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GELLERSTEDT PROBLEM FOR A PARABOLIC-HYPERBOLIC EQUATION WITH DEGENERACY OF TYPE AND ORDER

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This paper deals with the formulation and study of a boundary value Gellerstedt problem for parabolic-hyperbolic equation with degeneracy of type and order within a domain which is equivalently reduced to integral equations.

Key words: Gellerstedt problem, parabolic-hyperbolic equation, degeneracy of type and order, integral equations, integral-differential operator of fractional order, modified Cauchy problem

Statement of the problem

We consider an equation

$$0 = \begin{cases} xu_{xx} + \alpha_0 u_x - u_y, & (x;\ y) \in \Omega_0, \\ xu_{xx} + (-x)^n u_{yy} + \alpha_1 u_x, & (x;\ y) \in \Omega_{1j}, \ \left(j = \overline{1,3}\right), \end{cases} \tag{1}$$

where $\alpha_0, \alpha_1, n = const$ and

$$0 < \alpha_0 < 1, \tag{2}$$

here Ω_0 is a domain bounded by the lines $_0$: $x = 0$, $0 \le y \le 1$, $_1$:$0 \le x < +\infty$, $y = 0$, $_2$: $0 \le x < +\infty$, $y = 1$; Ω_{11} is a characteristic triangle bounded by the segment AE of y axis and two characteristics

$$AC_1\colon\ y - \frac{2}{n+1}(-x)^{\frac{n+1}{2}} = 0,\ EC_1\colon\ y + \frac{2}{n+1}(-x)^{\frac{n+1}{2}} = y_0,$$

equations (1), starting at the points $A\,(0;\ 0)$ and $E\,(0;\ y_0)$, meeting at the point $C_1\left(-\left(\frac{n+1}{4}y_0\right)^{\frac{2}{n+1}};\ \frac{y_0}{2}\right)$, $y_0 \in [0;\ 1]$; Ω_{12} is a characteristic triangle bounded by the segment E of axis y and two characteristics

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$$EC_2\colon\ y-\frac{2}{n+1}(-x)^{\frac{n+1}{2}}=y_0,\ BC_2\colon\ y+\frac{2}{n+1}(-x)^{\frac{n+1}{2}}=1,$$

equations (1), starting at the points $E\,(0;\ y_0)$ and $B\,(0;\ 1)$, meeting at the point $C_2\left(-\left(\frac{n+1}{4}(1-y_0)\right)^{\frac{2}{n+1}};\ \frac{1+y}{2}\right.$

Ω_{13} is a characteristic quadrilateral bounded by the characteristics EC_1, EC_2 and

$$C_1C\colon\ y-\frac{2}{n+1}(-x)^{\frac{n+1}{2}}=0,\ C_2\colon\ y+\frac{2}{n+1}(-x)^{\frac{n+1}{2}}=1,$$

equations (1), meeting at the points E, C_1, C_2 and $C\left(-\left(\frac{n+1}{4}\right)^{\frac{2}{n+1}};\ \frac{1}{2}\right)$.

Let us introduce the following notations: $\Omega_1=\Omega_{11}\cup\Omega_{12}\cup\Omega_{13}\cup EC_1\cup EC_2$, $I=\{(x,\ y):\ x=0,\ 0<y<1\}$, $\Omega=\Omega_0\cup\Omega_1\cup I$, $I_1=\{(x,\ y):\ x=0,\ 0<y<y_0\}$,$I_2=\{(x,\ y):\ x=0,\ y_0<y<1\}$.

In the domain Ω we study the following task for equation (1).

Task Γ_1. *Find a function $u(x,\ y)$ with the following properties:*

1) $u(x,\ y)$ is bounded for all $0\le x<+\infty$, $0\le y\le 1$ and is continuous right up to the boundary of the domain Ω;

2) $u(x,\ y)\in C^{2,1}_{x,y}(\Omega_0)\cap C^{2,2}_{x,y}(\Omega_{11}\cup\Omega_{12}\cup\Omega_{13})$ satisfies the equation (1) in the domains Ω_0, and Ω_{1j}, $\left(j=\overline{1,3}\right)$;

3) the condition of gluing is fulfilled on I

$$\lim_{x\to-0}(-x)^{\alpha_1}u_x=f(y)\lim_{x\to+0}x^{\alpha_0}u_x+g(y)\,\textit{is uniform when}\,0<y<1;\tag{3}$$

