Bulletin KRASEC. Phys. & Math. Sci. 2016. V. 13. no. 2. pp. 72-81. ISSN 2313-0156

DOI: 10.18454/2313-0156-2016-13-2-72-81

MSC 97I99

HOW TO TEACH TO SOLVE PROBLEMS WITH PARAMETERS?

O. K. Zhdanova

Vitus Bering Kamchatka State University, 683031, Petropavlovsk-Kamchatsky, Pogranichnaya st., 4, Russia
E-mail: kafmat@mail.ru

This article presents the materials of a seminar held in Vitus Bering Kamchatka State University for mathematics teachers and all those interested in similar problems. The seminar was devoted to solving the problems with parameters and the high level problems of the Unified State Exam.

Key words: Unified State Exam, problems with parameters, graphical methods of solution

Statistics of the Unified State Exam results within the recent years shows that problems with parameters are the rarest problems which the graduates try to solve.

What is the reason for that?

It is not a secret that problems with parameters mainly involve the theoretical knowledge available to any pupil, but they are practically not solved within the school course and are not considered during the preparation for end-of-course assessment.

One more reason, from the point of view of the Unified State Exam experts, is that the teachers themselves are not ready to solve such problems and, consequently, to teach the pupils.

Thus, a series o methodological seminars has been organized for mathematics teachers housed by Vitus Bering Kamchatka State University.

The paper suggests materials of the second seminar on the subject "Solution of problems with parameters by geometric interpretation "variable-parameter.

To solve equations and inequalities with parameters, it is appropriate to use a geometric interpretation of equation roots. A geometric interpretation "variable-value"was discussed at the previous seminar. For easier comparison of advantages and disadvantages of the considered interpretations, we shall solve the same problems which were solved at the previous seminar.

We consider a geometric interpretation in which we apply the representation of a dependence $f(x;a)=0$ given in an equation on a coordinate plane xOa or aOx.

When choosing a coordinate plane, we take the following reasons as guide: if it is easier to express a in terms of x from the equality $f(x;a)=0$ relating x and a, then we use the plane xOa, if x in terms of a, then we use the plane aOx.

Then we take different values of a on the ordinate axis and draw horizontal straight lines with ordinate a (Fig.2,a,b,c). If some of these lines do not cross the graph, there are no roots, if

Zhdanova Olesya Konstantinovna – Senior Lecturer, Dep. of Mathematics and Physics, Vitus Bering Kamchatka State University, Petropavlovsk-Kamchatsky, Russia.

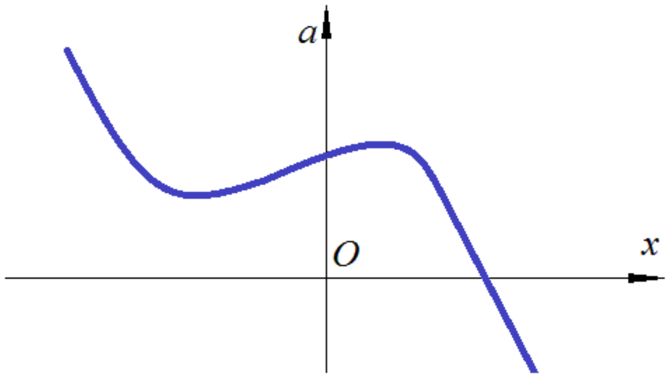
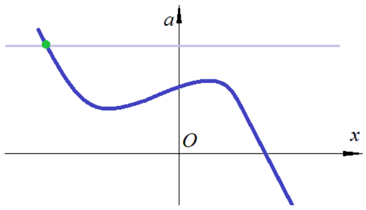
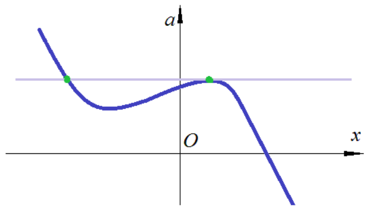


Fig. 1

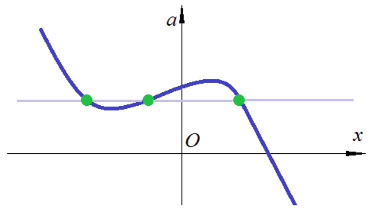
one does, the equation roots are the abscissas of the cross points (Fig.2,d,e,f).



