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PROBLEM WITH A SHIFT FOR A MIXED-TYPE EQUATION OF THE SECOND KIND IN AN UNBOUNDED DOMAIN

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The paper investigates a non-local problem in the mixed area the elliptical part of which is a vertical half-strip. The non-local conditions in this problem connect pointwise the values of a fractional derivative of an unknown function at the points of a boundary characteristic.

Key words: problem with a shift, mixed-type equation of the second kind, uniqueness and existence of a solution, singular integral equation, unbounded domain

Introduction

After the well known publications by Karol I.L. [1],[2] in 1953, the interest to investigate boundary-value mixed-type problems of the second kind arose. Similar Tricomi problems for elliptic-hyperbolic equations of the second kind in bounded domains were considered in the papers by M.S. Salakhitdinova, S.S. Isamukhedova [3], M.M. Smirnova [4], Yu.M. Krikunova [5], Zh. Oramova etc. Problems with shifts on the characteristics of different families for elliptic-hyperbolic equation of the second kind in a bounded domain are considered in the paper by G.A. Ivashkina [8].

This paper describes a problem with a shift on the characteristics of one family for elliptic-hyperbolic equation of the second kind in an unbounded domain.

Statement of the problem

We consider a mixed-type equation of the second kind:

$$u_{xx} + sign y|y|^m u_{yy} = 0, 0 < m < 1, \tag{1}$$

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in an unbounded mixed domain $\Omega=\Omega_1\cup AB\cup\Omega_2$. Here $\Omega_1=\{(x,y):0<x<1,y>0\}$, $AB=\{(x,y):0<x<1,\ y=0\}$, a Ω_2 is the finite domain of a half-plane $y<0$ bounded by the segment $\overline{AB}$ and two characteristics:

$$AC: x-[2/(2-m)]\,(-y)^{(2-m)/2}=0,$$

$$BC: \eta=x+[2/\,(2-m)]\,(-y)^{(2-m)/2}=1.$$

Equation (1), starting from the points $A(0,0)$ and $\upsilon(\xi,0)>0\ (<0)$. We introduce the notations: $\beta=\dfrac{m}{2\,(m-2)}$, $k=const>1$, $a=2/\,(1+k)$,

$$\theta_0(x_0)=\left(\frac{x_0}{2},-\left[\frac{2-m}{2}\cdot\frac{x_0}{2}\right]^{\frac{2}{2-m}}\right),\theta_{0k}(x_0)=\left(\frac{x_0}{k+1},-\left[\frac{2-m}{2}\cdot\frac{x_0}{k+1}\right]^{\frac{2}{2-m}}\right).$$

Here $\theta_0(x_0)$ and $\theta_{0k}(x_0)$ are the points of intersections of the characteristic AC of equation (1) with the lines $l_i: x+\dfrac{2i}{2-m}(-y)^{\frac{2}{2-m}}=x_0$ (when $i=1,2$).

We consider equation (1) in the domain Ω.

Task $\mathbf{T}^{\infty}$. *Find the function $u(x,y)$ having the following properties:*

1) $u(x,y)\in C\left(\bar{\Omega}\right)\cap C^1\left(\Omega\right)\cap C^2\left(\Omega_1\cup\Omega_2\right)$, and $u_y(x,0)$ can go into infinity of the order less than $1-2\beta$, when $x\to 1$;

2) $u(x,y)$ is regular in Ω_1 and generalized from the class R_2 in Ω_2 by solution of equation (1) [2];

3) $u(x,y)$ satisfies the following conditions:

$$u(0,y)=\varphi_1(y),\ \ u(1,y)=\varphi_2(y),\ \ 0\le y<+\infty, \tag{2}$$

$$\lim_{y\to+\infty}u(x,y)=0, uniformly\ in x\in[0,1], \tag{3}$$

$$D_{0x}^{1-\beta}u\left[\theta_0\left(x\right)\right]+\omega\left(x\right)D_{0x}^{1-\beta}u\left[\theta_{0k}\left(x\right)\right]=\delta(x),\ \ 0<x<1, \tag{4}$$

$$u(x,+0)=u\left(x,-0\right),u_y\left(x,+0\right)=-u_y\left(x,-0\right), \tag{5}$$

where $\varphi_i(y)$ $(i=1,2)$, $\omega(x)$, $\delta(x)$ are the given functions, at that $\varphi_i(y)\in C[0,+\infty)$, and when y are large enough, it satisfies the inequality: $|\varphi\left(y\right)|\le M_1y^{-1-m/2}$; $\omega(x)$, $\delta(x)\in C\left[0,1\right]$; $\max\limits_{[0,1]}|\omega\left(x\right)|=M$, $0<M<a^{2\beta-1}$.