4) $u(x,\ y)$ satisfies the edge conditions

$$u(x,\ y)|_{y=0}=\varphi(x),\tag{4}$$

$$u(x,\ y)\,|_{AC_1}=\psi_1(y)\,,0\le y\le\frac{y_0}{2}u(x,\ y)\,|_{EC_2}=\psi_2(y)\,,y_0\le y\le\frac{y_0+1}{2},\tag{5}$$

where $\varphi(x)$, $\psi_1(y)$, $\psi_2(y)$, $f(y)$, $g(y)$ are the given functions and

$$\varphi(x)\ [0;\ +\ \infty)\,,\tag{6}$$

$$\psi_1(0)=\varphi(0)\,,\tag{7}$$

$$\psi_2(y)\in C^1\left[y_0;\ \frac{y_0+1}{2}\right]\cap C^3\left(y_0;\frac{y_0+1}{2}\right),\tag{8}$$

$$f(y)\,,g(y)\in C\left(\bar{I}\right)\cap C^2(I)\,,f(y)\neq 0,\forall\,(0;\ y)\in\bar{I}.\tag{9}$$

The following theorem is true.

Theorem. *If the conditions (2), (6)-(9) are fulfilled, then there is a unique solution of problem Γ_1 in the domain Ω.*

Proof.

Solutions of a modified Cauchy problem with initial data

$$\lim_{x\to-0}u(x;\ y)=\tau(y)\,,\tag{10}$$

for equation (1) in domain Ω_1 is determined by the formula [1, p. 110-111]

$$u(x,y)=\gamma_1\int\limits_0^1\tau\left(y+\frac{2}{n+1}(-x)^{\frac{n+1}{2}}(2t-1)\right)(t(1-t))^{\beta-1}dt-$$

$$-(-x)^{1-\alpha_1}\gamma_2\int\limits_0^1\nu_2\left(y+\frac{2}{n+1}(-x)^{\frac{n+1}{2}}(2t-1)\right)(t(1-t))^{-\beta}dt, \tag{11}$$

where $\gamma_1=\dfrac{\Gamma(2\beta)}{\Gamma^2(\beta)}$, $\gamma_2=\dfrac{\Gamma(2-2\beta)}{(1-\alpha_1)\Gamma^2(1-\beta)}$, $\beta=\dfrac{n-1+2\alpha_1}{2n+2}$, and $0<\beta<\dfrac{1}{2}$.

Satisfying (11) the first condition from (5), we have

$$\psi_1\left(\frac{y}{2}\right)=\gamma_1\int\limits_0^1\tau(ty)(t(1-t))^{\beta-1}dt-$$

$$-\gamma_2\left(\tfrac{n+1}{4}y\right)^{1-2\beta}\int\limits_0^1\nu_2(ty)(t(1-t))^{-\beta}dt,\ 0\le y\le y_0.$$

Hence, changing $ty=z$, we obtain

$$\psi_1\left(\frac{y}{2}\right)=\gamma_1 y^{1-2\beta}\int\limits_0^y\tau(z)(z(y-z))^{\beta-1}dz-\gamma_2\left(\frac{n+1}{4}\right)^{1-2\beta}\int\limits_0^y\nu_2(z)(z(y-z))^{-\beta}dt.$$

Owing to the definition of integral-differential operator of fractional order [1, P. 19] from the latest equality, we have

$$D_{0y}^{-\beta}\tau(y)y^{\beta-1}-\frac{\gamma_2\Gamma(1-\beta)}{\gamma_1\Gamma(\beta)}\left(\frac{n+1}{4}\right)^{1-2\beta}y^{2\beta-1}D_{0y}^{\beta-1}\nu_2(y)y^{-\beta}=$$

$$=\frac{1}{\gamma_1\Gamma(\beta)}y^{2\beta-1}\psi_1\left(\frac{y}{2}\right). \tag{12}$$

Applying the operator D_{0y}^{β} to the both parts of equality (12) and taking into account the identities

$$D_{0y}^{\beta}y^{2\beta-1}D_{0y}^{\beta-1}y^{-\beta}f(y)=y^{\beta-1}D_{0y}^{2\beta-1}f(y),$$

$$D_{0y}^{\beta}D_{0y}^{-\beta}f(y)=f(y),$$

we obtain a functional accuracy between $\tau(y)$ and $\nu_2(y)$ taken from the domain Ω_{11} on I_1

$$\tau(y)=\frac{\gamma_2\Gamma(1-\beta)}{\gamma_1\Gamma(\beta)}\left(\frac{n+1}{4}\right)^{1-2\beta}D_{0y}^{2\beta-1}\nu_2(y)+$$

$$+\frac{1}{\gamma_1\Gamma(\beta)}y^{1-\beta}D_{0y}^{\beta}y^{2\beta-1}\psi_1\left(\frac{y}{2}\right). \tag{13}$$