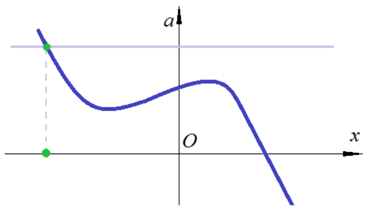
2,а



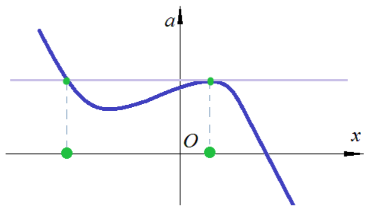
2,б



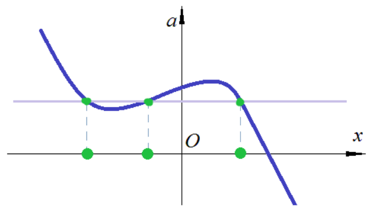
2,в



2,г



2,д



2,е

Fig.2

To solve an inequality with parameters applying a coordinate plane "variable – parameter we can use two approaches.

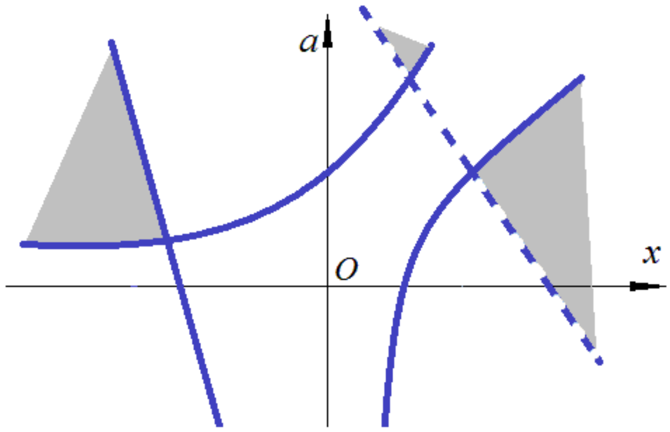
The first one is to represent a set $\{(x;a)|a(x)>0(a(x)<0)\}$. For this purpose, we appeal to hypographs and epigraphs of the function $a(x)=0$.

The second approach is to represent a set $\{(x;a)|a(x)=0\}$. In the result we obtain some lines on a plane which divide the whole plane into some parts. Inside each of them one of the inequalities $a(x)>0$ or $a(x)<0$ is satisfied. Then we choose a domains in which the required inequality is realized.

Fig.3

In order to find out which inequality is satisfied, it is enough to take some point in each of the domains and to check which inequality is satisfied in this point. At that, we can be sure that the same inequality holds for all the points of the domain where we chose a concrete point.

Now we take different points on the vertical axis a and draw horizontal lines (Fig. 4,a,b). If such a line crosses the specified domain, then there are solutions of an inequality for the chosen a value. If it does not cross the domain, there are no solutions.



The solution set for a fixed value a is the projection on abscises axis of the horizontal line part which falls within the specified domain (Fig. 4.c,d).