On account of inversability of the operator D_{sx}^{δ} from the problem $\mathbf{T}^{\infty}$ in a special case when $\omega\left(x\right)\equiv 0$ follows the Tricomi problem for equation (1) in the domain Ω.

Assume that $u\left(x,y\right)$ is the solution of the problem T^{∞}. Then in the domain Ω_2, it may be represented as follows [2]:

$$\begin{gathered}u\left(x,y\right)=\int\limits_0^1 H\left\{\left[x-\left(1-2\beta\right)\left(-y\right)^{1/(1-2\beta)}\right]t\right\}\left[x+\left(1-2\beta\right)\left(-y\right)^{1/(1-2\beta)}-\right.\\ \left.-xt+\left(1-2\beta\right)\left(-y\right)^{1/(1-2\beta)}t\right]^{-\beta}\left[x-\left(1-2\beta\right)\left(-y\right)^{1/(1-2\beta)}\right]^{1-\beta}\left(1-t\right)^{-\beta}dt+\\ +\frac{\left[2\left(1-2\beta\right)\right]^{1-2\beta}}{2\cos\pi\beta}\int\limits_0^1 H\left[x-\left(1-2\beta\right)\left(-y\right)^{1/(1-2\beta)}\left(2t-1\right)\right]\left(-y\right)t^{-\beta}\left(1-t\right)^{-\beta}dt-\\ -\frac{\Gamma\left(2-2\beta\right)}{\Gamma^2\left(1-\beta\right)}\int\limits_0^1 v\left[x-\left(1-2\beta\right)\left(-y\right)^{1/(1-2\beta)}\left(2t-1\right)\right]\left(-y\right)t^{-\beta}\left(1-t\right)^{-\beta}dt\end{gathered} \tag{6}$$

were $\nu(x) = u_y(x,-0)$, $u(x,0) = \tau(x) = \Gamma(1-2\beta) D_{0x}^{2\beta-1} H(x)$.

From (6) we have:

$$u[\theta_0(x)] = \frac{\Gamma(1-\beta)}{2\cos\pi\beta} D_{0x}^{\beta-1} H(x) x^{-\beta} - \frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}} D_{0x}^{\beta-1} \nu(x) x^{-\beta} \tag{7}$$

$$u[\theta_{0k}(x)] = \frac{a^{1-\beta}\Gamma(1-\beta)}{2\cos\pi\beta} D_{0x}^{\beta-1} H(ax)(ax)^{-\beta} - \frac{a^{1-\beta}\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}} D_{0x}^{\beta-1} \nu(ax)(ax)^{-\beta}.$$

Substituting (7) into (4), we obtain a functional equation of the following form:

$$\Phi(x) + a^{1-2\beta}\omega(x)\Phi(ax) = \delta_1(x),\ 0 < x < 1, \tag{8}$$

where

$$\Phi(x) = \gamma_1 H(x) - \gamma_2 \nu(x), \tag{9}$$

$$\Phi(ax) = \gamma_1 H(ax) - \gamma_2 \nu(ax), \tag{10}$$

$$\delta_1(x) = x^{\beta}\delta(x), \gamma_1 = \frac{\Gamma(1-\beta)}{2\cos\pi\beta}, \gamma_2 = \frac{\Gamma(2-2\beta)}{[2(1-2\beta)]^{1-2\beta}\Gamma(1-\beta)},$$

We seek the function $\Phi(x)$ in a class of functions bounded at the point $x = 0$. Applying the iteration method [8] to the solution of the functional equation (8), for n-th iteration, we have:

$$\Phi(x) = (-a^{1-2\beta})^n A_n(x)\Phi(a^n x) + \sum_{j=0}^{n-1} (-a^{1-2\beta})^j A_j(x)\delta_1(a^j x), \tag{11}$$

where $A_n(x) = \omega(x)\,\omega(ax)...\omega(a^{n-1}x)$, $A_0(x) = 1$.

