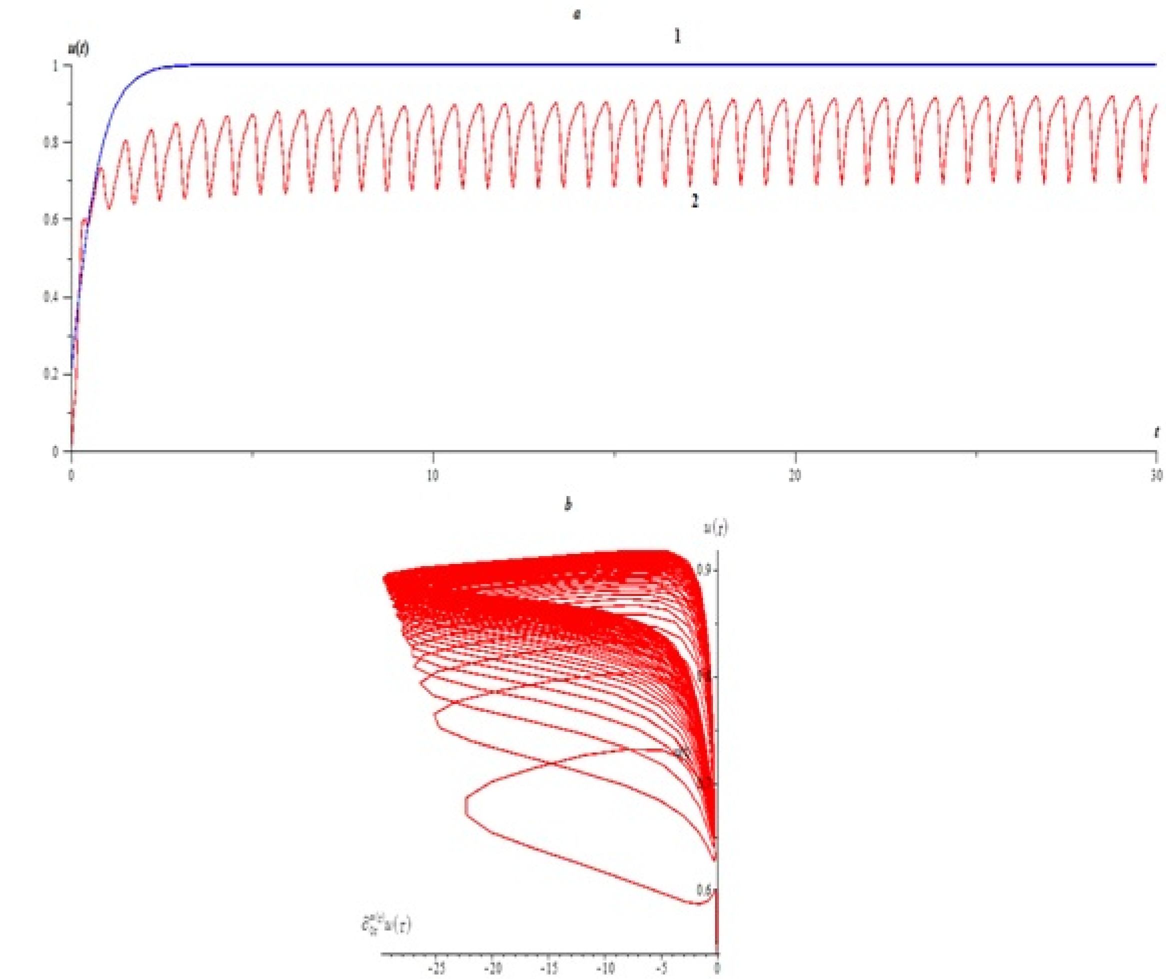

Fig. 3. Modeling results for example 3: a) calculated curves: 1 – classic solution of Riccati equation; 2 – solution of system (8); c) phase trajectory.

We can conclude from the modeling results presented in Fig. 3 that if we choose the parameter $\alpha(t)$ in the form of a trigonometric function, solution of the Cauchy problem (5) and (6) describes an oscillatory regime. The oscillatory regime illustrated in Fig. 3a (curve 2) is similar to one of the oscillatory regimes of Van der Pol self-excited oscillators that is of great application interest when modeling nonlinear oscillators. It is clear from Fig. 3a that at first oscillations occur with amplitude increase and then the amplitude reaches the steady state. This fact is clearly sean in Fig. 2b. Phase trajectory reaches the boundary cycle, some stable trajectory. This example shows that it is possible to model different oscillatory regimes by hereditary Riccati equation with fractional variable derivative although $0 < \alpha(t) < 1$.

