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FINDING OF A SUBGROUP INDEX AND THE OCCURRENCE PROBLEM

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For separate classes of groups, the relation between two algorithmic problems, the problem of calculation of a subgroup index and the occurrence problem, is revealed

Key words: group, subgroup, subgroup index, algorithmic problem, free product, direct product, occurrence problem.

Introduction

The problem of index for a finitely defined group G is the question of existence of an algorithm allowing us to determine by any finite set of elements $h_i (i = 1, 2, \ldots, m)$ of group G if subgroup $H = \mathrm{gr}(h_1, h_2, \cdots, h_m)$ generated by this set has a finite or an infinite index in G.

There is only a finite number of groups for each given finite index in a finitely generated group. Thus, if the occurrence problem and the index problem are solvable in group G, then having obtained the information that the index of subgroup H in G is finite, this index may be accurately estimated by a simple exhaustion of the finite index subgroups to the finite number of steps (for example, see details in the paper [1], Chapter 1, § 8, pp. 42–44).

A special case of estimation of an index is the determination of an index of an identity subgroup of group G, i.e. finding of the order of group G. The property of "being finite"for a group is a Markov property. Thus, in a class of all finitely defined, there is no algorithm for detection if this group is finite or infinite (considered in the paper by S.Adyan [**?**] for the first time).

Examples of groups with solvable problem of index

For a concrete finitely defined group G, estimation of its order is not a mass problem. However, if G is finite, then it contains subgroups of infinite and finite indexes. Such indexes have, for example, trivial subgroups. It is possible, that there are other subgroups of both finite and infinite indexes in G.

If G is an infinite simple finitely defined group, then the solvability of index problem follows from the solvability of occurrence problem in G as long as there is only one subgroup of a finite index, G itself, in such a group.

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However, a reversed situation with simple groups is significantly more complicated. Each countable group is isomorphically embeddable into a two-generated simple group (considered in the paper [3] for the first time). In particular, a finitely defined group S with unsolvable equality problem (consequently, with unsolvable occurrence problem) is also isomorphically embedded into some simple two-generated group G. It was determined in the paper by Kyznetsov [4] that the equality problem is solvable in every recursively defined group. It means that a two-generated simple group containing such S is not only finitely defined, but it cannot be recursively presented.

Practically, the occurrence problem is discussed within the course of linear algebra. The compatibility criterion for a linear equation system is the solution of the occurrence problem in finitely dimensional vector spaces. An algorithm based on this criterion is the calculation of the dimension of two additional subspaces. This dimension is determined by sequential change of generating sets of subspaces.

In order to find an index of subgroup H in group G, we can change the generating set for H. The generating set of the subgroup is modified by transformations analogous to elementary transformations of a generating set of vector space subspace. The transformations in the group are the following:

– change of element x by x^{-1};

– change of element x by element xy, where $x \neq y$;

– removal of an identity element.

For example, if H is a subgroup of an infinite cyclic group $F_1 = < a >$, and $H = \mathrm{gr}(a^{m_1}, a^{m_2}, \cdots, a^{m_k})$, where $m_1, m_2, \cdots, m_k \in Z$, then by elementary transformations it is possible to find a unique generating the subgroups H which is equal to a^s, where $s = \mathrm{GCD}(m_1, m_2, \cdots, m_k)$. The index is equal to s. Thus, the problem of finding a subgroup index of an infinite cyclic group is reduce to the finding of the greatest common divisor for a finite set of integer numbers. In other words, the problem of index for an infinite cyclic group is algorithmically solvable.

An infinite cyclic group is a special case of a free group. Group $< a >= F_1$ is a free group of rank 1. The problem of calculation of an arbitrary subgroup index of a free group F_r of any rank r can also have an algorithmic solution.

Assume that $H = \mathrm{gr}(h_1, h_2, \cdots, h_m)$ is a finitely generated subgroup of a free group F_n. We can obtain free generators for subgroup H, so finding the rank of H, by transformations of a generating set into the finite number of steps. Such a way of obtaining of free generators of a free group is generally called *Nielsen method* (for example, see details in [5], Chapter 1, Section 2, pp. 16–21).

