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ON SOLVABILITY OF THE BOUNDARY-VALUE PROBLEM FOR ONE EVEN-ORDER EQUATION

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The paper considers a boundary-value problem for one even-order equation. A unique solvability of the problem under additional conditions and conditions for a domain is proved.

Key words: partial differential equations of higher order, boundary-value problem, method of separation of variables, continued fractions

Statement of the problem

For the equation

$$Lu = f(x,t), \tag{1}$$

where

$$Lu \equiv \frac{\partial^{2k} u}{\partial x^{2k}} - \frac{\partial^{2p} u}{\partial t^{2p}}, \ \ k,p \in N \tag{2}$$

in the domain $\Omega = \{(x,t) : 0 < x < a, \ 0 < t < T\}$, the following boundary-value problem is under investigation.

Task. *In the domain* Ω*, we find a solution* $u(x,t)$ *of equation (1) satisfying the edge conditions*

$$\frac{\partial^{2m} u}{\partial x^{2m}}(0,t) = \frac{\partial^{2m} u}{\partial x^{2m}}(a,t) = 0, 0 \le t \le T, m = \overline{0,k-1} \tag{3}$$

and

$$\frac{\partial^{2m} u}{\partial t^{2m}}(x,0) = \frac{\partial^{2m} u}{\partial t^{2m}}(x,T) = 0, 0 \le x \le a, m = \overline{0,p-1}. \tag{4}$$

We should note that the problem correctness depends on the values of k,p. So, if $(k-p)$ is even, the problem is ill-conditioned. The stability of such a problem was studied in the paper [1]. If $k = p = 1$, the problem (1),(3), (4) is a Dirichlet problem for string vibration equation the solvability of which was investigated in [3]. The case when $k \in N$, $p = 1$ was considered in [2, 7], and the case when $k = p$ was discussed in [4, 5].

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The case of odd $(k-p)$

Lemma 1. *We assume that $u(x,t)$ is a regular solution of problem (1),(3), (4) and $(k-p)$ is odd, then the following calculation is true*

$$\|u\|_{W_2^{k,p}(\Omega)} \leq C\|Lu\|_{L_2(\Omega)}, \tag{5}$$

where $C>0$ is a constant depending on the size of the domain Ω and k,p, but independent on the function $u(x,t)$.

Proof. We multiply the both parts of equation (2) by $(-1)^k u$ and integrate over the domain Ω

$$\int_0^T\int_0^a \left[\sum_{i=0}^{k-1}(-1)^{k+i}\frac{\partial}{\partial x}\left(\frac{\partial^{2k-1-i}u}{\partial x^{2k-1-i}}\cdot\frac{\partial^i u}{\partial x^i}\right)+\left(\frac{\partial^k u}{\partial x^k}\right)^2-\right.$$
$$\left.-\sum_{i=0}^{p-1}(-1)^{k+i}\frac{\partial}{\partial t}\left(\frac{\partial^{2p-1-i}u}{\partial t^{2p-1-i}}\cdot\frac{\partial^i u}{\partial t^i}\right)-(-1)^{k+p}\left(\frac{\partial^p u}{\partial t^p}\right)^2\right]dxdt=\iint\limits_{\Omega}(-1)^k uLudxdt.$$

Applying the inequality $2|ab|\leq\varepsilon a^2+\dfrac{b^2}{\varepsilon}$ $(\varepsilon=const>0)$ to the right part of the latest equality and taking the conditions (3) and (4) into account, we have

$$\int_0^a\int_0^T\left(\left(\frac{\partial^k u}{\partial x^k}\right)^2+\left(\frac{\partial^p u}{\partial t^p}\right)^2\right)dxdt\leq\frac{\varepsilon}{2}\int_0^a\int_0^T u^2(x,t)\,dxdt+\frac{1}{2\varepsilon}\int_0^a\int_0^T(Lu)^2dxdt,$$

we obtain

$$\left\|\frac{\partial^k u}{\partial x^k}\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial^p u}{\partial t^p}\right\|^2_{L_2(\Omega)}\leq\frac{\varepsilon}{2}\|u\|^2_{L_2(\Omega)}+\frac{1}{2\varepsilon}\|Lu\|^2_{L_2(\Omega)} \tag{6}$$

By definition, the norm $\left\|\dfrac{\partial^m u}{\partial x^m}\right\|^2_{L_2(\Omega)}$, $m=1,\cdots,k-1$ equals

$$\left\|\frac{\partial^m u}{\partial x^m}\right\|^2_{L_2(\Omega)}=\int_0^T\int_0^a\frac{\partial^{m+1}u}{\partial x^{m+1}}\cdot\frac{\partial^{m-1}u}{\partial x^{m-1}}dxdt.$$

