Bulletin KRASEC. Phys. & Math. Sci. 2016. V. 13. no. 2. pp. 10-15. ISSN 2313-0156

DOI: 10.18454/2313-0156-2016-13-2-10-15

MSC 34L99

LINEAR INVERSE PROBLEM FOR TRIKOMI EQUATION IN THREE-DIMENSIONAL SPACE

S. Z. Djamalov

Institute of Mathematics, National University of Uzbekistan, Tashkent, 100125, Academgorodok, Do'rmon yo'li, 29 str.
E-mail: siroj63@mail.ru

In the present work the problems of correctness of a linear inverse problem for Trikomi equation in three-dimensional space are considered. For this problem, the theorems on existence and uniqueness of the solution in a certain class are proved by "ε-regularization Galerkin's methods and by the method of successive approximations.

Key words: Trikomi equations, linear inverse problem, correctness of solution, Galerkin's method, « ε-regularization» method, method of successive approximations

Introduction

Close relation of the problems with nonlocal boundary conditions and inverse problems was discovered in the process of study of nonlocal problems. By the present, inverse problems for parabolic, elliptical and hyperbolic equations have been studied quite well. [1,2,5,8]. Inverse problems for mixed equations (in particular for Trikomi equation) are significantly less investigated. [4,6,7]. We shall try to fill this gap within the framework of this paper.

Statement of the problem

In a domain

$$Q=(-1,1)\times(0,T)\times(0,\ell)=Q_1\times(0,\ell)=$$
$$=\{(x,t,y);\ -1<x<1,\ 0<t<T<+\infty,\ 0<y<\ell<+\infty\}$$

we consider Trikomi equation.

$$Lu=xu_{tt}-\Delta u+\alpha(x,t)\,u_t+c(x,t)\,u=\psi(x,t,y), \tag{1}$$

where $\Delta u=u_{xx}+u_{yy}$ is Laplace operator in a plane. We assume that the coefficients of equation (1) are sufficiently smooth functions.

Djamalov Sirojiddin Zuhriddinovich – Ph.D.(Phys & Math), Senior Researcher of department Differential equations, Institute of mathematics, National university of Uzbekistan, Tashkent, Republic of Uzbekistan.

Task 1. *(Nonlocal boundary value problem.) Find a generalized solution of equation (1) from S.L. Sobolev's space where* $W_2^l(Q)$*,* $2 \leq l$ *is an integer satisfying nonlocal boundary conditions*

$$\gamma D_t^p u|_{t=0} = D_t^p u|_{t=T}, \tag{2}$$

$$\eta D_x^p u|_{x=-1} = D_x^p u|_{x=1}, \tag{3}$$

$$u(x,t,0) = u(x,t,\ell) = 0, \tag{4}$$

where $p = 0,1, \gamma$ *u* $\eta - const \neq 0$, $D_t^p u = \dfrac{\partial^p u}{\partial t^p}$, $D_t^0 u = u$.

We should note that in the paper [3] the authors proved correctness of the solution of problem (2)-(4) from S.L. Sobolev's space $W_2^l(Q)$, when $2 \leq l$ is an integer under certain conditions on the equation coefficients and the right part of equation (1).

In this paper, under additional condition, the solution of equation (1) is written in certain classes, both the solution and the right part of the equation.

Assume $\psi(x,t,y) = g(x,t,y) + h(x,t) \cdot f(x,t,y)$ where $g(x,t,y)$ and $f(x,t,y)$ are given functions.