O. Schreier operated not only the generating elements of a subgroup, but also the representatives of cosets. He ascertained the relation between a subgroup index of a free group, the rank of this group and the rank of the initial free group (for example, see [5], Chapter 1, Section 3, pp. 33–34). If a subgroup H of rank k has a finite index in a free noncyclic group of rank r, then this index is equal to

$$\frac{k-1}{r-1}.$$

Based on Schreier formula, the index problem is reduced to the calculation of a subgroup rank which can be found by Nielsen method (see in detail in the paper by Karrass and Solitar [6]). Thus, the problem of index in free groups is *algorithmically solvable*.

Examples of groups with unsolvable problem of index

We will show that the problem of index is algorithmically solvable not in every finitely defined group.

Theorem 1. *The problem of index finiteness is unsolvable in a direct product of two free noncyclic groups of the same rank.*

Proof. Assume that group G is the direct product of two free groups $A =< a_1, a_2, \cdots, a_m >$ and $B =< b_1, b_2, \cdots, b_m >$, where $m \geq 2$.

We consider an arbitrary finitely defined group R given by the presentation

$$R =< r_1, r_2, \cdots, r_k; w_1(r_i), \cdots, w_n(r_i) > .$$

Assume that a number s satisfies the inequality

$$s \geq \frac{k-1}{m-1}.$$

There are subgroups of any finite index in group A. We choose a subgroup P of index s in A. Based on Scheier's formula, subgroup P rank is equal to $s(m-1)+1$, and

$$s(m-1)+1 \geq k.$$

If the subgroup P rank turns to be strictly greater than k, the presentation of group R is enlarged $s-k$ by generating elements and these elements are set to a unit. Without loss of generality, we may consider that it has been done, i.e. $s = k$. Assume that P elements of $p_1, p_2, \cdots, p_k$ generate the subgroup P freely:

$$P =< p_1, p_2, \cdots, p_k > .$$

In group B we choose a subgroup Q of rank k, of index s in B and with free generating elements $q_1, q_2, \cdots, q_k$:

$$Q =< q_1, q_2, \cdots, q_k > .$$

The group G subgroup generated by subgroups P and Q is isomorphic to the direct product $P \times Q$ and has a finite index in group G.

We consider a normal closure of elements $w_1(p_i), \cdots, w_n(p_i)$ in group P:

$$H_1 = \mathrm{gr}(w_1(p_i), \cdots, w_n(p_i), p_1q_1, \cdots, p_kq_k).$$

The elements r_i, q_i are in different direct factors of group G, so they are commutative:

$$r_iq_i = q_ir_i.$$

That means that for any word φ the following equality holds:

$$\varphi(p_iq_i) = \varphi(p_i)\varphi(q_i)$$

that is why for any $w_j(p_i)$ we have the equality:

$$\varphi^{-1}(p_iq_i)w_j(p_i)\varphi(p_iq_i) = \varphi^{-1}(q_i)\varphi^{-1}(p_i)w_j(p_i)\varphi(p_i)\varphi(q_i) = \varphi^{-1}(p_i)w_j(p_i)\varphi(p_i).$$

Thus, subgroup N_1 is contained in subgroup H_1:

$$N_1 \subset H_1 \cap P.$$

On the other side, if $\varphi(w_n(p_i), p_iq_i)$ belongs to P, the sum of powers in the word φ for each q_i equals zero and that means that $\varphi(w_j(p_i), p_iq_i) \in N_1$. Thus,

$$N_1 = H_1 \cap P.$$

Now we make similar constructions in the second direct factor. Assume that N_2 is a normal closure of elements $w_1(q_i), \cdots, w_n(q_i)$ in group Q:

$$N_2 =< w_1(q_i), \cdots, w_n(q_i) >$$

and

$$H_2 = \mathrm{gr}(w_1(q_i), \cdots, w_n(q_i), p_1q_1, \cdots, p_kq_k).$$

Just like for groups H_1, N_1 and P, we obtain for groups H_2, N_2 and Q:

$$N_2 = H_2 \cap Q.$$

For any $j = 1, 2, \cdots, n$ we have:

$$w_j(p_iq_i) = w_j(p_i)w_j(q_i),$$

and, consequently,

$$w_j(q_i) = w_j^{-1}(p_i)w_j(p_iq_i).$$

That means that $H_2 \subset H_1$. By the same reasoning, the reverse inclusion is true: $H_1 \subset H_2$. Groups H_1 and H_2 coincide. We symbolize them by one letter H:

$$H = H_1 = H_2.$$

Intersections with subgroups P and Q,

$$H \cap P = N_1, H \cap Q = N_2.$$

If group R is finite, indexes N_1 and N_2 in subgroups P and Q are finite but P and Q are the subgroups of a finite index in direct factors and, consequently, index $[G : H]$ is finite.

And vice versa, if index H in group G is finite, then index N_1 in subgroup P is finite and, consequently, group R is finite.

Thus, the problem of index in group G is equivalent to the problem of finiteness in the class of all finitely defined groups. The problem of finiteness is unsolvable and that means that the problem of index for a finitely defined group is also algorithmically unsolvable. □

Algorithmic unsolvability of a problem means, in particular, that there is no computer solution for such a problem. For example, no one computer will ever be able to answer always unambiguously by a single program if an index is finite or infinite in an arbitrary defined finitely generated subgroup in the group

$$G = F_2 \times F_2 =< a, b, c, d; aca^{-1}c^{-1}, ada^{-1}d^{-1}, bcb^{-1}c^{-1}, bdb^{-1}d^{-1} > .$$

However, we should note, that in some cases, estimation of a subgroup index in a finitely defined group can be carried out by a computer. For example, a package of symbolic calculations Maple 18 allows us sometimes to estimate a subgroup index in a finitely defined group when this index is finite and does not exceed 128000. Though, computation results require additional "manual"check. Examples of such calculations and manual checks are described in the papers [7] – [10].

S. Mikhailova showed in the paper [11] that for group $F_2 \times F_2$ the occurence problem is unsolvable. The proof (§ 2, theorem 1, pp. 242–244 in [11]) is easily applied in a more general case of direct product $F_r \times F_r$ of two free groups of rank $r \geq 2$. Thus, an infinite series of finitely defined groups arises for which the occurrence problem and the problem of index turned to be equivalent (both unsolvable).

We should note that the occurrence problem and the problem of index are solvable in free groups, however the direct product maintained neither.

Free product, in contrast to the direct one, maintains the solvability of the occurrence problem. Solvability of occurrence problem in a free product follows from the solvability of occurrence problem in the factors A, B (Mikhailova, [12]).

The relation of the problem of index and the occurrence problem for freely decomposed groups

D. Moldavanskiy in the paper [13] refined the results of Mikhailova on the solution of occurrence problem in a free product by a method close to Nielsen method for free groups.

Assume that W is some set of words from a free product $G=A*B$. We extend the set W to set $W^{\pm1}$ closed in relation to the conversion:

$$W^{\pm1}=\{g\mid g\in W \text{ or } g^{-1}\in W\}.$$

The initial segment of element g from W is called isolated in W if it is not an initial segment of any other element from $W^{\pm1}$. Assume that $W_v(X)$ is a set of all elements from W having the form vxv^{-1} where $x\in X(X=A$ or $X=B)$. The pair (v,X) is called *a transform type* from $W_v(X)$. The symbol $S(v,X)$ denotes a set of all elements from the factor X which are $(l(v)+1)$ a syllable of some element g from the set $(W\setminus W_v(X))^{\pm1}$ and the initial segment of element g is v, i.e. the irreducible form g takes the form: $g=vsz$ where $s\in S(v,X)$.