Applying to the right part of the latest equality the following inequality

$$|ab|\leq\frac{1}{2}\left(a^2+b^2\right),$$

we find

$$\left\|\frac{\partial^m u}{\partial x^m}\right\|^2_{L_2(\Omega)}\leq\frac{1}{2}\left\|\frac{\partial^{m-1}u}{\partial x^{m-1}}\right\|^2_{L_2(\Omega)}+\frac{1}{2}\left\|\frac{\partial^{m+1}u}{\partial x^{m+1}}\right\|^2_{L_2(\Omega)}. \tag{7}$$

Summing the inequalities (7) over m from 1 to $k-1$, we obtain

$$\left\|\frac{\partial u}{\partial x}\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial^{k-1}u}{\partial x^{k-1}}\right\|^2_{L_2(\Omega)}\leq\|u\|^2_{L_2(\Omega)}+\left\|\frac{\partial^k u}{\partial x^k}\right\|^2_{L_2(\Omega)}. \tag{8}$$

Applying (7) to inequality (8), we find

$$\left\|\frac{\partial u}{\partial x}\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial^{k-1}u}{\partial x^{k-1}}\right\|^2_{L_2(\Omega)}\leq\frac{\varepsilon}{2}\|u\|^2_{L_2(\Omega)}+\frac{1}{2\varepsilon}\|Lu\|^2_{L_2(\Omega)}. \tag{9}$$

Now summing (7) over m from 2 to $k-2$, according to (9), we have

$$\left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial^{k-2} u}{\partial x^{k-2}}\right\|^2_{L_2(\Omega)}\le\frac{\varepsilon}{2}\|u\|^2_{L_2(\Omega)}+\frac{1}{2\varepsilon}\|Lu\|^2_{L_2(\Omega)}. \tag{10}$$

If we do the same further, we obtain estimates for all $\left\|\frac{\partial^m u}{\partial x^m}\right\|^2_{L_2(\Omega)}$, $m=1,...,k-1$, summing up which with (6), (9), (6), we obtain

$$\sum_{i=1}^{k}\left\|\frac{\partial^i u}{\partial x^i}\right\|^2_{L_2(\Omega)}\le\frac{\varepsilon k}{4}\|u\|^2_{L_2(\Omega)}+\frac{k}{4\varepsilon}\|Lu\|^2_{L_2(\Omega)}. \tag{11}$$

If we do the same, substituting x by t, and k by p, we find

$$\sum_{i=1}^{p}\left\|\frac{\partial^i u}{\partial t^i}\right\|^2_{L_2(\Omega)}\le\frac{\varepsilon p}{4}\|u\|^2_{L_2(\Omega)}+\frac{p}{4\varepsilon}\|Lu\|^2_{L_2(\Omega)}. \tag{12}$$

We estimate $\|u\|^2_{L_2(\Omega)}$

$$u^2(x,t)=\int_0^x\frac{\partial}{\partial\xi}\left(u^2(\xi,t)\right)d\xi\le 2\int_0^a|u|\,|u_x|\,dx,$$

$$\int_0^T u^2(x,t)dt\le 2\int_0^T\int_0^a|u|\,|u_x|\,dxdt\le 2\|u\|_{L_2(\Omega)}\cdot\|u_x\|_{L_2(\Omega)}.$$

Integrating the latest inequality over x from 0 to p and dividing by $\|u\|_{L_2(\Omega)}$, we obtain

$$\|u\|^2_{L_2(\Omega)}\le 4a^2\|u_x\|^2_{L_2(\Omega)}.$$

According to (9) we find

$$\|u\|^2_{L_2(\Omega)}\le 4a^2\left(\frac{\varepsilon}{2}\|u\|^2_{L_2(\Omega)}+\frac{1}{2\varepsilon}\|Lu\|^2_{L_2(\Omega)}\right).$$

Summing up the latest inequality with (11) and (12) and choosing

$$\varepsilon=\frac{2}{k+p+8a^2},$$

we obtain

$$\sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial x^i}\right\|^2_{L_2(\Omega)}+\sum_{i=1}^{p}\left\|\frac{\partial^i u}{\partial t^i}\right\|^2_{L_2(\Omega)}\le C^2\|Lu\|^2_{L_2(\Omega)}. \tag{13}$$

where $C^2=\frac{\left(k+p+8a^2\right)^2}{8}$. Extracting the root from (13), we obtain

$$\|u\|_{W_2^{k,p}(\Omega)}\le C\|Lu\|_{L_2(\Omega)}.$$

□

CONSEQUENCE. From Lemma 1 follows uniqueness and continuous dependence of the problem regular solution on the right part $f(x,t)$ when $(k-p)$ values are odd.