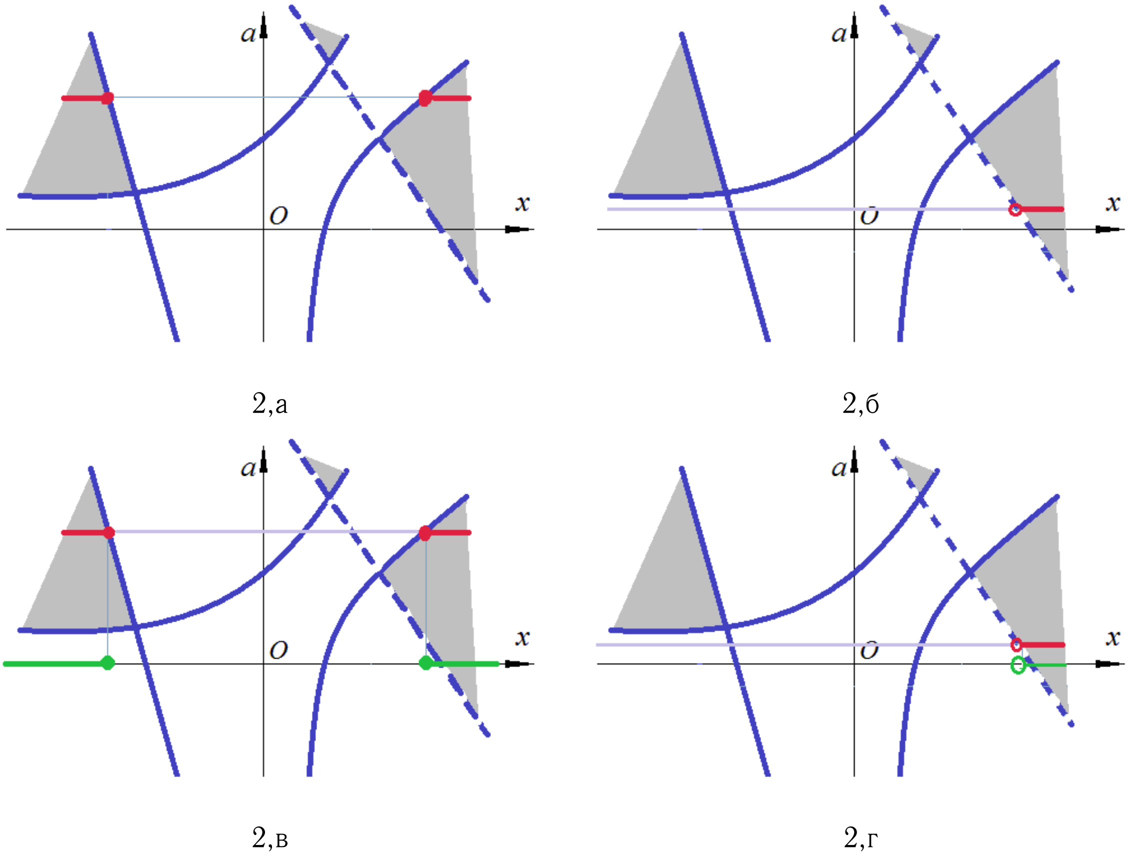


Fig. 4

Let's consider some examples of problem solutions.

1. For which a are the both roots of equation $x^2+(a+1)x-3=0$ negative?

Soltuion. We express a through x, as long as a enter into the first-order equation:

$a=a(x)=\dfrac{-x^2-x+3}{x}=-x-1+\dfrac{3}{x}$.

We study the behavior of this function.

When x tends to a large negative number, a will increase without limit, i.e. when $x\to-\infty$ $a\to+\infty$;

When x tends to a large positive number, a will decrease without limit, i.e. when $x \to +\infty$ $a \to -\infty$;

When x tends to 0 on the left, a will decrease without limit, i.e. when $x \to 0-0$ $a \to -\infty$;

When x tends to 0 on the right, a will increase without limit, i.e. when $x \to 0+0$ $a \to +\infty$.

Thus, $x = 0$ is a vertical asymptote.

As long as the function $a(x)$ is the sum of two decreasing functions everywhere in the domain, $a(x)$ is decreasing in each interval $(\infty;0),(0;+\infty)$.

We calculate several points and make a plot on the plane xOa.

$a(-3) = 1, a(-2) = -0,5, a(-1) = -3,$ $a(1) = 1, a(2) = -1,5, a(3) = -3.$

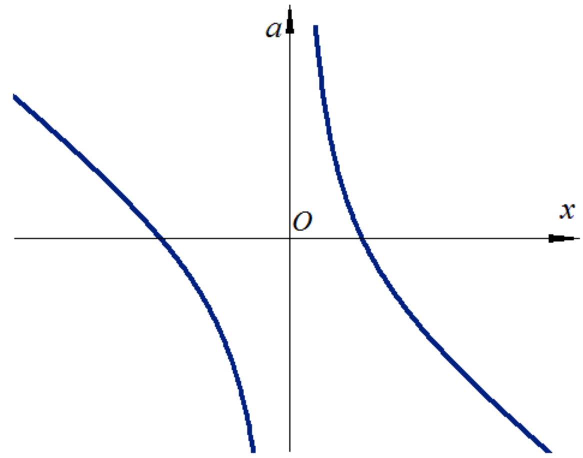


Fig. 5

Constructing the lines $a = a_0$, we obtain the number of equation solutions and see that one root is always positive and another one is always negative. Thus, there is no such a value of parameter a that the equation would have two negative roots.