According to [9], we call the set of elements W from a free product *the Nielsen set*, if:

– a large initial segment of each element from $W^{\pm1}$ is isolated in W;

– the left half of each element of even length from $W^{\pm1}$ is isolated in W;

– for each type (v,X) the set $S(v,X)$ does not contain elements from subgroup $v^{-1}\mathrm{gr}(W_v(X))v$, and set $S(v,X)$ is composed of the representatives of different right cosets of the group with respect to subgroup $v^{-1}\mathrm{gr}(W_v(X))v$.

– left half of each element from $W^{\pm1}$ which is not a transform is isolated in W;

Just like for free groups, Nielsen transformations of set M of elements of the free product G is:

– the change of some element x from M by element x^{-1},

– the change of some element x by element xy^{ε}(where $y\in M, y\neq x, \varepsilon=\pm1$).

– deletion of a unit.

It is determined by the induction on the total length of all the words of set W that any finite set W may be turned into Nielsen set by a finite sequence of transformations. The procedure of transformation is effective if the occurrence problem is solvable in free factors. The properties of Nielsen set mean that the obtained generating groups H are the generators of Kurosh-MacLean decomposition of this subgroup (Moldavanskiy [13] and [14]).

Hence:

– if in groups A and B, the occurrence problem is solvable, there is an efficient procedure of transition which converts any set of elements W of group G into Nielsen set W_1;

– solvability of occurrence problem in group G follows from the solvability of occurrence problem in groups A and B;

– if W_1 is a Nielsen set of generators for subgroup H of group G, then H is a free product of the groups generated by transforms of one type and infinite cyclic groups generated by elements from W_1 which are not transforms. This decomposition for H is the Kurosh-MacLean decomposition.

Theorem 2. *If in groups A, B, the occurrence problem is solvable, the problem of index in a free product $G=A*B$ is solvable .*

Proof.

Assume that W is some finite set of elements from group G; and H is a subgroup generated by set W. As long as there is an effective procedure converting each finite set into a Nielsen one, we can consider W to be a Nielsen set.

Assume that $v_iA_iv_i^{-1}$, $i=1,2,\cdots,m$, and $w_jB_jw_j^{-1}$, $j=1,2,\cdots,n$ are subgroups generated by the types of transforms (v_i,A_i) and (w_j,B_j), respectively, and F is a free group generated by elements from W which are not transforms. Then according to Kurosh theorem on the subgroups of a free product, there are such systems of representatives of double cosets $s_A(HgA)$ and $s_B(HgB)$ that

– $s_A(HA) = s_B(HB) = 1$;

– if $HgA \neq HA$, then $s_A(HgA)$ is ended by B-syllable, and if $HgB \neq HB$, then $s_A(HgB)$ is ended by A- syllable;

– group H is a free product;

$$H = F * \prod_{X \in \{A,B\}, g \in G} * s_X(HgX)X\left[s_X(HgX)^{-1}\right] \cap H,$$

where F is a free group not containing a conjugation from groups A, B.

It was shown in the paper [1] that if G is a nontrivial free product $A * B$, and H is a finitely decomposable group in G, the index of decomposition $[G : (H,A)]$ in double module is finite only when the index $[G : H]$ is finite ([1], Chapter 1, Paragraph 2, pp. 15 – 17).

We consider decomposition of group G in double module

$$G = Hg_1A + Hg_2A + \cdots$$

If set $\{g, g_2, \cdots\}$ forms a complete system of representatives for decomposition $G \mod (H,A)$, then there is such a subset $\{g_{\alpha_1}, \cdots, g_{\alpha_m}\}$ in this set that for $i = 1, 2, \cdots, m$

$$g_{\alpha_i} = v_i \mod (H,A).$$

Moreover, for each element g from G, if g is comparable in $\mod (H,A)$ with some element from the difference

$$Y = \{g, g_2, \cdots\} \setminus \{g_{\alpha_1}, \cdots, g_{\alpha_m}\},$$

then the intersection $H \cap gAg^{-1}$ is equal to a unit subgroup. Similar statement is applied for subgroup B. Now we consider three cases.