Task 2. *(Linear inverse problem.) Find the functions* $(u(x,t,y),\ h(x,t))$ *satisfying equation (1) in a domain* Q *so that the function* $u(x,t,y)$ *satisfies boundary conditions (2),(3),(4) and the additional condition*

$$u(x,t,\ell_0) = \phi(x,t), 0 < \ell_0 < \ell < +\infty, \tag{5}$$

Theorem 1. *Assume that the conditions given above are fulfilled for the coefficients of equation (1), moreover, assume that* $2\alpha + \lambda x \geq \delta_1 > 0$*;* $c(x,0) = c(x,T)$*;* $\alpha(x,0) = \alpha(x,T)$*,* $\lambda c - c_t \geq \delta_2 > 0$*, where* $\lambda = \frac{2}{T}\ln\gamma$ *is such that* $\gamma \in (1,\infty)$*.*

Then let

$$(1 + D_y^3)g \in W_2^1(Q), \gamma g(x,0,y) = g(x,T,y),$$

$$g(x,t,\ell_0) = g_0(x,t) \in W_2^1(Q_1),$$

$$(1 + D_y^3)f \in W_2^2(Q), \gamma f(x,0,y) = f(x,T,y),$$

$$f(x,t,\ell_0) = f_0(x,t) \in W_2^2(Q_1),\ |f_0(x,t)| \geq \tau > 0.$$

We assume that the given function $\phi(x,t) \in W_2^2(Q_1)$ *is the solution of the following problem*

$$L_0\phi = x\phi_{tt} - \phi_{xx} + \alpha(x,t)\phi_t + c(x,t)\phi = g_0(x,t),$$

$$\gamma D_t^p \phi|_{t=0} = D_t^p \phi|_{t=T},$$

$$\eta D_x^p \phi|_{x=-1} = D_x^p \phi|_{x=1}.$$

the unique solvability of which was studied in the paper [3] and assume that there is such a positive number ν *that* $\delta_0 - 6\nu^{-1} \geq \delta_* > 0$*,* $2\rho \equiv M \cdot \sum\limits_{s=0}^{\infty} (1 + \mu_s^6) \| f_s \|^2_{W_2^1(Q_1)} < \delta_*$*, where* $\delta_0 = \min\{\delta_1, \delta_2, \lambda\}$*;* $M = const(\delta_0; \eta; \nu)$*,* $\mu_s = \dfrac{s\pi}{l}$*.*

Then the functions

$$u(x,t,y) = \sum_{s=0}^{\infty} u_s(x,t)\sin\mu_s y,$$

$$h(x,t) = \frac{1}{f_0}\sum_{s=0}^{\infty} \mu_s^2 u_s(x,t)\sin\mu_s\ell_0$$

are the solutions of the linear inverse problem (1)-(5) from the class

$$U = \{(u,h)|\ u \in W_2^2(Q); h \in W_2^2(Q_1); D_y^3\{u_{xx}, u_{tx}, u_{tt}\} \in L_2(Q), D_y^4 u \in L_2(Q)\},$$

where the functions $u_s(x,t);\ s=0,1,2,3,...$ *are the solutions in the domain* Q_1 *of the corresponding nonlinear loaded problems [2,5].*

$$Lu_s = L_0 u_s + \mu_s^2 u_s = g_s + \frac{f_s}{f_0} \cdot \sum_{m=0}^{\infty} \mu_m^2 u_m \sin \mu_m \ell_0 \equiv F_s(u_s) \tag{6}$$

$$\gamma D_t^p u_s|_{t=0} = D_t^p u_s|_{t=T} \tag{7}$$

$$\eta D_x^p u_s|_{x=-1} = D_x^p u_s|_{x=1} \tag{8}$$

where

$$f_s = \frac{2}{\ell} \int_0^{\ell} f(x,t,y) \sin \mu_s y dy, g_s = \frac{2}{\ell} \int_0^{\ell} g(x,t,y) \sin \mu_s y dy.$$

