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MATHEMATICS

MSC 34M10, 35M20

LINEAR INVERSE PROBLEM FOR A MIXED SECOND ORDER EQUATION OF THE SECOND KIND WITH NONLOCAL BOUNDARY CONDITIONS IN THREE-DIMENSIONAL SPACE

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The paper considers the problems of correctness of a linear inverse problem for a mixed second order equation of the second kind in three-dimensional space. The theorems on the existence and the uniqueness of the solution in a certain class are proved by «ε-regularization», Galerkin and successive approximation methods.

Keywords: linear inverse problem, solution correctness, Galerkin's method, «ε - regularization» method, method of successive approximations.

Introduction

In the course of study of nonlocal problems, a close relation of the problems with nonlocal boundary conditions with inverse problems was detected. By the present time the inverse problems for parabolic, elliptical and hyperbolic equations have been studied quite well [1,2,8,9,12]. The inverse problems for mixed equations are less investigated [6,10,11].

Within this paper, we are trying to fill partly this gap.

Problem statement

We consider a differential second order equation in the domain $Q = (0,1) \times (0,T) \times (0,\ell) = Q_1 \times (0,\ell)$.

$$Lu = K(x,t)\,u_{tt} - \Delta u + \alpha(x,t)\,u_t + c(x,t)\,u = \psi(x,t,y) \tag{1}$$

where $\Delta u = u_{xx} + u_{yy}$ is the Laplace operator in the domain. We assume that the equation (1) coefficients are quite smooth functions and let $K(x,0) \le 0 \le K(x,T)$. Equation (1) refers to mixed equations of the second kind as long as no restrictions are imposed on the function $K(x,t)$ sign with respect to t inside the domain Q [3].

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Task 1. *(Nonlocal boundary problem) Find a solution of equation (1) satisfying the conditions.*

$$\gamma u(x,0,y) = u(x,T,y) \tag{2}$$

$$D_x^p u|_{x=0} = D_x^p u|_{x=1} \tag{3}$$

$$D_y^p u\big|_{y=0} = D_y^p u\big|_{y=\ell}, p = 0,1, \tag{4}$$

where $\gamma - const \neq 0$ *such that* $\gamma \in (1,\infty)$. *We should note that in the papers [4,5] in case* $K(x,0) \leq 0 \leq K(x,T)$ *under certain conditions on the equation coefficients and the right part of equation (1), the solution correctness of problems (2)-(4) from S.L. Sobolevs space* $W_2^l(Q)$, *when* $2 \leq l$ *is a whole number, was proved.*

In this paper, under additional condition, we seek the solution of equation (1) in the definite classes. Assume that $\psi(x,t,y) = g(x,t,y) + h(x,t) \cdot f(x,t,y)$, where $g(x,t,y)$ and $f(x,t,y)$ are defined functions.

Task 2. *(Linear inverse problem) Find the functions* $(u(x,t,y), h(x,t))$ *satisfying equation (1) in the domain* Q *such that the function* $u(x,t,y)$ *satisfies the boundary conditions (2)-(4) and the additional condition*

$$u(x,t,\ell_0) = \phi(x,t), \tag{5}$$

where $0 < \ell_0 < \ell < +\infty$.

Theorem 1. *Assume that the conditions mentioned above for equation (1) coefficients are fulfilled. Moreover assume that* $2\alpha - |K_t| + \lambda K \geq \delta_1 > 0$; $\lambda c - c_t \geq \delta_2 > 0$; *where* $\lambda = \frac{2}{T}\ln\gamma$ *such that* $\gamma \in (1,\infty)$, $\alpha(x,0) = \alpha(x,T)$, $c(x,0) = c(x,T)$, *and let* $(1+D_y^3)g \in W_2^1(Q)$; $\gamma g(x,0,y) = g(x,T,y)$; $g(x,t,\ell_0) = g_0(x,t) \in W_2^1(Q_1)$ $(1+D_y^3)f \in W_2^2(Q)$; $\gamma f(x,0,y) = f(x,T,y)$; $f(x,t,\ell_0) = f_0(x,t) \in W_2^2(Q_1)$; $|f_0(x,t)| \geq \eta > 0$.