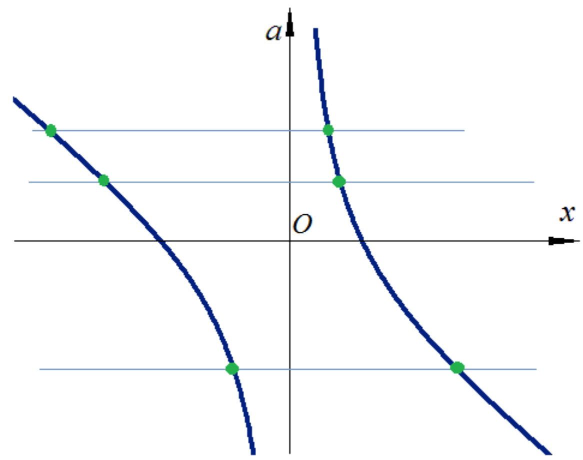


Fig. 6

Now we consider this problem in another form.

2. For which values of the parameter a does the equation $x^2+(a+1)x-3=0$ have a root satisfying the inequality $x^2-3x+2\leq 0$?

Solution. This problem can be reformulated as follows: for which values of the parameter a does the root of the equation $x^2+(a+1)x-3=0$ lie in the interval $[1;2]$ (the interval is the solution of the inequality $x^2-3x+2\leq 0$)?

To solve it, we apply a graphic interpretation of "variable-parameter".

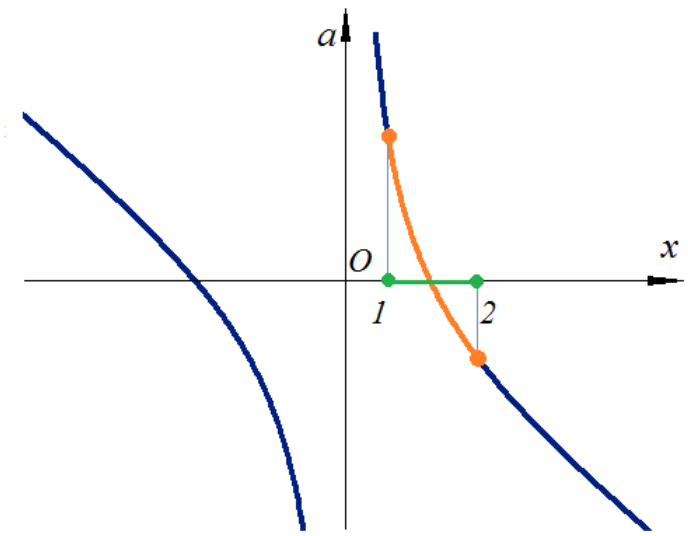


Fig. 7

As long as the function $a(x)$ is decreasing in the interval $x\in[1;2]$, the required parameter is within the interval $a\in[a_1;a_2]$, where $a_1=a(2)$, $a_2=a(1)$.

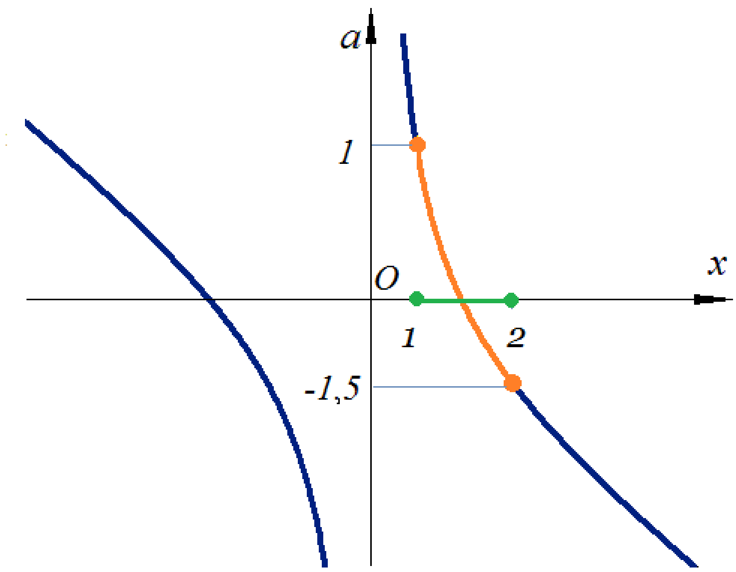


Fig. 8

$a(2) = -1,5$, $a(1) = 1$, thus, we obtain that for $a \in [-1,5;1]$ the root of the equation $x^2 + (a+1)x - 3 = 0$ satisfies the inequality $x^2 - 3x + 2 \leq 0$.

