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NUMERICAL ANALYSIS SOME OSCILLATION EQUATIONS WITH FRACTIONAL ORDER DERIVATIVES

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The paper presents a mathematical model of non-classical dynamic systems. A numerical method of difference schemes, depending on various parameters of the system were found numerical solutions of models. The phase trajectory.

Key words: operator Gerasimov-Caputo, numerical solution, finite difference scheme, the phase trajectories

Introduction

Construction of mathematical models, considering fractal properties of different environments, is of great theoretical and practical importance. For example, in a porous geological environment (geo-environment), its fractal dimension, which affects the process intensity, is interesting for the investigation due to the pore inhomogeneity and complicated topology. These processes are generally called non-local ones or the process with memory [1].

Non-local processes are described by mathematical modeling, by differential equations of fraction orders. Derivative fraction orders are closely associated with environment fractal dimension [2], and their functional dependence may be determined experimentally.

Environment fractal dimension may change depending on time and spatial value. Thus, fraction derivative order is, generally, some function on time and spatial value, and, consequently, it significantly complicates the equations describing non-local processes. Solutions of such equations are found by numerical methods which may be implemented in different computer software tools [3].

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Definition of the problem

It is necessary to find the bias function $u(t)$ Investigation of oscillations in a fractal environment will be carried out according to the following equation of the variable fraction order:

$$\partial_{0t}^{\alpha(t)}u(\eta)+A(t)u(t)=0,\ u(0)=u_0,\ u'(0)=0,\quad 1<\alpha(t)<2, 0<t<T, \tag{1}$$

where $\partial_{0t}^{\alpha(t)}u(\eta)=\dfrac{1}{\Gamma(2-\alpha(t))}\displaystyle\int_0^t\dfrac{u''(\eta)\,d\eta}{(t-\eta)^{\alpha(t)-1}}$ is the Gerasimov-Caputo fraction order derivative, $A(t)$ is some function, u_0 is the known value.

The dynamic system (1) when $\alpha(t)=\alpha-const$ may have, for example, one of the following forms:

1) The equation (1) when $A(t)=\omega^{\alpha}$ transforms into the fractal oscillator equation [4] and in terms of Gerasimov-Caputo fractional differentiation operator [5].

2) In the case when in the equation (1), the relation $A(t)=a+b\cos(\omega t)$ is satisfied, we come to the equation of fractal parametric oscillator [6]-[7].

3) If (1) is put into the equation $A(t)=t$, we come to the equation of Airy fractal oscillations [8].

In a more general case, when $\alpha=\alpha(t)$, the equation (1) may be solved by the finite difference method. We introduce τ as a sample spacing, and $t_j=j\tau$, $j=1,2,..,N$, $N\tau=T$, $u(j\tau)=u_k$. Then fraction order derivative in the equation (1) may be approximated as follows [3]

$$\partial_{0t}^{\alpha(t)}u(\eta)=\frac{\tau^{-\alpha_j}}{\Gamma(3-\alpha_j)}\sum_{k=0}^{j-1}\left[(k+1)^{2-\alpha_j}-k^{2-\alpha_j}\right]\left(u_{j-k+1}-2u_{j-k}+u_{j-k-1}\right). \tag{2}$$

Substituting the formula (2) into the equation (1) and after some transformations we obtain a clear difference scheme:

$$u_{j+1}=\left[2-A_j/B_j\right]u_j-u_{j-1}-\sum_{k=1}^{j-1}b_k\left(u_{j-k+1}-2u_{j-k}+u_{j-k-1}\right), \tag{3}$$

where $b_k=(k+1)^{2-\alpha_j}-k^{2-\alpha_j}$, $B_j=\dfrac{\tau^{-\alpha_j}}{\Gamma(3-\alpha_j)}$, $u_1=u_0$.

Numerical modeling

For simplicity, assume $u_0=1$. Consider some dynamic systems according to (1).

1.**Fractal oscillator.** Consider the case $\alpha(t)=\alpha-const$ and $A(t)=A-const$**.** On the basis of Maple system, simulations for (3) and solutions are constructed.