Assume that the defined function $\phi(x,t) \in W_2^2(Q_1)$ *is the solution of the following problem*

$$L_0\phi = K(x,t)\phi_{tt} - \phi_{xx} + \alpha(x,t)\phi_t + c(x,t)\phi = g_0(x,t)$$
$$\gamma \cdot \phi(x,0) = \phi(x,T); \quad D_x^p\phi\big|_{x=0} = D_x^p\phi\big|_{x=1}, p = 0,1.$$

the unique solvability and smoothness of which was studied in [4,5], and let there be a positive number ν *such that* $\delta_0 - 6\nu \geq \delta_* > 0$

$$2\rho \equiv M \cdot \sum_{s=0}^{\infty} (1+\mu_s^6)\, \| f_s \|^2_{W_2^1(Q_1)} < \delta_*, \delta_0 = \min\{\delta_1, \delta_2, \lambda\}; M - const\,(\delta_0; \eta; \nu^{-1}; ||g_0||; mes(Q_1))$$

Then the functions

$$u(x,t,y) = \sum_{s=0}^{\infty} u_s(x,t) Y_s(y), h(x,t) = \frac{1}{f_0}\sum_{s=0}^{\infty} \mu_s^2 u_s(x,t) Y_s(\ell_0)$$

are the solution of linear inverse problem (1)-(5) from the class

$$U = \{(u,h)|\ u \in W_2^2(Q); h \in W_2^2(Q_1); D_y^3\{u_{xx}, u_{tx}, u_{tt}\} \in L_2(Q);\ D_y^4 u \in L_2(Q)\}$$

where the functions $Y_k(y) = \{\frac{1}{\sqrt{\ell}}, \sqrt{\frac{2}{\ell}}\cos\mu_k y, \sqrt{\frac{2}{\ell}}\sin\mu_k y\}$, $\mu_s^2 = (\frac{2\pi s}{\ell})^2$, $k \in N_0 = N \cup \{0\}$, N *are the sets of natural numbers and are the solutions of spectral Sturman-Liouville problem with periodic conditions. It is known that the eigen function system* $\{Y_k(y)\}$ *is fundamental in space* $L_2(Q)$ *and forms an orthonormal basis in it [14], and the functions* $u_s(x,t)$; $s = 0, 1, 2, 3, \ldots$ *are the solutions in the domain* Q_1 *of corresponding loaded problems.*

$$Lu_s = L_0 u_s + \mu_s^2 u_s = g_s + \frac{f_s}{f_0} \cdot \sum_{m=0}^{\infty} \mu_m^2 u_m Y_m(\ell_0) \equiv F_s(u_s) \tag{6}$$

$$\gamma u_s(x,0) = u_s(x,T) \tag{7}$$

$$D_x^p u_s|_{x=0} = D_x^p u_s|_{x=1},\ p = 0,1 \tag{8}$$

where

$$f_s = \sqrt{\frac{2}{\ell}} \cdot \int\limits_0^\ell f(x,t,y)Y_s(y)dy; \quad g_s = \sqrt{\frac{2}{\ell}} \cdot \int\limits_0^\ell g(x,t,y)Y_s(y)dy.$$

A loaded equation is the equation with partial derivatives containing the values of various functionals from equation solution in their coefficients [7,8,9].

Proof. Prove theorem 1 step-by-step. First we show that the function $u(x,t,y)$ satisfies the additional condition (5), i.e. $u(x,t,\ell_0) = \phi(x,t)$. Assume the opposite. Let $u(x,t,\ell_0) = v(x,t) \neq \phi(x,t)$, then for the function $z(x,t) = v(x,t) - \phi(x,t)$ in the domain Q_1 from (6)-(8) we obtain

$$L_0 z = K(x,t)z_{tt} - z_{xx} + \alpha(x,t)z_t + c(x,t)z = 0 \tag{9}$$

$$\gamma \cdot z(x,0) = z(x,T); \quad D_x^p z|_{x=0} = D_x^p z|_{x=1}, p = 0,1 \tag{10}$$

It follows from the uniqueness of problem (9), (10) solution [4,5] that $z(x,t) = 0$, i.e. $v(x,t) = \phi(x,t)$. Further we shall need the following notations and additional lemmas to prove theorem 1. Let $u_{s,\varepsilon} \in W_2^2(Q_1)$, then determine the spaces $W_i(Q_1); i = 0,1,2$ with the corresponding norm

$$\langle u_{s,\varepsilon} \rangle_i^2 = \sum_{s=0}^{\infty} (1+\mu_s^6) \|u_{s,\varepsilon}\|^2_{W_2^i(Q_1)}; i = 0,1,2$$

for $i = 0$; $W_0(Q_1) = L_2(Q_1)$. It is obvious that the spaces $W_i(Q_1); i = 0,1,2$ with the defined norm are Banach ones [13]. It follows from the Sobolev inclusion theorem that

$$W_2(Q_1) \subset W_1(Q_1) \subset W_0(Q_1).$$

Theorem 2. *Assume that all the conditions of theorem 1 mentioned above are fulfilled, than there is a unique solution of problem (6)-(8) from the space $W_2(Q_1)$.*