Just in the same way, satisfying (11) the second condition from (5), we have

$$\tau(y)=\frac{\gamma_2\Gamma(1-\beta)}{\gamma_1\Gamma(\beta)}\left(\frac{n+1}{4}\right)^{1-2\beta}D_{y_0y}^{2\beta-1}\nu_2(y)+$$

$$+\frac{1}{\gamma_1\Gamma(\beta)}(y-y_0)^{1-\beta}D_{y_0y}^{\beta}(y-y_0)^{2\beta-1}\psi_2\left(\frac{y+y_0}{2}\right). \tag{14}$$

Solution of the second boundary value problem for equation (1) in the domain Ω_0, meeting the conditions (4) and

$$\lim_{x\to+0} x^{\alpha_0}u_x = \nu_1(y), \tag{15}$$

has the following form [2]:

$$u(x,y) = -\frac{1}{\Gamma(\alpha_0)}\int_0^y e^{-\frac{x}{y-t}}(y-t)^{-\alpha_0}\nu_1(t)dt + \int_0^\infty E(x,t,y,\alpha_0)\varphi(t)dt, \tag{16}$$

where

$$E(x,\xi,y-\eta,\alpha_0) = (y-\eta)^{-1}(x\xi)^{\frac{1-\alpha_0}{2}} I_{\alpha_0-1}\left(\frac{2\sqrt{x\xi}}{y-\eta}\right)e^{-\frac{x+\xi}{y-\eta}}\xi^{\alpha_0-1}$$

is the functional solution of equation (1), $I_\chi(z)$ is a modified Bessel function [3].

In (16), turning to the boundary for $x\to+0$ and taking into account the condition 1) of the problem $_1$, we obtain a functional accuracy between $\tau(y)$ and $\nu_1(y)$ on the degeneracy line I taken from the domain D_1

$$\tau(y) = -\frac{\Gamma(1-\alpha_0)}{\Gamma(\alpha_0)}D_{0y}^{\alpha_0-1}\nu_1(y) + F(y), \tag{17}$$

where

$$F(y) = \frac{1}{\Gamma(\alpha_0)}y^{-\alpha_0}\int_0^\infty t^{\alpha_0-1}e^{-\frac{t}{y}}\varphi(t)dt. \tag{18}$$

On the strength of (2), (6), and taking into account the formula [4, P. 4]:

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt\text{из}\,(\mathrm{Re}\, z>0),$$

(18) it follows that

$$|F(y)| \le const. \tag{19}$$

Eliminating $\tau(y)$ from relations (13), (14) and (17), taking into account (3), (10), (15), we obtain the equalities

$$-\frac{\Gamma(1-\alpha_0)}{\Gamma(\alpha_0)}D_{0y}^{\alpha_0-1}\nu_1(y) + F(y) = \frac{1}{\gamma_1\Gamma(\beta)}y^{1-\beta}D_{0y}^{\beta}y^{2\beta-1}\psi_1\left(\frac{y}{2}\right) +$$
$$+\left(\frac{n+1}{4}\right)^{1-2\beta}\frac{\gamma_2\Gamma(1-\beta)}{\gamma_1\Gamma(\beta)}D_{0y}^{2\beta-1}\left(f(y)\nu_1(y)+g(y)\right), 0\le y\le y_0, \tag{20}$$

$$-\frac{\Gamma(1-\alpha_0)}{\Gamma(\alpha_0)}D_{0y}^{\alpha_0-1}\nu_1(y) + F(y) = \frac{1}{\gamma_1\Gamma(\beta)}(y-y_0)^{1-\beta}D_{y_0y}^{\beta}(y-y_0)^{2\beta-1}\psi_2\left(\frac{y+y_0}{2}\right) +$$
$$+\left(\frac{n+1}{4}\right)^{1-2\beta}\frac{\gamma_2\Gamma(1-\beta)}{\gamma_1\Gamma(\beta)}D_{y_0y}^{2\beta-1}\left(f(y)\nu_1(y)+g(y)\right), y_0\le y\le 1. \tag{21}$$

Then, to find $\nu_1(y)$, we consider the following cases.