Proof. We prove the theorem step-by-step. First, we shall show that the function $u(x,t,y)$ satisfies the additional condition (5), i.e. $u(x,t,\ell_0) = \phi(x,t)$. Assume the contrary. Let $u(x,t,\ell_0) = v(x,t) \neq \phi(x,t)$, for the function $z(x,t) = v(x,t) - \phi(x,t)$ in the domain Q_1 from (6)-(8) we obtain

$$L_0 z = x z_{tt} - z_{xx} + \alpha(x,t) z_t + c(x,t) z = 0, \tag{9}$$

$$\gamma D_t^p z|_{t=0} = D_t^p z|_{t=T}, \tag{10}$$

$$\eta D_x^p z|_{x=-1} = D_x^p z|_{x=1}. \tag{11}$$

It follows from the solution uniqueness of problem (9)-(11) [3] that $z(x,t) = 0$, i.e. $v(x,t) = \phi(x,t)$. To prove theorem 1, we shall further need the following notations and preparatory lemmas.

Assume that $u_s \in W_2^2(Q_1)$, then determine the spaces $W_i(Q_1); i = 0,1,2$ with the norm

$$\langle u_s \rangle_i^2 = \sum_{s=0}^{\infty} (1+\mu_s^6) \|u_s\|^2_{W_2^i(Q_1)}; i = 0,1,2,$$

when $i = 0$; $W_2^0(Q_1) = L_2(Q_1)$.

It is obvious that the spaces $W_i(Q_1)$ with the given norm are Banach ones [9]. As long as Q_1 is a bounded domain with piecewise-smooth boundary, the following embeddings are fulfilled:

$$W_2(Q_1) \subset W_1(Q_1) \subset W_0(Q_1).$$

Theorem 2. *Assume that all the above mentioned conditions of theorem 1 are fulfilled, then there is a unique solution of problem (6)-(8) from the space* $W_2(Q_1)$.

Proof. First we prove the solvability of problem (6)-(8) by the methods of «ε-regularization», successive approximations and Galerkin's one [2,3,9]. In particular, we consider the family of nonlinear loaded equations

$$L_\varepsilon u_{s,\varepsilon}^{(\theta)} = -\varepsilon \frac{\partial^3}{\partial t^3} u_{s,\varepsilon}^{(\theta)} + L_0 u_{s,\varepsilon}^{(\theta)} + \mu_s^2 u_{s,\varepsilon}^{(\theta)} =$$

$$g_s + \frac{f_s}{f_0} \cdot \sum_{m=0}^{\infty} \mu_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \mu_m \ell_0 \equiv F_s(u_{s,\varepsilon}^{(\theta-1)}) \tag{12}$$

$$\gamma D_t^p u_{s,\varepsilon}^{(\theta)}\Big|_{t=0} = D_t^p u_{s,\varepsilon}^{(\theta)}\Big|_{t=T}, \tag{13}$$

$$\eta D_x^p u_{s,\varepsilon}^{(\theta)}\Big|_{x=-1} = D_x^p u_{s,\varepsilon}^{(\theta)}\Big|_{x=1}, \tag{14}$$

where $\varepsilon > 0$, $\theta = 0,1,2,...$; γ and $\eta - consts \neq 0$ are such that $\gamma \in (1,\infty)$; $\eta \in [1,\infty)$.

Lemma 1. *Assume that all the conditions of theorem 2 are fulfilled, then to solve problem (12)-(14), the following estimates are true*

$$I)\ \frac{\varepsilon}{\delta_*}\left\langle \frac{\partial^2}{\partial t^2}u_{s,\varepsilon}^{(\theta)}\right\rangle_0^2+\left\langle \frac{\partial^2 u_{s,\varepsilon}^{(\theta)}}{\partial t\,\partial x}\right\rangle_1^2+\left\langle u_{s,\varepsilon}^{(\theta)}\right\rangle_1^2\leq const(\not\theta),$$

$$II)\ \frac{\varepsilon}{\delta_*}\left\langle \frac{\partial^3 u_{s,\varepsilon}^{(\theta)}}{\partial t^3}\right\rangle_0^2+\left\langle u_{s,\varepsilon}^{(\theta)}\right\rangle_2^2\leq const(\not\theta).$$

Hereafter, the symbol const $(\not\theta)$ *denotes the constant independent of* θ.