3. Solve the inequality $x^2 + x + a > 0$.

Solution. We rearrange the inequality in the form $a > -x^2 - x$ and consider the function $a(x) = -x^2 - x = -(x+0,5)^2 + 0,25$. The graph of this function is a parabola with a vertex at the point $(-0,5;0,25)$ and downward branches.

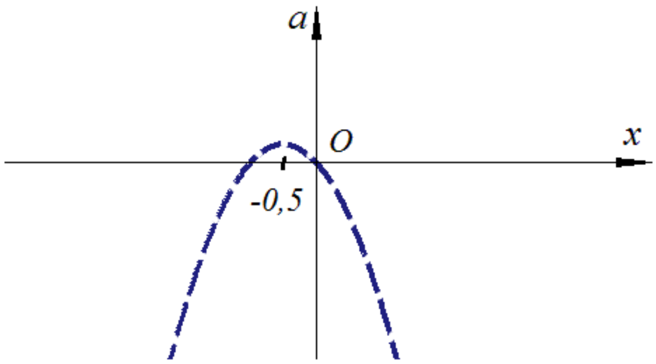


Fig. 9

The inequality is described by an epigraph of the function $a(x)$.

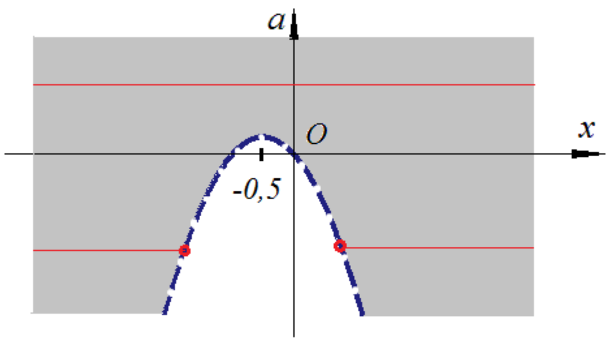


Fig. 10

For $a \leq 0,25$, we obtain $x \in (-\infty; -0,5 - \sqrt{0,25-a}) \cup \cup (-0,5 - \sqrt{0,25-a}; +\infty)$.

For $a > 0,25$, we obtain $x \in R$.

4. Solve the inequality $|x-a| > x+1$.

Solution. We consider the cases $x - a \geq 0$ and $x - a < 0$ and obtain the following:

I. $\begin{cases} x - a \geq 0, \\ x - a > x + 1, \end{cases}$

whence

$\begin{cases} a \leq x, \\ a < -1, \end{cases}$

the first inequality defines a set of points determined by the hypograph of the function $a(x) = x$. The second one is determined by the hypograph of the function $a(x) = -1$. We obtain the domain

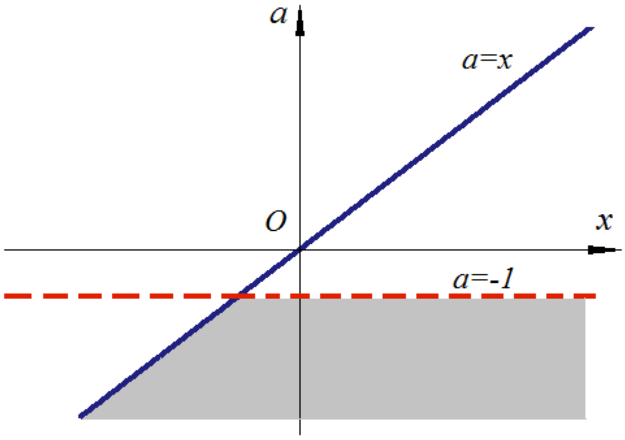


Fig.11

II. $\begin{cases} x-a<0, \\ -x+a>x+1, \end{cases}$

whence

$\begin{cases} a>x, \\ a>2x+1, \end{cases}$

the first inequality defines a set of points determined by the epigraph of the function $a(x)=x$. The second inequality is determined by the epigraph of the function $a(x)=2x+1$. We obtain the domain