Method error and designed accuracy

We consider the change of absolute error ε and the designed accuracy order $p = \dfrac{\ln(|\varepsilon|)}{\ln(\tau)}$ of the scheme (8), when the step k changes. To calculate the absolute error ε, we consider the difference between the exact and numerical solutions in example 1. In the examples 2 and 3, we apply the Runge rule [10]:

$$\varepsilon = \max\left(\frac{|u_{2N}[2i-1] - u_N[i]|}{2^{p_{aprior}-1}}\right), \tag{9}$$

where $i = 1...N$. We assume a priori accuracy p_{aprior} of the solution in this method to be equal to 1. That follows from the scheme approximation total ordering defined in net boundary nodes and equal to 1, although problem (5), (6) approximation of the difference scheme (8) gives the second order.

Table 10

Study of numerical scheme (8)

N	$k=T/M$	ε	p
Example 1. $\alpha = 0.9999$ Calculation time $T = 3$			
65	0.04615	0.0080057826	1.569552780
131	0.02290	0.0040108135	1.461309925
263	0.01141	0.0020088245	1.388207803
527	0.005693	0.0010069838	1.335141320
1055	0.002844	0.0005117944	1.292511738
2111	0.001421	0.0002509894	1.264446995
N	$k=T/M$	ε	p
Example 2			
65	0.3077	-	-
131	0.1527	0.177863110	0.9187311426
263	0.07605	0.1719669330	0.6832938221
527	0.03795	0.1879743500	0.5109173073
1055	0.01896	0.1659797310	0.4528711939
2111	0.00947	0.1538727525	0.4017074643
Example 3			
65	0.3077	-	-
131	0.1527	0.157214319	0.9843999572
263	0.07605	0.093047735	0.9216824185
527	0.03795	0.047894435	0.9288660948
1055	0.01896	0.024516910	0.9351487980
2111	0.00947	0.01227877	0.9443462189

It follows from Table 1 that the absolute error ε reduces when step k decreases. In the first and the third cases, the error reduces proportionally of step decrease, however, in the second case $\varepsilon \to 0$ slowly and nonuniformly. As it can be seen in Fig. 4, is likely to be associated with incidential distribution of $\alpha(t)$.

The designed accuracy order p tends to 1 in the first case. However, in the second case, the accuracy order decreases sharply for small N, and slows down for large values. In the third case, the accuracy order for small N also decreases but not so sharply. However, for $N > 263$ it

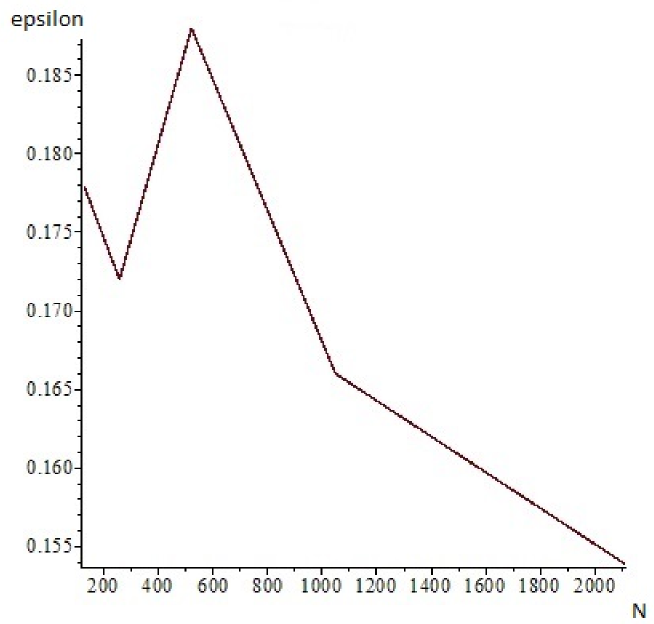

Fig. 4. Error behavior in example 2 depending on N

increases again (Fig. 5) and, probably, will also tend to 1. Such a behavior in examples 2 and 3 is likely to be explained by logarithm properties when calculating p.