I. If $\alpha_0 < 2\beta$ (i.e. $\alpha_0 - 1 < \frac{2(\alpha_1-1)}{n+1}$), then, applying the operators $D_{0y}^{1-2\beta}$ and $D_{y_0y}^{1-2\beta}$ to the both parts in equalities (20) (21), respectively, we obtain integral Volterra equation of the second kind

$$\nu_1(y) + \int_0^y K_1(y,\ t)\,\nu_1(t)\,dt = \Phi_1(y), \tag{22}$$

$$\nu_1(y)+\int\limits_0^y K_2(y,\ t)\,\nu_1(t)\,dt=\Phi_2(y), \tag{23}$$

where $K_1(y,\ t)=\frac{\mu_1}{f(y)}(y-t)^{2\beta-\alpha_0-1}$, $K_2(y,\ t)=\begin{cases} K_{21}(y,\ t),\ 0\le t\le y_0,\\ K_{22}(y,\ t),\ y_0\le t\le y,\end{cases}$

$$K_{21}(y,\ t)=\frac{\mu_1}{f(y)}(2\beta-\alpha_0)(y-t)^{-\alpha_0}(y-y_0)^{2\beta-1}\frac{y-y_0}{y-t}F\left(2\beta,\ \alpha_0,\ 2\beta+1;\ \frac{y-y_0}{y-t}\right)+$$

$$+\frac{\mu_1}{f(y)}2\beta(y-y_0)^{2\beta-1}(y_0-t)^{-\alpha_0}\frac{y_0-t}{y-t},$$

$$K_{22}(y,\ t)=\frac{\mu_1}{f(y)}(2\beta-\alpha_0)(y-t)^{2\beta-\alpha_0-1},$$

$$\Phi_1(y)=\frac{\mu_2}{f(y)}D_{0y}^{1-2\beta}F(y)-\frac{\mu_3}{f(y)}D_{0y}^{1-2\beta}y^{1-\beta}D_{0y}^{\beta}y^{2\beta-1}\psi_1\left(\frac{y}{2}\right)-\frac{g(y)}{f(y)},$$

$$\Phi_2(y)=\frac{\mu_2}{f(y)}D_{y_0y}^{1-2\beta}F(y)-\frac{g(y)}{f(y)}-$$

$$-\frac{\mu_3}{f(y)}D_{y_0y}^{1-2\beta}(y-y_0)^{1-2\beta}D_{y_0y}^{\beta}(y-y_0)^{2\beta-1}\psi_2\left(\frac{y+y_0}{2}\right),$$

and μ_j, $(j=1,2,3)$ are the constants.

Kernels and the right parts of integral equations (22) and (23) admit the following estimates, respectively:

$$|K_1(y,\ t)|\le c_{11}(y-t)^{2\beta-\alpha_0-1}, |\Phi_1(y,\ t)|\le c_{12}y^{2\beta-1}, c_{11},\ c_{12}=const, \tag{24}$$

$$|K_{21}(y,\ t)|\le c_{11}(y_0-t)^{-\alpha_0}(y-y_0)^{2\beta-1}, |K_{22}(y,\ t)|\le c_{22}(y-t)^{2\beta-\alpha_0-1}, \tag{25}$$

$$|\Phi_2(y,\ t)|\le c_{23}(y-y_0)^{2\beta-1},\ \ c_{2j}=const,\ \left(j=\overline{1,3}\right).$$

Owing to the estimate (24) and (25), equations (22) and (23) are the integral Volterra equations of the second kind with weak singularities. According to the theory of integral Volterra equations [4], we conclude that the integral equation (22) is uniquely solvable in the class $C^2(0,\ y_0)$, and the function $\nu_1(y)$ may have a singularity one order less than $1-2\beta$ when $y\to 0$, and when $y\to y_0$ it is bounded and its solution is given by the formula

$$\nu_1(y)=\Phi_1(y)-\int\limits_0^y R_1(y,\ t)\,\Phi_1(t)\,dt,\quad 0\le y\le y_0,$$

where $R_1(y,\ t)$ is the resolvent of the kernel $K_1(y,\ t)$. Just in the same way, solving equation (23) we obtain

$$\nu_1(y)=\Phi_2(y)-\int\limits_0^y R_2(y,\ t)\,\Phi_2(t)\,dt,\quad y_0\le y\le 1,$$

where $R_2(y,\ t)$ is the resolvent of the kernel $K_2(y,\ t)$. $\nu_1(y)\in C^2(y_0,\ 1)$, and may have the singularity one order less than $1-2\beta$ when $y\to y_0$, and when $y\to 1$, it is bounded. Consequently, the problem $_1$ for $\alpha_0<2\beta$ is uniquely solvable since it is equivalent to the integral Volterra equations of the second kind (22) and (23).

Solution of the problem $_1$ for $\alpha_0<2\beta$ may be restored in the domain Ω_0 as the solution of the second boundary value problem (16); and in Ω_{11} and Ω_{12}, as the solution of a modified Cauchy problem (11); and in Ω_{13}, as the solution of Goursat problem for equation (1) (Riemann method).

II. By a similar method described above, we prove the unique solvability of the problem Γ_1 in the case when $\alpha_0>2\beta$, and $\alpha_0=2\beta$. $\square$

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