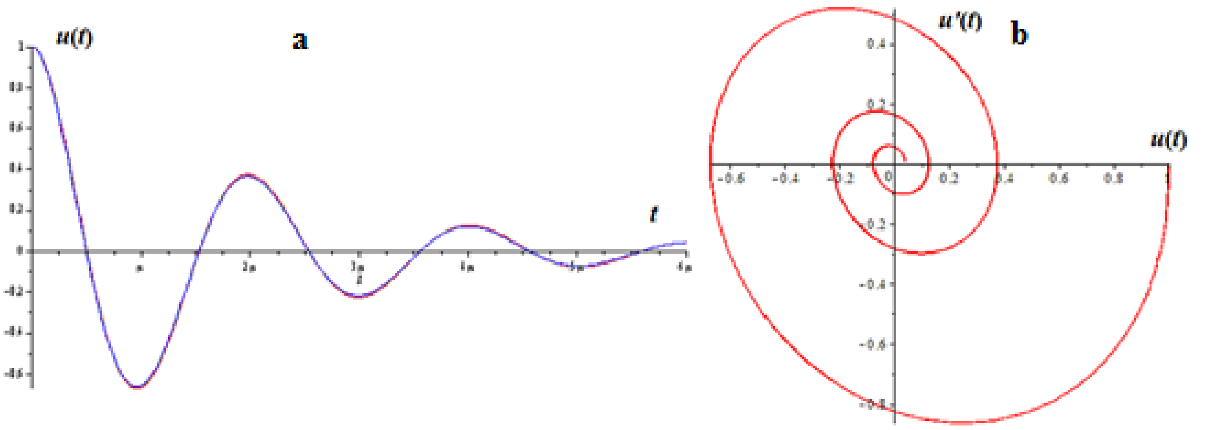

Fig. 1. a) simulations calculated by the formula (3) (red line) and exact solution – (blue line) for parameter values $A = 1, \alpha = 1.8, t \in [0, 6\pi]$. b) phase trajectory

It is clear from Fig. 1a that the scheme of (3) approximates well the exact solution of (4), and the oscillations attenuate. It is confirmed by the phase trajectory which has a stable focus (Fig. 1b).

2. **Fractal oscillator with the variable parameter α.** Consider the case when in the solution of (3) $\alpha(t) = \frac{(1-\varepsilon-\delta)\cos(t \cdot m) + \varepsilon - \delta + 3}{2}$ and $A(t) = 1$, where δ and ε determine the range for parameter $\alpha(t)$ change: $1 + \varepsilon < \alpha < 2 - \delta$, and $\delta + \varepsilon < 1, \delta, \varepsilon \geq 0$, m is the arbitrary number [5].

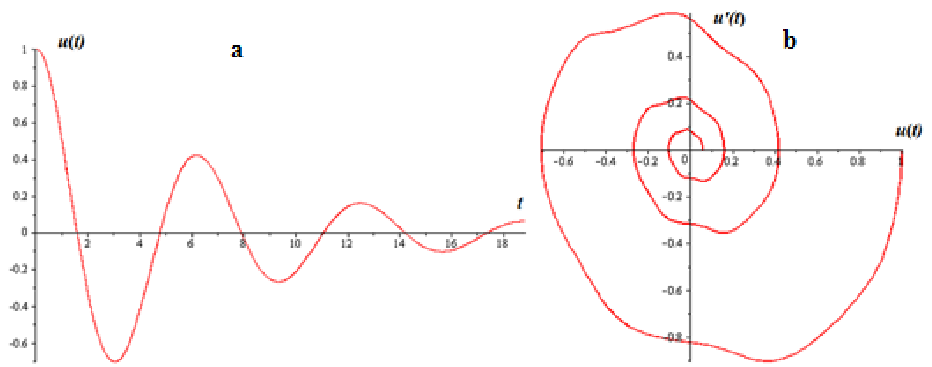

Fig. 2. a) simulation of the equation (1) solution when $\alpha(t) = \frac{(1-\varepsilon-\delta)\cos(t \cdot m) + \varepsilon - \delta + 3}{2}$; b) phase trajectory

It is clear from Fig. 2a that the oscillation process attenuates, but in comparison to the previous case (Fig. 1b), phase trajectories are deformed (Fig. 2b). This fact indicates that the process attenuates slower in this case.

3. **Airy fractal oscillator.** Consider the case when

$$\alpha(t) = \frac{(1-\varepsilon-\delta)\cos(t \cdot m) + \varepsilon - \delta + 3}{2}, A(t) = t.$$

The following parameters were chosen for the problem: $m = 7$, $\varepsilon = 0.69$, $\delta = 0.003$, $a = 1.8$, $\omega = 3$.