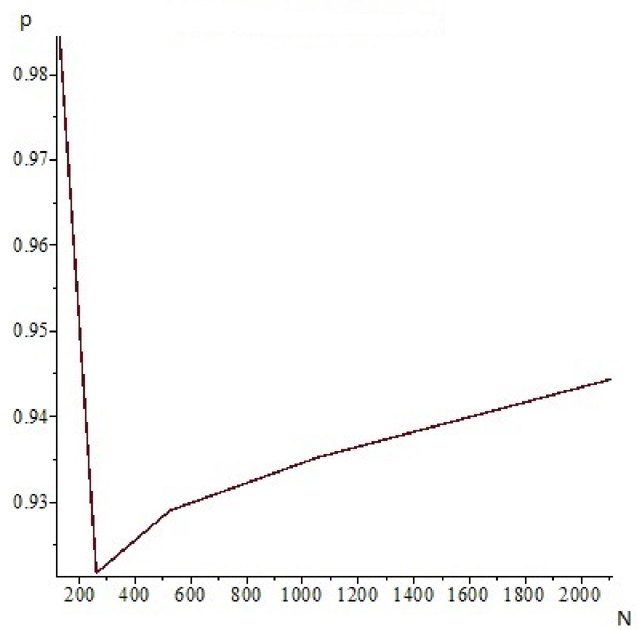

Fig. 5. Error behavior in example 2 depending on N

Conclusions

Summarizing the results of the modeling in the paper, we can make the following conclusions. Introduction of an additional fractional parameter $\alpha(t)$ into the Riccati equation results in the appearance of new distribution curves which characterize the solutions of Cauchy problems (5) and (6). Thus, we can model oscillatory regimes and construct models of different signals that deserves attention for applied problem solutions [12].

The investigation results of absolute error and the designed accuracy order for example 1 allow us to assume that the numerical scheme (8) is applicable to this problem.

The possible course of the study of Riccati heredity equation is associated with applied problems, for example, in economy [11], and in the solution of a reverse problem of $\alpha(t)$ parameter estimation by known experimental data.

I appreciate the help of my academic advisor, Parovik Roman Ivanovich (Cand. Sci. (Phys.-Math.)), for valuable advices on the study plan and recommendations on the paper layout.

References

1. Uchaykin V. V. Metod drobnykh proizvodnykh [Fractional derivative method]. Ul'yanovsk. Artishok. 2008. 512 p.

2. Petras I. Fractional-order nonlinear systems: modeling, analysis and simulation. Springer Science & Business Media. 2011. 218 p.

3. Volterra V. Sur les 'equations int´egro-diff´erentielles et leurs applications. Acta Mathematica. 1912. vol. 35. no. 1. pp. 295–356.

4. Nakhushev A. M. Drobnoe ischislenie i ego primenenie [Fractional calculus and its application]. Fizmatlit. 2003. 272 p.

5. Riccati J. Animadversiones in aequationes differentiales secundi gradus. Actorum Eruditorum Supplementa. 1724. vol. 8. pp. 66–73.

6. Sweilam N. H., Khader M. M., Mahdy A. M. S. Numerical studies for solving fractional Riccati differential equation. Applications and Applied Mathematics. 2012. vol. 7. no. 2. pp. 595–608.

7. Feder E. Fraktaly [Fractals]. Moscow. Mir. 1991. 254 p.

8. Parovik R. I. Matematicheskoe modelirovanie lineynykh ereditarnykh ostsillyatorov [Mathematical modeling of linear heredity oscillators]. Petropavlovsk-Kamchatskiy. KamGU imeni Vitusa Beringa. 2015. 178 p.

9. Parovik R. I. Finite-differential schemes for fractal oscillator with variable fractional orders. Bulletin KRASEC. Physical and Mathematical Sciences. 2015. vol. 11. no. 2. pp. 85–92.

10. Berezin I. S., Zhidkov N. P. Metody vychisleniy [Calculus methods]. Moscow. Fizmatlit. 1962

11. Makarov D. V., Parovik R. I. Modeling of the economic cycles using the theory of fractional calculus. Journal of Internet Banking and Commerce. 2016. vol. 21. S6.

12. Tverdyy D. A. Uravnenie Rikkati s proizvodnoy drobnogo peremennogo poryadka [Riccati equation with fractional variable derivative]. Mezhdunarodnyy studencheskiy nauchnyy vestnik 2017 – International Student Scientific Bulletin. 2017. no. 2. pp. 42-42. https://www.eduherald.ru/ru/article/view?id=16889(reference date: 22.04.2017).

For citation: Tvyordyj D. A. Riccati equation with variable heredity. *Bulletin KRASEC. Physical and Mathematical Sciences* 2017, vol. **16**, issue **1**, 61-68. DOI: 10.18454/2313-0156-2017-16-1-61-68

Original article submitted: 22.03.2017