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ON THE UNIQUENESS OF SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A MIXED EQUATION WITH HYPERBOLIC DEGENERATION OF ORDER

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The paper proves the uniqueness of solution of a boundary value problem for a mixed elliptic-hyperbolic equation of the second order.

Key words: mixed equation, extremum principle

Introduction

The paper studies the uniqueness of solution of a boundary value problem for a linear mixed elliptic-hyperbolic equation of the second order in a two-dimensional strip.

Papers of many authors, in particular, [1]–[6] plaid a significant role in the development of modern theory of mixed equations and its applied aspects.

Uniqueness of a boundary value problem for a mixed elliptic-hyperbolic equation with hyperbolic degeneration of order

Assume that D is a simply connected domain of a complex plane $z = x + iy$ bounded by Jordan curve σ located in a half plane $\mathrm{Im}\, z > 0$ with the ends at the points $A = (0,0)$ and $B = (1,0)$, by the segments AA_δ, BB_δ of straight lines $x = 0$, $x = 1$, $-\delta \leq y \leq 0$, $\delta = \mathrm{const} > 0$ and by segments $A_\delta C : 0 \leq x \leq 1/2$ and $B_\delta C : 1/2 \leq x \leq 1$ of straight lines $x + y = -\delta$ and $x - y = 1 + \delta$, respectively (see the figure).

In a domain D we consider the equation

$$0 = \begin{cases} u_{xx} + u_{yy} + a_1(z)u_x + b_1(z)u_y + c_1(z)u, & y > 0, \\ u_x - u_y + c_0(z)u, & -\delta < y < 0, \\ u_{xx} - u_{yy} + a_2(z)u_x + b_2(z)u_y + c_2(z)u, & y < -\delta. \end{cases} \quad (1)$$

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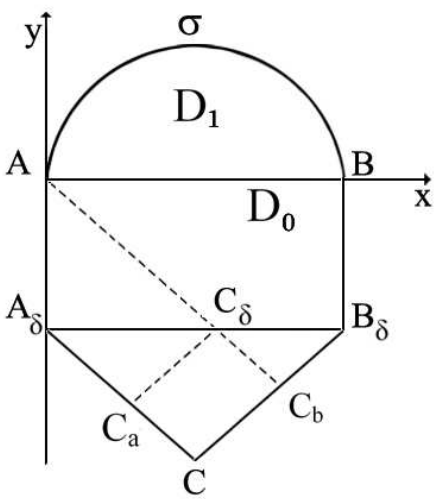

Figure

We denote the parts of the domain D by D_1, D_0 and D_2 where $y > 0$, $-\delta < y < 0$ and $y < -\delta$, respectively. Assuming that coefficients $a_j(z)$, $b_j(z)$, $c_j(z)$ belong to the class $C(\overline{D}_j)$, $\overline{D}_j$ is a closure of D_j, $j = 0,1,2$.

Equation (1) in the domain D is a mixed equation with hyperbolic degeneration of order in the domain D_0. It is an elliptic equation in the domain D_1, and a hyperbolic equation in the domain D_2. In the domain D_0, equation (1) has one family $\xi = x+y$ of real characteristics, and in the domain D_2 it has two families $\xi = x+y$, $\eta = x-y$.

The aim of this section is to study the unique solvability of the following mixed problem.

Task. *Find a solution $u(z) = u(x,y)$ of equation (1) which is regular everywhere in the domain D, possibly, with the exception of characteristic segments $AC_b : x+y = 0,\ 0 < x < 1/2+\delta/2, C_\delta C_a : x-y = 2\delta, \delta/2 < x < \delta$ which belongs to a class $C(\overline{D}) \cap C'(D \setminus AC_b \setminus C_\delta C_a)$ and satisfies the boundary conditions*

$$u(z) = \varphi(s)\ \forall z \in \sigma \quad 0 \le s \le l, \tag{2}$$

$$u(iy) = \psi(y)\ \forall y \in [-\delta, 0], \tag{3}$$

where $\varphi(s)$ and $\psi(y)$ are given continuous functions, l is the length of a curve σ, measured from a point B.

Theorem. *Assume that $c_1(z) \le 0$ in D_1; $c_0(x) \ge 0$ for $0 \le x \le 1$; $\dfrac{\partial a_2(z)}{\partial x}$ and $\dfrac{\partial b_2(z)}{\partial y}$ belong to $C(D_2)$. Then the problem may not have more than one solution.*

Proof.

The proof is considerably based on the following extremum principle. Assume that $c_1(z) \le 0$ in D_1, $c_0(x) \ge 0$ for $0 \le x \le 1$. Then the positive maximum (negative minimum) of the solution $u(z)$ of problem (1)-(3) on a compact $\overline{D}_1$ is achieved only on σ.

Really, assume that $u(z)$ is the solution of problem (1)-(3) and $\tau(x) = u(x)$, $\nu(x) = u_y(x)$. Then from the equation

$$u_x - u_y + c_0(z)u = 0 \tag{4}$$

we conclude that

$$\nu(x) = \tau'(x) + c_0(x)\tau(x). \tag{5}$$

Assume that $\max\limits_{\overline{D}_1} u(z) = u(\zeta)$. It follows from the Hopf principle that $\zeta \overline{\in} D_1$. The assumption that $\zeta \equiv \xi \in]0,1[$ owing to (5) results in the inequality $\nu(\xi) \geq 0$ that discords with Zaremba-Zhiro principle [1] asserting that $\nu(\xi) < 0$. Inclusion $\zeta \in \sigma$ remains.

Now we prove that the uniform problem corresponding to problem (1)-(3), that is problem (1)-(3) for $\varphi(z) \equiv 0$, $\psi(y) \equiv 0$, has only zero solution $u(z) = 0$. Let $u(z)$ be the solution of a uniform problem. It follows from the extremum principle that $u(z) \equiv 0$ in a closed domain $\overline{D}_1$. Then, the function $u(z)$ should be the solution of uniform Cauchy problem $u(x) = 0$, $0 \leq x \leq 1$ for equation (4). It follows from the uniqueness of this problem solution that $u(z) = 0$ in the part D_0^+ of the domain D, ling in the characteristic strip $0 \leq x + y \leq 1$. In the domain $D_0^- = D \setminus D_0^+$ $u(z)$ is the solution of uniform Cauchy problem $u(iy) = 0, -\delta \leq y \leq 0$ for equation (4). Thus, $u(z) = 0$ both in D_0^-, and in $\overline{D}_0$ in whole. In the domain D_2 the function $u(z)$ as the solution of the equation

$$u_{xx} - u_{yy} + a_2(z)u_x + b_2(z)u_y + c_2(z) = 0, \tag{6}$$

should satisfy the uniform Cauchy problem $u(x - i\delta) = 0$ for $0 < x < 1$ and, as it follows from equation (4), for $y = -\delta$ $u_y(x, -\delta) = 0$ for $0 < x < 1$. We conclude from the uniqueness of Cauchy problem solution for (6) that $u(z) = 0$ in $\overline{D}_2$. This completes the proof of the solution uniqueness of problem (1)-(3). □

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