Proof. Applying the methods of induction, Galerkin, priori estimates, S.L. Sobolev's imbedding theorems to the identities

$$2(L_\varepsilon u_{s,\varepsilon}^{(\theta)}-F_s(u_{s,\varepsilon}^{(\theta-1)}),\exp(-\lambda t-\mu x)\frac{\partial}{\partial t}u_{s,\varepsilon}^{(\theta)})_0=0, \tag{15}$$

$$-2(L_\varepsilon u_{s,\varepsilon}^{(\theta)}-F_s(u_{s,\varepsilon}^{(\theta-1)}),\exp(\frac{-(\lambda t+\mu x)}{2})\frac{\partial^2 \ell u_{s,\varepsilon}^{(\theta)}}{\partial t^2})_0=0, \tag{16}$$

where $(\cdot,\cdot)_0$ is a standard scalar product in $L_2(Q_1)$, $\Delta w=w_{tt}+w_{xx}$ is Laplace operator with respect to the variables t и x,, $\mu=\ln\eta,\eta\in[1,+\infty)$

$$\frac{\partial^2\ell w}{\partial t^2}=\exp(\frac{-(\lambda t+\mu x)}{2})\left[\frac{\partial^3 w}{\partial t^3}-\lambda\ w_{tt}+\frac{\lambda^2}{4}w_t\right].$$

After integration we obtain the first and the second estimates, respectively. Lemma 1 has been proved. □

We introduce a new function from $W_2(Q_1)$ by the formula $v_{s,\varepsilon}^{(\theta)}=u_{s,\varepsilon}^{(\theta)}-u_{s,\varepsilon}^{(\theta-1)},\varepsilon>0,s=0,1,2,\ldots,\theta=1,2,3,\ldots$.

Then the following lemma is true for it.

Lemma 2. *Assume that all the conditions of theorem 2 and lemma 1 are fulfilled. Then for the function* $\left\{v_{s,\varepsilon}^{(\theta)}\right\}\in W_2(Q_1)$ *the following estimates are true:*

$$III)\ \frac{\varepsilon}{\delta_*}(\left\langle \frac{\partial^2}{\partial t^2}v_{s,\varepsilon}^{(\theta)}\right\rangle_0^2+\left\langle \frac{\partial^2 u_{s,\varepsilon}^{(\theta)}}{\partial t\partial x}\right\rangle_0^2)+\left\langle v_{s,\varepsilon}^{(\theta)}\right\rangle_1^2\leq\left(\frac{\rho}{\delta_*}\right)^{(\theta)}const(\not\theta),$$

$$IV)\ \frac{\varepsilon}{\delta_*}\left\langle \frac{\partial^3 v_{s,\varepsilon}^{(\theta)}}{\partial t^3}v_{s,\varepsilon}^{(\theta)}\right\rangle_0^2+\left\langle v_{s,\varepsilon}^{(\theta)}\right\rangle_2^2\leq\left(\frac{\rho}{\delta_*}\right)^{(\theta)}const(\not\theta).$$

Proof. As long as for the estimates I),II) are true for the function $\left\{u_{s,\varepsilon}^{(\theta)}\right\}\in W_2(Q_1)$, then repeating the proof for lemma 1, we obtain the statement for lemma 2. □

Lemma 3. *Assume that all the statements of theorem 2 and lemmas 1 and 2 are fulfilled. Then problem (12)-(14) is uniquely solvable in* $W_2(Q_1)$, *so that* $\varepsilon\cdot\dfrac{\partial^3 u_{s,\varepsilon}^{(\theta)}}{\partial t^3}\in W_0(Q_1)$.

