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DUFFING OSCILLATOR WITH EXTERNAL HARMONIC ACTION AND VARIABLE FRACTIONAL RIEMANN-LIOUVILLE DERIVATIVE CHARACTERIZING VISCOUS FRICTION

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The paper suggested generalization of Duffing oscillator with viscous hereditary friction which is represented by the operator of a variable fractional derivative in the sense of Riemann-Liouville. Explicit finite-difference scheme was derived to calculate approximate solutions, and phase trajectories for different values of control parameters were plotted.

Key words: Riemann-Liouville derivative, Grunwald-Letnikov derivative, heredity, Duffing oscillator, phase trajectory

Introduction

At the present time, one of the research areas of nonlinear dynamics, fractional dynamics, is extensively developing. The essence of it is the consideration of memory and consequence effects characterized by fractional derivatives. The issues of study of such systems are described in detail in the book by I. Petras [1]. Elements of hereditary dynamics, which includes fractional dynamics as a special case, are considered in the book by V.V. Uchaykina [2].

A model of Duffing oscillator with fractal friction was suggested in the paper [3]. Fractal friction has the viscous friction properties due to the power series in the integral operator («heavy tails»). Thus, such an oscillator may have different applications, for example, [4] or [5].

In this paper we continue to study the Duffing oscillator with fractal friction, but the fractional derivative order represents the function of time but not a constant as in the paper [3]. Then we built a numerical explicit finite-difference scheme to calculate an approximate solution of the corresponding Cauchy problem. Based on the numerical solution, we plot and study phase trajectories.

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Statement of the problem

Find a solution $x(t)$ where $t \in [0,T]$ of the following Cauchy problem:

$$\ddot{x}(t) + \alpha D_{0t}^{q(t)} x(\tau) - x(t) + x^3(t) = \delta \cos(\omega t),\ x(0) = x_0, \dot{x}(0) = y_0, \quad (1)$$

where $D_{0t}^{q(t)} x(\tau) = \frac{1}{\Gamma(q(t))} \int\limits_0^t \frac{x(\tau) d\tau}{(t-\tau)^{1-q(t)}}$ is Riemann-Liouville derivative of variable fractional order $0 < q(t) < 1$, α is the viscous friction coefficient, δ and ω are the amplitude and the frequency of external periodic force, x_0 и y_0 are the given constants, initial conditions.

The case when $q(t)$ is a constant was discussed in the paper [3] in detail. The paper [6] considered the model for hereditary Duffing oscillator with Gerasimov-Caputo derivative.

It is convenient to represent the Cauchy problem (1) in the form of differential equation system

$$\begin{cases} \dot{x}(t) = y(t), \\ D_{0t}^{q(t)} x(\tau) = w(t), \\ \dot{y}(t) = x(t) + \alpha w(t) - x^3(t) + \delta \cos(\omega t), \\ x(0) = x_0,\ y(0) = y_0. \end{cases} \quad (2)$$

Owing to its nonlinearity, system (2) does not have an exact solution, thus we shall seek an approximate solution by the finite-difference scheme theory [7]-[9]. We divide the segment $[0,T]$ into N equal parts with a step h. Solution of the differential problem $x(t)$ turns into an approximate mesh solution $x(t_k)$, $t_k = kh$, $k = 1,...,N$. The fractional derivative in system (2) is approximated by a differential analogue, generalized Grunwald-Letnikov derivative [10]

$$D_{0t}^{q(t)} x(\tau) \approx \frac{1}{h^{q_{k-1}}} \sum_{j=0}^{k-1} c_j^{(q_{k-1})} x_{k-j} = \frac{x_k}{h^{q_{k-1}}} + \sum_{j=1}^{k-1} c_j^{(q_{k-1})} x_{k-j}, \quad (3)$$

$$c_0^{(q_{k-1})} = 1, c_j^{(q_{k-1})} = \left(1 - \frac{1+q_{k-1}}{j}\right) c_{j-1}^{q_{k-1}},$$

and the integer derivatives

$$\dot{x}(t) \approx \frac{x_k - x_{k-1}}{h}, \dot{y}(t) \approx \frac{y_k - y_{k-1}}{h}. \quad (4)$$

Substituting (3) and (4) into system (2), we come to the following approximate solution of the Cauchy problem:

$$\begin{cases} x_k = x_{k-1} + h y_{k-1}, \\ x_k = w_{k-1} h^{q_{k-1}} - \sum\limits_{j=1}^{k-1} c_j^{(q_{k-1})} x_{k-j}, \\ y_k = y_{k-1} + \left[\alpha w_{k-1} - x_{k-1}^3 + \delta \cos(\omega(k-1))\right] h. \end{cases} \quad (5)$$

On the basis of the paper [10] we may note that approximation (5) of the differential problem (2) has the first order. We shall not investigate the stability and the convergence of the explicit scheme (5). Explicit schemes are stable, as a rule, i.e. there is a limit for the step h.