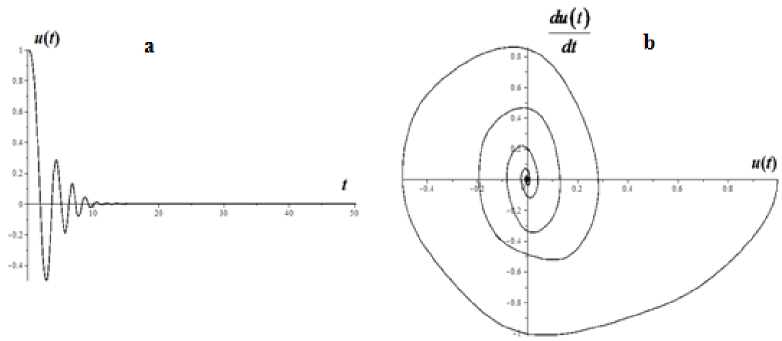

Fig. 3. a) simulation b) phase trajectory

It is clear in Fig. 3a that the oscillation process attenuates fast, and the phase trajectory (Fig. 3b) is bent, but close to a stable focus. Such a behavior of the solution agrees well with the results obtained earlier [7].

When hereditarity properties are taken into the account in the model equation (1), it results in fast dissipation of oscillating system energy and time decrease to its full relaxation.

4. **Fractal parametric oscillator.** Consider the case when $\alpha(t) = \frac{(1-\varepsilon-\delta)\cos(t\cdot m)+\varepsilon-\delta+3}{2}$ and $A(t) = a + b\cos(\omega t)$. The following parameters were chosen for the problem: $m = 7$, $\varepsilon = 0.69$, $\delta = 0.003$, $a = 1.8$, $\omega = 3$.

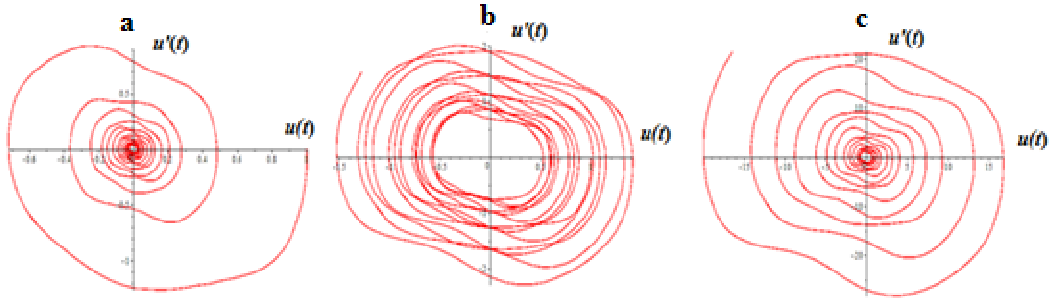

Fig. 4. Phase trajectories a) - $b = 0.9$; b) - $b = 1$; c) - $b = 1.2$

We should note that phase trajectories in Fig. 4 confirm the possibility of application of fractional calculus in the description of non-linear effects. Actually, the phase trajectory in Fig. 4a twists clockwise, and a singular point is a stable focus.

In Fig. 4b, phase pattern changes; first, the trajectory twists clockwise, and then it untwists from some moment of time. Culmination of such a transition is the phase trajectory in Fig. 4c, on which it is clear that the phase trajectory untwists, i.e. it is an unstable focus.

5. **Generalized fractal parametric oscillator.** Consider the case when $\alpha(t) = \dfrac{(1-\varepsilon-\delta)\cos(t\cdot m)+\varepsilon-\delta+3}{2}$ and $A(t) = a + b\cos_\beta(\omega t)$, $\cos_\beta(\omega t) = E_{\beta,1}\left[-(\omega t)^\beta\right] = \sum_{k=0}^{\infty} \dfrac{(-1)^k (t\omega)^{\beta k}}{\Gamma(\beta k+1)}$ is Mittag-Leffler type function. The following parameters were chosen for the problem: $m = 7$, $\varepsilon = 0.69$, $\delta = 0.003$, $a = 1.8$, $\omega = 3$, $\beta = 1.2$.

It is clear in Fig. 5 how the phase pattern changes in comparison to other cases. Besides the non-linear effects as in case 4, phase trajectory compresses significantly.

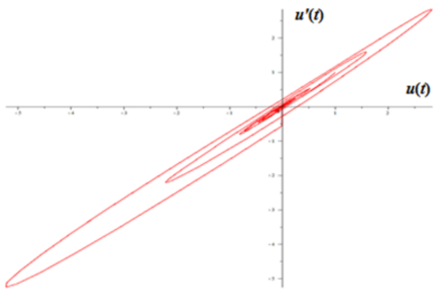

Fig. 5. Phase trajectory (case 5)

Conclusions

We should note, that in contrast to the paper [5], phase trajectories in this paper are built in plane $[u(t), u'(t)]$, not in plane $[u(t), \partial_{0t}^{\alpha-1}u(\eta)]$. That is why we do not observe here the effects associated with multiple reversal point in the center of phase plane.

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