Assume that $\max\limits_{[0,1]} |\omega(x)| = M_0$ and $0 < M_0 < a^{2\beta-1}$. Then the following inequality is true:

$$|A_n(x)| \le M_0^n. \tag{12}$$

Passing in (11) to the limit when $n \to \infty$ and taking into account $0 < a < 1$, inequality (12) and the boundedness of the unknown function $\Phi(x)$, we obtain:

$$\Phi(x) = F_1(x),\ \ 0 \le x \le 1, \tag{13}$$

where

$$F_1(x) = \sum_{j=0}^{\infty} \left(-a^{1-2\beta}\right)^j A_j(x)\delta_1(a^j x). \tag{14}$$

Owing to $0 < a < 1$, (12) and the conditions imposed on $\delta_1(x)$, the series in the right part of equality (14) converge uniformly and $F_1(x) \in C[0,1]$, $F_1(0) = 0$.

Taking into account (9), from (13) we obtain a functional relation between $\tau(x)$ and $\nu(x)$ on AB taken from the domain Ω_2:

$$\nu(x) = \frac{\gamma_1}{\gamma_2} H(x) - \frac{1}{\gamma_2} F_1(x), 0 < x < 1. \tag{15}$$

Theorem. *The problem* $\mathbf{T}^{\infty}$ *cannot have more than one solution.*

Proof. Assume that $u(x,y)$ is the solution of a homogeneous problem $\mathbf{T}^{\infty}$. We have $F_1(x) \equiv 0$. Thus, the relation (15) takes the form

$$\nu(x) = \frac{\gamma_1}{\gamma_2} H(x), 0 < x < 1, \tag{16}$$

We prove that $u(x,y) \equiv 0$ in $\overline{\Omega}$. We suppose the opposite. Then there exists a domain $\Omega_{1\rho} = \{(x,y):\ 0<x<1,\ 0<y<\rho\}$ in which $u(x,y)0$. Consequently, $\sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| > 0$ and this value is attained at some point $(\xi,\eta) \in \overline{\Omega}_{1\rho}$.

We introduce a notation: $\partial\Omega_{1\rho} = AB \cup BD \cup DP \cup PA$ where

$$AB = \{(x,y):\ 0<x<1,\ y=0\},\ BD = \{(x,y):\ x=1,\ 0<y<\rho\},$$

$$DP = \{(x,y):\ 0<x<1,\ y=\rho\},\ PA = \{(x,y):\ x=0,\ 0<y<\rho\}.$$

On account of the extremum principle for elliptic equations [9] $(\xi,\eta) \notin \Omega_{1\rho}$. In consequence of the condition (2) and $\varphi_1(y) \equiv \varphi_2(y) \equiv 0$ it follows that $(\xi,\eta) \notin \overline{BD} \cup \overline{PA}$. Then $(\xi,\eta) \in AB \cup \overline{DP}$. Assume that $(\xi,\eta) \in AB$, i.e. $\sup\limits_{\bar{\Omega}_{1\rho}} |u(x,y)| = \sup\limits_{\overline{AB}} |u(x,y)| = |u(\xi,0)| > 0,\ 0<\xi<1$.

Then if $u(\xi,0) > 0$ (<0), $(\xi,0)$ is the point of positive maximum (negative minimum) of the function $u(x,y)$. If we reason by analogy with the papers [4],[7], we may prove that $u_y(\xi,0) > 0$ (<0). On the other side, owing to Zaremba-Zhiro principle [9], $u_y(\xi,0) < 0$ (>0). It follows from the obtained contradiction that $(\xi,\eta) \notin AB$.

Consequently, $(\xi,\eta) \in \overline{DP}$, i.e. $\sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| = \sup\limits_{0\le x\le 1} |u(x,\rho)| > 0$.