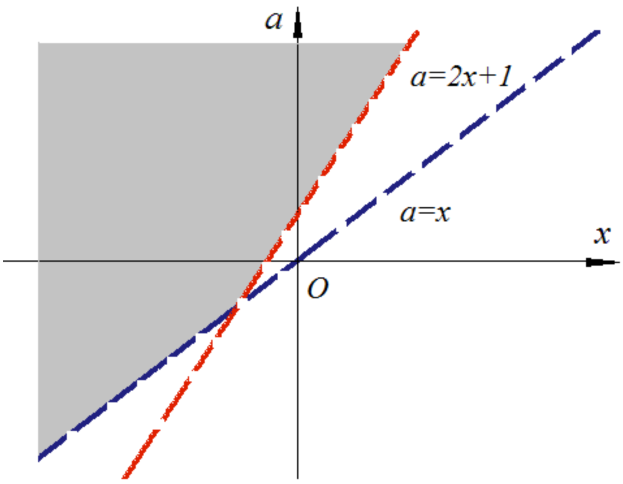


Fig. 12

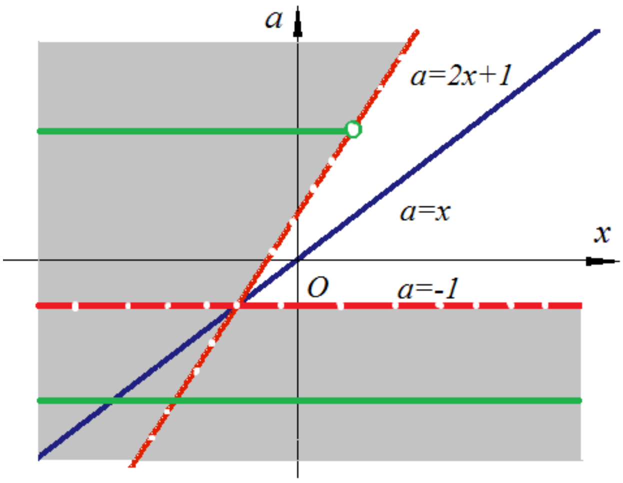
In the result, we obtain

Fig. 13

and constructing the curves a parallel to the axis Ox, we obtain that for $a<-1$ any value of x is the inequality solution. And for $a\geq -1$, the interval $x\in\left(-\infty;\dfrac{a-1}{2}\right)$ is the solution.

We again confirm the solution obtained before.

5. Solve the inequality $\log_2 x(x-4)+\log_2\dfrac{x-2}{x-4}\geq a$



Solution. We rearrange the inequality to the following form:

$$\begin{cases} x(x-4) > 0, \\ \dfrac{x-2}{x-4} > 0, \\ \log_2 x(x-2) \geq a. \end{cases}$$

whence $\begin{cases} x \in (-\infty;0) \cup (4;+\infty), \\ \log_2 x(x-2) \geq a. \end{cases}$

Thus, we need to solve the inequality $\log_2 x(x-2) \geq a$ under condition $x \in (-\infty;0) \cup (4;+\infty)$.

Consider the function $a(x) = \log_2(x^2 - 2x)$. This function is determined on the set $x \in (= \infty;0) \cup (2;+\infty)$.

When x is approaching to 2 on the right and to 0 on the left, the argument of the logarithm tends to 0 and, consequently, the value of the function $a(x)$ tends to large negative values.

When $x_0 = 1 \pm \sqrt{2}$, the argument of the logarithm equals 1 and, consequently, $a(x_0) = 0$.

For very large values of x (both positive and negative), the argument of the logarithm increases without limit, consequently, the value of the function $a(x)$ increases.

Construct a graph of the function $a(x)$ on "variable-parameter"plane

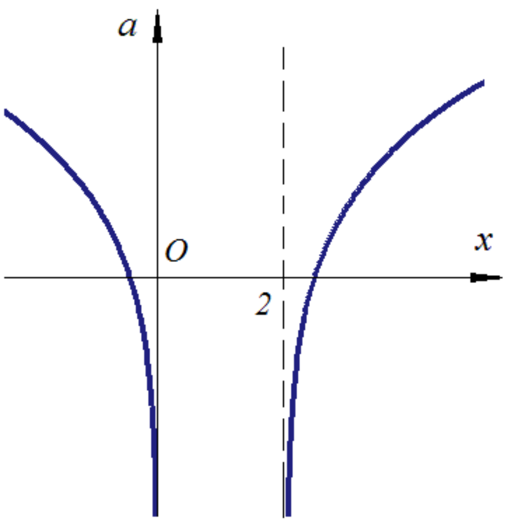


Fig. 14

Taking into account the limitations $x \in (-\infty;0) \cup (4;+\infty)$, we obtain