Theorem 1. *Assume that $(k-p)$ is odd and the function $f(x,t)\in C^{k+1,p+1}_{x,\,t}(\Omega)$ satisfies the following edge conditions*

$$\frac{\partial^{2m}f(0,t)}{\partial x^{2m}}=\frac{\partial^{2m}f(a,t)}{\partial x^{2m}}=0,\forall t\in[0;T],$$

$m=0,1,\cdots,\frac{k}{2}$, *if k is even and* $m=0,1,\cdots,\frac{k-1}{2}$, *if k is odd*

$$\frac{\partial^{2m} f(x,0)}{\partial t^{2m}}=\frac{\partial^{2m} f(x,T)}{\partial t^{2m}}=0, \forall x \in [0;a],$$

$m=0,1,\cdots,\frac{p}{2}$, *if p is even and* $m=0,1,\cdots,\frac{p-1}{2}$, *if p is odd. Than the solution of problem (1), (3),(4) exists.*

Proof. The function satisfying the conditions (3), (4) may be represented in the form

$$u(x,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2u_{nm}}{\sqrt{aT}}\sin\frac{\pi n}{a}x\cdot\sin\frac{\pi m}{T}t. \tag{14}$$

Substituting into (1), we have

$$\left[(-1)^k\left(\frac{\pi n}{a}\right)^{2k}-(-1)^p\left(\frac{\pi m}{T}\right)^{2p}\right]u_{nm}=f_{nm}, \tag{15}$$

where $f_{nm}=\iint\limits_{\Omega}\frac{2}{\sqrt{aT}}\sin\left(\frac{x\pi n}{a}\right)\sin\left(\frac{t\pi m}{T}\right)f(x,t)\,dxdt$. From (14) with even k and odd p we find

$$u_{nm}=\frac{f_{nm}}{\left[\left(\frac{\pi n}{a}\right)^{2k}+\left(\frac{\pi m}{T}\right)^{2p}\right]},$$

and with odd k and even p

$$u_{nm}=-\frac{f_{nm}}{\left[\left(\frac{\pi n}{a}\right)^{2k}+\left(\frac{\pi m}{T}\right)^{2p}\right]}.$$

Thus, the formal solution of problem (1),(3), (4) is

$$u(x,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2}{\sqrt{aT}}\cdot\frac{(-1)^k f_{nm}}{\left[\left(\frac{\pi n}{a}\right)^{2k}+\left(\frac{\pi m}{T}\right)^{2p}\right]}\sin\frac{\pi n}{a}x\cdot\sin\frac{\pi m}{T}t. \tag{16}$$

Now we have to prove the uniform convergence of series (16) and that of the series

$$\frac{\partial^{2k}u}{\partial x^{2k}}(x,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2}{\sqrt{aT}}\cdot\frac{f_{nm}}{\left[\left(\frac{\pi n}{a}\right)^{2k}+\left(\frac{\pi m}{T}\right)^{2p}\right]}\left(\frac{\pi n}{a}\right)^{2k}\sin\frac{\pi n}{a}x\cdot\sin\frac{\pi m}{T}t, \tag{17}$$

$$\frac{\partial^{2k}u}{\partial t^{2p}}(x,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2}{\sqrt{aT}}\cdot\frac{-f_{nm}}{\left[\left(\frac{\pi n}{a}\right)^{2k}+\left(\frac{\pi m}{T}\right)^{2p}\right]}\left(\frac{\pi m}{T}\right)^{2p}\sin\frac{\pi n}{a}x\cdot\sin\frac{\pi m}{T}t \tag{18}$$

On account of the estimation

$$\left(\frac{\pi n}{a}\right)^{2k}+\left(\frac{\pi m}{T}\right)^{2p}\geq 2\frac{\pi^{k+p}n^k m^p}{a^k T^p},$$

we find

$$|u_{nm}|\leq\frac{a^kT^p}{2\pi^{k+p}n^k m^p}|f_{nm}|.$$

Then the series majorant to series (16) takes the form

$$\frac{a^kT^p}{\pi^{k+p}\sqrt{aT}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|f_{nm}|n^k}{m^p},$$

Owing to the conditions imposed on the function $f(x,t)$, it fallows that series (16), (17) and (18) have uniform convergence. The theorem has been proved. □

The case of even $(k-p)$.