Taking an arbitrary number $\rho_1 > \rho$, we obtain the following by the same method:

$$\sup_{\overline{\Omega}_{1\rho_1}} |u(x,y)| = \sup_{0\le x\le 1} |u(x,\rho_1)| > 0.$$

As long as $\Omega_{1\rho} \subset \Omega_{1\rho_1}$, then $\sup\limits_{\overline{\Omega}_{1\rho_1}} |u(x,y)| \ge \sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| > 0$, i.e. $\sup\limits_{0\le x\le 1} |u(x,\rho_1)| \ge \sup\limits_{0\le x\le 1} |u(x,\rho)| > 0$. This implies $\lim\limits_{y\to\infty} u(x,y) \neq 0$, that contradicts to condition (3). Consequently, $u(x,y) \equiv 0$, $(x,y) \in \overline{\Omega}_1$. As long as $u(x,0) = \tau(x) \equiv 0$, then it follows from (16) that $\nu(x) \equiv 0$. Then, according to formula (6), $u(x,y) \equiv 0$ in $\overline{\Omega}_2$. Consequently, $u(x,y) \equiv 0$, $(x,y) \in \overline{\Omega}$. The theorem has been proved.

We prove the existence of a solution of problem $\mathbf{T}^\infty$ by the method of integral equations.

Solving a problem N in the domain Ω_1 by the method of Green functions, we obtain a functional relation between $\tau(x)$ and $\nu(x)$, brought on AB from the elliptic Ω_1 part of a mixed domain Ω which has the following form:

$$\tau(x) = -\int_0^1 \nu(t)\, G(x,t)dt + F_2(x), \tag{17}$$

$$G(x,t) = k_1 \Bigg[|x-t|^{-2\beta} - (x+t)^{-2\beta} +$$

$$+\sum_{n=1}^{\infty} \left[(2n-x+t)^{-2\beta} - (2n-x-t)^{-2\beta} + (2n+x-t)^{-2\beta} - (2n+x+t)^{-2\beta} \right] \Bigg],$$

$$F_2(x) = \int_0^\infty \eta^m \varphi_1(\eta)\, G_\xi(0,\eta;x,0)d\eta - \int_0^\infty \eta^m \varphi_2(\eta)\, G_\xi(1,\eta;x,0)d\eta.$$

Taking into account the conditions of gluing, (5), excluding the function $\tau(x)$ in (15) and (17) we obtain a singular integral equation in relation to an unknown function $\nu(x)$ in the following form:

$$\nu(x) + \gamma_3 \int_0^1 \nu(t) K(x,t)dt = F_3(x), \tag{18}$$

$$K(x,t)=\left(\frac{x}{t}\right)^{2\beta}\left[\frac{1}{t-x}+\frac{1}{t+x}-\sum_{n=1}^{\infty}\left(\frac{t}{2n-t}\right)^{2\beta}\left(\frac{1}{2n-t+x}+\frac{1}{2n-t-x}\right)\right]-$$

$$-\left(\frac{x}{t}\right)^{2\beta}\left[\left(\frac{t}{2n+t}\right)^{2\beta}\left(\frac{1}{2n+t-x}+\frac{1}{2n+t+x}\right)\right],$$

$$F_3(x)=\frac{1}{\gamma_2(1+\sin\pi\beta)}\left\{\gamma_1 D_{0x}^{1-2\beta}[F_2(x)]-F_1(x)\right\},$$

$$\gamma_3=\cos\pi\beta/[\pi(1+\sin\pi\beta)].$$

Equation (18) is reduced to a singular integral equation with Cauchy type kernel. Applying the well-known Carleman-Vekua regularization method [10], we arrive at the equivalent in the sense of solvability Fredholm equation of the second kind the unconditional solvability of which follows from the uniqueness of the problem solution. □

Conclusions

The results of the work were obtained by extremum principle method, integro-differential operators, Green function method, and integral equation theory methods.

An analogue of the problem with shift with the conditions of A.M. Nakhushev for equation (1) in a mixed domain were considered. The nonlocal condition is defined on a characteristic of one family. The elliptic part of the domain under analysis is a vertical half-strip. Green function of problem N for this domain was constructed. One-valued solvability of the assigned problems was proved.

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