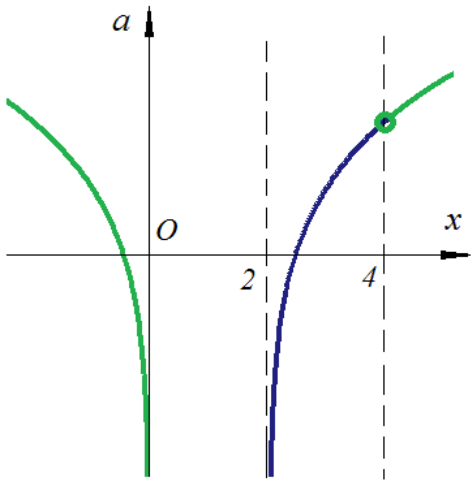


Fig. 15

We obtained some domains and hatch those which satisfy the inequality $\log_2 x(x-2) \geq a$.

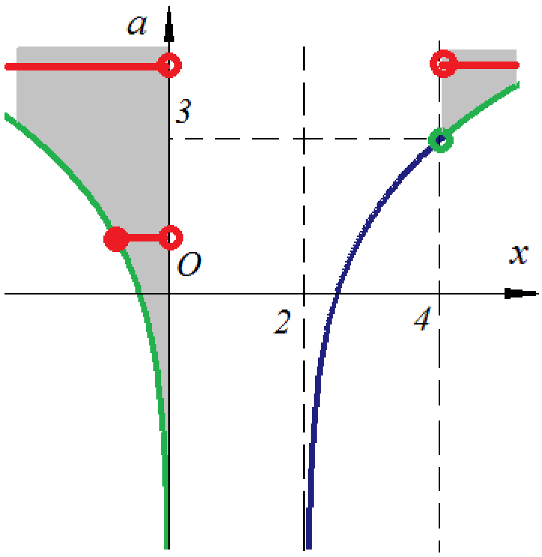


Fig. 16

For different values of a we obtain the following:

1) for $a \leq 3$ the function $a(x)$ intersects the set at one interval, i.e. $x \in [1-\sqrt{1+2^a};0)$;

2) for $a > 3$ the function $a(x)$ intersects the set at two intervals, i.e. $x \in [1-\sqrt{1+2^a};0) \cup (4;1+\sqrt{1+2^a}]$.

The obtained solution confirmed the solution obtained at the previous lesson.

Drawing the conclusions, the advantages to the considered method applicable to many tasks are obvious.

References

1. Aleksandr Larin, http://www.alexlarin.net.

2. Dyatlov V. N. Kak nauchit' reshat' zadachi s parametrami: lektsii 1-4 [How to teach to solve the problems with parameters: lectures 1-4]. Moscow. Pedagogicheskiy universitet "Pervoe sentyabrya". 2014. 80 p.

3. Dyatlov V. N. Kak nauchit' reshat' zadachi s parametrami: lektsii 5-8 [How to teach to solve the problems with parameters: lectures 5-8]. Moscow. Pedagogicheskiy universitet "Pervoe sentyabrya". 2014. 80 p.

4. Rossiyskoe obrazovanie [Russian Education] http://www.edu.ru/tests/course/testselect.php?subject=19.

5. Yashchenko I. V., Vysotskiy I. R., Kosukhin O. N., Semenov P. V., Semenov A. V., Trepalin A. S. Uchebno-metodicheskie materialy dlya predsedateley i chlenov regional'nykh predmetnykh komissiy po proverke vypolneniya zadaniy s razvernutym otvetom ekzamenatsionnykh rabot EGE 2015 goda. Matematika. V trekh chastyakh [Educational materials for chairmen and members of regional subject committees for checking the tasks with open answers of the Unified State Exam 2015. Mathematics. Three parts]. Mathematics. FIPI. 2015. 72 p.

6. Federal'nyy institut pedagogicheskikh izmereniy [Federal Institute of Educational Measurement], http://fipi.ru/.

7. Yashchenko I. V., Shestakov S. A., Trepalin A. S. Podgotovka k EGE po matematike v 2015 godu. Bazovyy i profil'nyy urovni. Metodicheskie ukazaniya [Preparation for Unified State Exam in Mathematics in 2015. Basic and profile levels. Methodological instructions]. Mathematics. MNTsMO. 2015. 288 p.

For citation: Zhdanova O. K. How to teach to solve problems with parameters?. *Bulletin KRASEC. Physical and Mathematical Sciences* 2016, vol. **13**, no **2**, 72-81. DOI: 10.18454/2313-0156-2016-13-2-72-81

Original article submitted: 08.06.2016