Case 1. Both free factors A and B are infinite groups.

Assume that the rank of a free group F in the decomposition for subgroup H is equal to r. The number r just like the numbers m, n are effectively calculable. We denote

$$k = m + n + r - 1.$$

At a fixed decomposition $G = A * B$, the number k is the invariant of all possible Kurosh decompositions for subgroup H. In particular, if

$$H = F_1 * \prod_{v} * H_v,$$

is the MacLean decomposition for the group H, the rank of subgroup F_1 also equals r.

It was determined in the paper by Kun [15] that MacLean decomposition has the following property: if subgroup H has a finite index in a free product $A * B$, the rank of a free part F_1 for decomposition H is equal to

$$[G : H] - [G : (H,A)] - [G : (H,B)] + 1.$$

We should note now that in the case under consideration, it follows from the finiteness of index H in G that $[G : (H,A)] = m$, and $[G : (H,B)] = n$.

We assume that $[G : (H,A)] > m$. Then set Y is not empty, consequently, there is such an element g from G that the intersection $H \cap gAg^{-1}$ is unique.

It follows from the infinity of group A that the index of group H in group G is infinite.

The same arguments may be applied to group B.

We should also note that we can consider that $k > 0$. Indeed, if it turned out that $k = 0$ then H is an infinite cyclic group or a subgroup from a factor conjugation and, consequently, index H in G is infinite. We consider the set $\mathfrak{R}$ of group G subgroups of index k in G,

$$\mathfrak{R} = K \leq G \mid [G:K] = k.$$

The set $\mathfrak{R}$ is finite an we may effectively find system sets generating for the subgroups from $\mathfrak{R}$. From the solvability of occurrence problem in group G follows the solvability of occurrence problem of a arbitrary finitely generated subgroup P from G into the set $\mathfrak{R}$. Now if H has a finite index in G, the following equality is fulfilled

$$r = [G:H] - m - n + 1,$$

i.e. H belongs to $\mathfrak{R}$. In other words, subgroup H has a finite index in G only when H belong to $\mathfrak{R}$. This is the end of discussion of Case 1.

Case 2. Group G is a free product $A*B$ of a finite group A and of infinite group B.

We consider a normal closure $\bar{B}$ of a free factor B in group G. Subgroup $\bar{B}$ is a free product of conjugations of the subgroup by the elements from A,

$$\bar{B} = \prod_{a\in A} *B^a,$$

i.e. it is a free product

$$\bar{B} = B * \prod_{a\in A\setminus\{1\}} *B^a,$$

where the both factors are infinite groups.

Denote by R a subgroup of group G, generated by subgroups H and $\bar{B}$, and by D we denote the intersection $\bar{B}\cap H$. Subgroup R has a finite index in G. As long as the occurrence problem is solvable in group G, the generators for R can be effectively found. Moreover, we can also effectively find the set $r_1, r_2, \cdots, r_s$ which is the complete system of representatives of right cosets for $R \mod \bar{B}$. The intersections $H \cap Dr_i, i = 1, \cdots, s$ are not empty. If we choose a system of elements $h_i, \cdots, h_s$ by one from each set, this set forms a complete system of representatives of right-sides decomposition $H \mod D$. From the solvability of occurrence problem in group G follows the existence of an effective procedure for finding the set $h_i, \ldots, h_s$.

Now we can find the generating elements for subgroup D. As long as subgroup $\bar{B}$ index in group G is finite, subgroup H has a finite index in G only when D has a finite index in $\bar{B}$. Thus, Case 2 is reduced to Case 1.

Case 3. Group G is a free product of non-identity finite groups A and B.

We consider $K = [A,B]$, mutual commutant A and B. Just like in Case 2, we can find the generating intersections $D = H \cap K$.