Proof. First we prove the solvability of problem (6)-(8) by the methods of ε – regularization, successive approximations and priori estimates [2,3,4,5,13], in particular, we consider the equation family

$$L_\varepsilon u_{s,\varepsilon}^{(l)} = -\varepsilon \tfrac{\partial}{\partial t} \Delta u_{s,\varepsilon}^{(l)} + L_0 u_{s,\varepsilon}^{(l)} + \mu_s^2 u_{s,\varepsilon}^{(l)} = g_s + \tfrac{f_s}{f_0} \cdot \sum_{m=0}^{\infty} \mu_m^2 u_{m,\varepsilon}^{(l-1)} Y_m(\ell_0) \equiv F_s(u_{s,\varepsilon}^{(l-1)}) \tag{11}$$

$$\gamma D_t^q u_{s,\varepsilon}^{(l)}(x,0) = D_t^q u_{s,\varepsilon}^{(l)}(x,T);\ q = 0,1,2 \tag{12}$$

$$D_x^p u_{s,\varepsilon}^{(l)}\Big|_{x=0} = D_x^p u_{s,\varepsilon}^{(l)}\Big|_{x=1},\ p = 0,1 \tag{13}$$

where $\varepsilon > 0$, $\quad l = 0,1,2,......$; $\gamma - const \neq 0$ such that $\gamma \in (1,\infty)$

Lemma 1. *Assume that all the conditions of theorem 2 are fulfilled, than the following estimates are true for the solution of problem (11)-(13)*

I) $\frac{\varepsilon}{\delta_*}\left(\left\langle \frac{\partial^2}{\partial t^2} u_{s,\varepsilon}^{(l)} \right\rangle_0^2 + \left\langle \frac{\partial^2 u_{s,\varepsilon}^{(l)}}{\partial t \partial x} \right\rangle_1^2\right) + \left\langle u_{s,\varepsilon}^{(l)} \right\rangle_1^2 \leq const(\not{l}),$

II) $\frac{\varepsilon}{\delta_*} \left\langle \frac{\partial \Delta u_{s,\varepsilon}^{(l)}}{\partial t} \right\rangle_0^2 + \left\langle u_{s,\varepsilon}^{(l)} \right\rangle_2^2 \leq const(\not{l}).$

The symbol const ($\not{l}$) denotes a constant independent of l.

Proof. Applying the results of the paper [2,4,5,6], the methods of induction, priori estimates and the Sobolev inclusion theorems to the identity

$$2(L_\varepsilon u_{s,\varepsilon}^{(l)} - F_s(u_{s,\varepsilon}^{(l-1)}), \exp(-\lambda t)\frac{\partial}{\partial t} u_{s,\varepsilon}^{(l)})_0 = 0, -2(L_\varepsilon u_{s,\varepsilon}^{(l)} - F_s(u_{s,\varepsilon}^{(l-1)}), \exp(-\frac{\lambda t}{2})\Delta \ell u_{s,\varepsilon}^{(l)})_0 = 0,$$

where $(\cdot,\cdot)_0$ is a usual scalar product in $L_2(Q_1)$, $\Delta w = w_{tt} + w_{xx}$ is the Laplace operator with respect to the variables t and x

$$\Delta \ell w = \exp(-\frac{\lambda t}{2})\left[\frac{\partial \Delta w}{\partial t} - \lambda \ w_{tt} + \frac{\lambda^2}{4} w_t\right],$$

and after integration we obtain the first and the second estimates, respectively. Lemma 1 has been proved. □

Now we introduce a new function from $W_2(Q_1)$ using the formula $v_{s,\varepsilon}^{(l)} = u_{s,\varepsilon}^{(l)} - u_{s,\varepsilon}^{(l-1)}; \varepsilon > 0; s = 0,1,2,\ldots;\ l = 1,2,3,\ldots$. Then the following lemma is fair for it.

Lemma 2. *Assume that all the conditions of theorem 2 and lemma 1 are fulfilled. Then the following estimates are fair for the function* $\left\{v_{s,\varepsilon}^{(l)}\right\} \in W_2(Q_1)$.

III) $\frac{\varepsilon}{\delta_*}(\left\langle \frac{\partial^2}{\partial t^2} v_{s,\varepsilon}^{(l)}\right\rangle_0^2 + \left\langle \frac{\partial^2 u_{s,\varepsilon}^{(l)}}{\partial t \partial x}\right\rangle_0^2) + \left\langle v_{s,\varepsilon}^{(l)}\right\rangle_1^2 \le \left(\frac{\rho}{\delta_*}\right)^{(l)} const(l\!\!/),$

IV) $\frac{\varepsilon}{\delta_*}\left\langle \frac{\partial}{\partial t}\Delta v_{s,\varepsilon}^{(l)}\right\rangle_0^2 + \left\langle v_{s,\varepsilon}^{(l)}\right\rangle_2^2 \le \left(\frac{\rho}{\delta_*}\right)^{(l)} const(l\!\!/).$