We can estimate the step h by a double counting method [11]. For the selected control parameters we can make an experiment to study the stability of the right part and the initial data. If the scheme is stable with the first order, than it converges with the same order according to Lax theorem.

Stationary point stability of system (2) can be made on the analogy of the paper [12]. We consider some results of modeling of Duffing oscillator with friction.

Results of modeling

Example 1. We consider a numerical solution of the Cauchy problem (2) with the following control parameters: $q(t) = q_0 - A\exp(-\sin(\omega t))$, $q_0 = 0.9$, $A = 0.08$, $\omega = 1$, $\delta = 0.3$, $\alpha = 0.15$, $x_0 = 0.3$, $y_0 = 1.51$, $N = 3300$, $T = 150$, $h = 0.0454$.

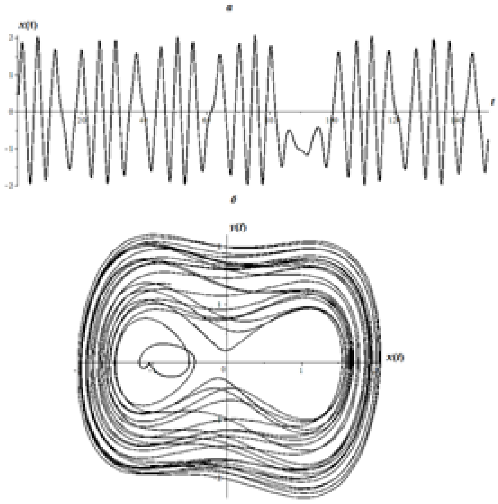

Fig. 1. Numerical solution curve - *a* and its phase trajectory - *b*

Fig. 1(a, b) show numerical solution curve and its phase trajectory for hereditary Duffing oscillator (5) with fractional exponent $q(t) = q_0 - A\exp(-\sin(\omega t))$. The phase trajectory comes out to a regular chaotic attractor.

Example 2. We consider a numerical solution of the Cauchy problem (2) with the following control parameters: $q(t) = q_0 - A/\exp(\cos(\omega t))$, $q_0 = 0.9$, $A = 0.05$, $\omega = 100$, $\delta = 30$, $\alpha = 0.15$, $x_0 = 0.3$, $y_0 = 1.51$, $N = 4000$, $T = 20$, $h = 0.005$.

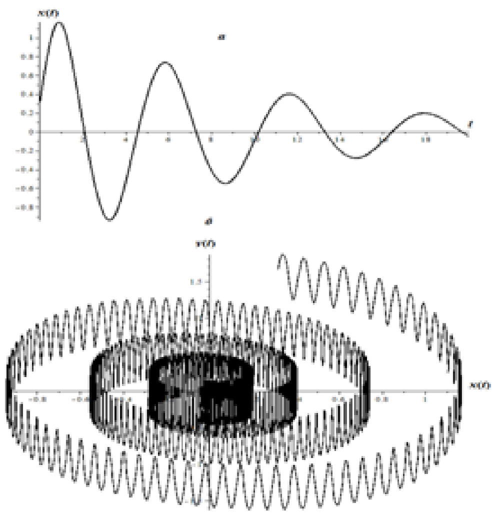

Fig. 2. Numerical solution curve - *a* and its phase trajectory - *b*

It is clear from Fib. 2(a, b) that, firstly, there are attenuating oscillations and then the phase trajectory comes out to the limit cycle.

Conclusions

The paper suggests a model of hereditary Duffing oscillator with variable fractional derivative. A finite-difference scheme was plotted for this model. Depending on different control parameters, oscillograms and phase trajectories were drawn by this scheme. It was shown that phase trajectories come out to limit cycles. It is interesting to investigate other nonlinear hereditary oscillators.

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