Thus, if group D rank is more than a unit, the situation is reduced to Case 1. If D rank equals a unit, G is an infinite dihedral group in which subgroup H has a finite index only when the order of H is more than two.

□

So, solvability of the occurrence problem in groups A and B is enough for the solvability of the problem of index in a free product $A*B$. We show that this sufficient condition is a necessary one.

Theorem 3. *If in a nontrivial free product $A*B$, the problem of index is solvable, the occurrence problem is solvable in free factors A, B.*

Proof. We will show that from the solvability of the problem of index in group $A*B$ follows the solvability of occurrence problem in group A.

Assume that A_1 is an arbitrary finitely generated subgroup of group A, and x is an arbitrary element from group A. We will find out by an algorithm solving the problem of index in group

G if the element x belongs to subgroup A_1 or not. We take b_0, a non-identity element from subgroup B. The algorithm solving the occurrence problem in group A depends on the fact if the order of subgroup B is equal to two or not.

Case 1. The order of group B is more than two. In group G we consider a subgroup

$$H_1 = \mathrm{gr}(A_1, B^x, A^{b_0}).$$

We will show that subgroup H_1 has a finite index in group G only when x belongs to A_1. That means solvability of occurrence problem for group A.

If x belongs to A, H_1 has a finite (equal to a unit) index in G. Assume that $x \notin A$, then H_1 is decomposed into a free product

$$H_1 = A_1 * B^x * A^{b_0}.$$

Assume that b is a non-identity element from B different from b_0. Then the subgroup generated by subgroups H_1 and A_b is their free product,

$$\mathrm{gr}(H_1, A^b) = H_1 * A^b$$

and, consequently, has an infinite index in group G. This is the end of the case when the order is more than two.

Case 2. The order of B is equal to two. Then in group G we take a subgroup

$$H_2 = \mathrm{gr}(A_1, A^{xb_0}, a_0^{-b} A a_0^{b_0}),$$

where a_0 is a non-identity element from group A. Subgroup H_2 is contained in the normal closure of factor A in group G and

$$\bar{A} = A * b_0^{-1} A b_0.$$

Thus, group $\bar{A}$ falls into the conditions of the previous case. Now group A^{b_0} represents group B and element $a_0^{b_0}$ represents element b_0. Subgroup H_2 in the group is constructed similarly to subgroup H_1 in group $A * B$. Thus, element x belongs to subgroup A_1 only when H_2 has a finite index in $\bar{A}$ and, consequently, a finite index in group G. Theorem 3 has been proved. □

It follows from theorems 2 and 3 that for each freely decomposed group, the problem of index is solvable only when the occurrence problem is solvable in this group.

We should note, that a group need not necessarily be freely decomposed for the equivalence of the index problem and the occurrence problem. It is obvious that a free product with amalgamated finite normal subgroup or a direct product of a free product and a finite group also have this property.

Conclusions

In the infinite series of the groups under consideration, the problem of index and the occurrence problem turned to be equivalent. Probably, this relation between two algorithmic problems is fulfilled for all finitely defined groups.

QUESTION 1. Is it right that in a class of finitely defined groups, the occurrence problem is equivalent to the problem of index? We should note, we did not need the solvability of the problem of index in groups A and B for the solvability of the problem of index in $A * B$. If the solvability of the problem of index in the factors was a necessary condition for the solvability of the problem of index in a fee product, than for any finitely defined group, the occurrence problem solvability would follow from the solvability of the problem of index.

On the other hand, is the solvability of the problem of index was a sufficient condition of the solvability of the problem of index in a free product, then the occurrence problem solvability would follow from the solvability of the problem of index. Thus, the question if the occurrence problem is equivalent to the problem of index, may be put in the form of the two following questions.

QUESTION 2. Is it right that the solvability of the problem of index in free factors follows from the solvability of the problem of index in a free product?

QUESTION 3. Is it right that the solvability of the problem of index in a free product follows from the solvability of the problem of index in free factors?

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