Proof. As long as the estimates I),II) are true for the function $\left\{u_{s,\varepsilon}^{(l)}\right\} \in W_2(Q_1)$ then, repeating the proof for lemma 1, we obtain the statement for lemma 2. □

Lemma 3. *Let all the statements of theorem 2 and lemmas 1 and 2 are fulfilled. Then problem (11)-(13) is uniquely solvable in* $W_2(Q_1)$, *such that*

$$\varepsilon \cdot \frac{\partial \Delta u_{s,\varepsilon}^{(l)}}{\partial t} \in W_0(Q_1),$$

Proof. We prove by the contracting mapping method [2,6,13,14]. Determine the operator in space $W_2(Q_1)$.

$$u_{s,\varepsilon}^{(l)} = L_\varepsilon^{-1} F_s(u_{s,\varepsilon}^{(l-1)}) \equiv P u_{s,\varepsilon}^{(l-1)}$$

1. *We show that the operator P maps the spaces* $W_2(Q_1)$ *into itself.*

Let $\left\{u_{s,\varepsilon}^{(l-1)}\right\} \in W_2(Q_1)$, then to solve the problems (11)-(13), the statement of lemma 1 is fair, i.e. estimate II) is true. This means that for any $l = 1,2,3...$, we obtain $\left\{u_{s,\varepsilon}^{(l)}\right\} \in W_2(Q_1)$. Thus, $P: W_2(Q_1) \to W_2(Q_1)$

2. *We show that P is a contracting operator.*

Let $\{u_{s,\varepsilon}^{(l)}\}$, $\{u_{s,\varepsilon}^{(l-1)}\} \in W_2(Q_1)$. We consider a new function $v_{s,\varepsilon}^{(l)} = u_{s,\varepsilon}^{(l)} - u_{s,\varepsilon}^{(l-1)}$. Lemma 2 statement is fair for it, i.e. estimate IV) is true, i.e.

$$\frac{\varepsilon}{\delta_*}\left\langle \frac{\partial}{\partial t}\Delta v_{s,\varepsilon}^{(l)}\right\rangle_0^2 + \left\langle v_{s,\varepsilon}^{(l)}\right\rangle_2^2 \le \left(\frac{\rho}{\delta_*}\right)^{(l)} const(l\!\!/).$$

Thus, P is a contracting operator. Based on the known principle of contracting mapping [2,13,14], problem (11)-(13) has a unique solution that belongs to the space $u_{s,\varepsilon} \in W_2(Q_1)$, such that $\varepsilon \cdot \frac{\partial \Delta u_{s,\varepsilon}}{\partial t} \in W_0(Q_1)$, for $\varepsilon > 0$. □

Now we prove theorem 2. Assume that $\{u_{s,\varepsilon}\} \in W_2(Q_1)$ for the constant $\varepsilon > 0$ is the unique solution of problem (11)-(13). Then for $\varepsilon > 0$ for any $s = 0,1,2,3,\ldots$, inequality IV) is true. Based on the theorem of weak compactness [3,13], from the bounded sequence $\{u_{s,\varepsilon}\}$, we can find weakly convergent subsequence $\{u_{s,\varepsilon_j}\}$, such that $u_{s,\varepsilon_j} \to u_s$ is weak in $W_2(Q_1)$ for $\varepsilon_j \to 0$. We show that the limiting function $u_s(x,t)$ satisfies equation (6) almost everywhere in $W_2(Q_1)$. Indeed, as long as the subsequence $\{u_{s,\varepsilon_j}\}$ converges weakly in $W_2(Q_1)$ and the operator $L-$ is linear, then for the constant s we have

$$Lu_s - F_s = \varepsilon_j \frac{\partial \Delta u_{s,\varepsilon_j}}{\partial t} + L_0(u_{s,\varepsilon_j} - u_s)$$

Proceeding to the limit for $\varepsilon_j \to 0$, we obtain $Lu_s = F_s$ almost everywhere. For the constant s the function $u_s(x,t)$ is a unique solution of problems (6)-(8) from $W_2(Q_1)$. To prove the uniqueness of problem (6)-(8) we consider the following identity

$$2(Lu_s - F_s, \exp(-\lambda t)\frac{\partial}{\partial t} u_s)_0 = 0$$

Applying the priori estimate method [4,5,13] when meeting the conditions of the theorem in $W_2(Q_1)$, we obtain the inequality $\langle u_s \rangle_1 \leq 0$. Hence, the solution of problem (6)-(8) is unique. Theorem 2 has been proved. □

Now we prove theorem 1. Since all the conditions of theorem 1,2, are fulfilled applying the Parseval-Steklov equalities [13,14], to solve the problem (6)-(8) we obtain the solution of problem (1)-(5) from the defined class U. Thus, theorem 1 has been proved. □

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