Proof. Lemma 3 will be proved by a contracting mapping method [9]. We determine an operator in the space $W_2(Q_1)$.

$$u_{s,\varepsilon}^{(\theta)}=L_\varepsilon^{-1}F_s(u_{s,\varepsilon}^{(\theta-1)})\equiv Pu_{s,\varepsilon}^{(\theta-1)}.$$

1. We show that the operator P maps the spaces $W_2(Q_1)$ into itself.

Assume that $\left\{u_{s,\varepsilon}^{(\theta-1)}\right\} \in W_2(Q_1)$, than to solve problems (12)-(14), the statement of lemma 1 is true, i.e. the estimate II) is true. It follows that for any $\theta = 1,2,3...$ we obtain $\left\{u_{s,\varepsilon}^{(\theta)}\right\} \in W_2(Q_1)$. Thus, $P : W_2(Q_1) \to W_2(Q_1)$

2. We show that P is a contraction operator.

Assume that $\left\{u_{s,\varepsilon}^{(\theta)}\}, \{u_{s,\varepsilon}^{(\theta-1)}\right\} \in W_2(Q_1)$. We consider a new function $v_{s,\varepsilon}^{(\theta)} = u_{s,\varepsilon}^{(\theta)} - u_{s,\varepsilon}^{(\theta-1)}$. The statement of lemma 2 is true for it, i.e. estimate IV) is true, that is

$$\frac{\varepsilon}{\delta_*}\left\langle \frac{\partial^3}{\partial t^3} v_{s,\varepsilon}^{(\theta)}\right\rangle_0^2 + \left\langle v_{s,\varepsilon}^{(\theta)}\right\rangle_2^2 \leq \left(\frac{\rho}{\delta_*}\right)^{(\theta)} const(\not\theta).$$

Thus, P is a contraction operator. According to the known principle of contracting mapping [9], problem (12)-(14) has a unique solution belonging to the space $W_2(Q_1)$, such that $\varepsilon \cdot \frac{\partial^3 u_{s,\varepsilon}^{(\theta)}}{\partial t^3} \in W_0(Q_1)$ when $\varepsilon > 0$. □

Now we shall prove theorem 2.

Assume that $\{u_{s,\varepsilon}\} \in W_2(Q_1)$ for fixed $\varepsilon > 0$ is the unique solution of problem (12)-(14). Then for $\varepsilon > 0$ for any $s = 0,1,2,3...$ the inequality IV) is true. According to the theorem on weak compactness [9,10], from the bounded sequence $\{u_{s,\varepsilon}\}$, it is possible to find weakly convergent subsequence of the function $\left\{u_{s,\varepsilon_j}\right\}$ so that $u_{s,\varepsilon_j} \to u_s$ is weak in $W_2(Q_1)$. We shall show that the limiting function $u_s(x,t)$ satisfies equation (6) almost everywhere in $W_2(Q_1)$. Indeed, as long as the subsequence $\left\{u_{s,\varepsilon_j}\right\}$ weakly converge in $W_2(Q_1)$, and the operator is linear, for a fixed s we have

$$Lu_s - F_s = \varepsilon_j \frac{\partial^3 u_{s,\varepsilon_j}}{\partial t^3} + L_0(u_{s,\varepsilon_j} - u_s). \tag{17}$$

Proceeding to the limit in (17) for $\varepsilon_j \to 0$, we obtain $Lu_s = F_s$. For fixed s, the function $u_s(x,t)$ is the unique solution of problem (6)-(8) from $W_2(Q_1)$.

Thus, theorem 2 has been proved. □

Now we prove theorem 1. As long as all the conditions of theorem 1,2 has been fulfilled applying the Parseval-Steklov equalities [9,10] to solve problem (6)-(8), we obtain the solution of problem (1)-(5) from the given class U. □

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For citation: Djamalov S. Z. Linear inverse problem for Trikomi equation in three-dimensional space. *Bulletin KRASEC. Physical and Mathematical Sciences* 2016, vol. **13**, no **2**, 10-15. DOI: 10.18454/2313-0156-2016-13-2-10-15

Original article submitted: 12.05.2016