Theorem 2. *Assume that $(k-p)$ is even and the function $f(x,t) \in C^{2k+5,\,2p+3}_{x,\quad t}(\Omega)$ satisfies the following edge conditions*

$$\frac{\partial^{2m} f(0,t)}{\partial x^{2m}} = \frac{\partial^{2m} f(a,t)}{\partial x^{2m}} = 0, \forall t \in [0;T],$$

$m = 0,1,\cdots,k+1$, if k is even and $m = 0,1,\cdots,k+2$, if k is odd,

$$\frac{\partial^{2m} f(x,0)}{\partial t^{2m}} = \frac{\partial^{2m} f(x,T)}{\partial t^{2m}} = 0, \forall x \in [0;a],$$

$m = 0,1,\cdots,p$ if p is even and $m = 0,1,\cdots,p+1$, if p is odd. If the numbers a and T are such, the number $\frac{a^k}{T^p \pi^{k-p}}$ is a literal number of $n \geq 2$ degree, then the solution of the problem (1), (3),(4) exists and it is unique.

Proof. Similar to theorem 1, we seek the solution in the form of (14). Substituting it into (1), we obtain (15).

From (15) when k and p are even, we find

$$u_{nm} = \frac{f_{nm}}{\left[\left(\frac{\pi n}{a}\right)^{2k} - \left(\frac{\pi m}{T}\right)^{2p}\right]},$$

and when k and p are odd,

$$u_{nm} = -\frac{f_{nm}}{\left[\left(\frac{\pi n}{a}\right)^{2k} - \left(\frac{\pi m}{T}\right)^{2p}\right]}.$$

Thus, the formal solution of problem (1), (3), (4), when $(k-p)$ values are even, takes the form

$$u(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{2}{\sqrt{aT}} \cdot \frac{(-1)^k f_{nm}}{\left[\left(\frac{\pi n}{a}\right)^{2k} - \left(\frac{\pi m}{T}\right)^{2p}\right]} \sin\frac{\pi n}{a} x \cdot \sin\frac{\pi m}{T} t \tag{19}$$

Now we have to prove the uniform convergence of the series (19), and that of the series obtained from (19) by differentiation $2k$ times with respect to x and $2p$ times with respect to t.

First, we estimate the difference

$$\left[\left(\frac{\pi n}{a}\right)^{2k} - \left(\frac{\pi m}{T}\right)^{2p}\right] = \left[\left(\frac{\pi n}{a}\right)^{k} - \left(\frac{\pi m}{T}\right)^{p}\right] \cdot \left[\left(\frac{\pi n}{a}\right)^{k} + \left(\frac{\pi m}{T}\right)^{p}\right] =$$

$$= \frac{\pi^k m^p}{a^k}\left[\frac{n}{m^p}^{k} - \frac{a^k}{T^p \pi^{k-p}}\right] \cdot \left[\left(\frac{\pi n}{a}\right)^{k} + \left(\frac{\pi m}{T}\right)^{p}\right].$$

In view of the fact that the number $\frac{a^k}{T^p \pi^{k-p}}$ is a literal number of $n \geq 2$ degree, it satisfies the following estimate

$$\left|\frac{n}{m^p}^{k} - \frac{a^k}{T^p \pi^{k-p}}\right| \geq \frac{C}{m^{2p}}.$$

Then we obtain,

$$\left[\left(\frac{\pi n}{a}\right)^{2k} - \left(\frac{\pi m}{T}\right)^{2p}\right] \geq \frac{C_0\sqrt{\left(\frac{\pi n}{a}\right)^{2k} + \left(\frac{\pi m}{T}\right)^{2p}}}{m^{\frac{p}{2}}}$$

or owing to the fact that the number $\frac{T^p\pi^{k-p}}{a^k}$ is also a literal number of $n \geq 2$ degree, we find

$$\left[\left(\frac{\pi n}{a}\right)^{2k} - \left(\frac{\pi m}{T}\right)^{2p}\right] \geq \frac{C_1\sqrt{\left(\frac{\pi n}{a}\right)^{2k} + \left(\frac{\pi m}{T}\right)^{2p}}}{n^{\frac{k}{2}}}.$$

That means that

$$\left[\left(\frac{\pi n}{a}\right)^{2k} - \left(\frac{\pi m}{T}\right)^{2p}\right] \geq \min\left(\frac{C_0\sqrt{\left(\frac{\pi n}{a}\right)^{2k} + \left(\frac{\pi m}{T}\right)^{2p}}}{m^{\frac{p}{2}}}; \frac{C_1\sqrt{\left(\frac{\pi n}{a}\right)^{2k} + \left(\frac{\pi m}{T}\right)^{2p}}}{n^{\frac{k}{2}}}\right).$$

Thus,

$$|u_{nm}| \leq 2\frac{Cn^k m^p}{\left(\frac{\pi n}{a}\right)^{2k} + \left(\frac{\pi m}{T}\right)^{2p}} |f_{nm}|.$$

On account of the function $f(x,t)$, the series (19) has the uniform convergence. The theorem has been proved. □

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