Bulletin KRASEC. Phys. & Math. Sci, 2014, V. 9, №. 2, pp. 18-22. ISSN 2313-0156

DOI: 10.18454/2313-0156-2014-9-2-18-22

MSC 35C05

ON ONE PROBLE FOR HIGHER-ORDER EQUATION

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In this paper not well posed problem for the even-order equation is studied. The stability of the problem is restored by additional conditions and conditions to domain.

Key words: partial differential equations of higher order, not well posed problem, method of separation of variables, simple continued fractions.

Problem definition

The present paper considers for the equation

$$\frac{\partial^{2k}u}{\partial x^{2k}} - \frac{\partial^2 u}{\partial t^2} = 0, \quad k = 2n+1,\, n \in N, \tag{1}$$

in the domain $D = \{(x,t) : 0 \le x \le \pi, 0 \le t \le 2\pi\}$ a problem with the following conditions:

$$\frac{\partial^{2m}u}{\partial x^{2m}}(0,t) = \frac{\partial^{2m}u}{\partial x^{2m}}(\pi,t) = 0, \quad m = 0,1,...,k-1, \quad 0 \le t \le 2\pi, \tag{2}$$

$$u(\alpha\pi,t) = f(t), \quad 0 \le t \le 2\pi, \tag{3}$$

where α is some constant from $(0,1)$ and $f(t)-$ is the given quite smooth function.

We shall show that if $\alpha$is an irrational number, then the theorem of solution uniqueness of the problem (1), (2), (3) is valid in the class $u \in C^{2k,2}_{x,t}(D)$.

Note that this problem is ill-posed, since a small change in the function $f(t)$ under the norm $C^s (s \in N)$ may cause arbitrary large change of the solution u under the norm L_2.

This problem may be regularized by a side condition, for example, by a priori estimate

$$\int_0^{\pi}\int_0^{2\pi} \left(\frac{\partial^k u}{\partial x^k}\right)^2 dtdx \le E^2, \quad 0 \le t \le 2\pi, \tag{4}$$

where E is the defined constant.

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The problem well-posedness

Assume that there is some function $u \in C^{2k,2}_{x,t}(D)$ which satisfies the conditions (1), (2), (3), then u may be presented in the form of a series

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \left(a_n \cos n^k t + b_n \sin n^k t \right), \tag{5}$$

and it follows from this representation that the function $f(t)$ should have the form

$$f(t) = \sum_{n=1}^{\infty} \sin n\alpha\pi \left(a_n \cos n^k t + b_n \sin n^k t \right). \tag{6}$$

Theorem 1. *If α is a irrational number, the problem (1), (2), (3) does not have more than one solution of $u \in C^{2k,2}_{x,t}(D)$.*

Proof. Indeed, if in (6) $f \equiv 0$, than $a_n = b_n = 0$. Consequently, и $u \equiv 0$. □

Remark. If α is a rational number, there is no uniqueness.

For example, let q be some natural number, then the function $u(x,t) = \sin qx \cos q^k t$ satisfies (1), (2) and

$$u(\frac{\pi}{q}, t) = 0, \quad 0 \le t \le 2\pi.$$

Defination. We shall indicate that the irrational number α have the order Ω, if Ω is the upper boundary of the numbers ω, satisfying the inequality

$$\left| \alpha - \frac{p}{q} \right| \frac{1}{q^{1+\omega}} \quad ,$$

for any $\frac{p}{q} \in Q$. It is known that almost all the numbers α have the order $\Omega = 1$ [3, **?**].

The next statement is associated with the question on the solution stability depending on α and f. Here is an example.

Theorem 2. *Let α be an irrational number. Then there is a sequence 2π of periodic functions $f_n \in C^\infty(R)$, uniformly vanishing, and it is such $u_n \in C^{2k,2}_{x,t}(D)$, satisfying (1), (2) and*

$$u_n(\alpha\pi, t) = f_n(t), \tag{7}$$

the following relation holds

$$\lim_{\to\infty} \|u_n\|_{L_2(D)} = +\infty. \tag{8}$$

Proof. Let

$$f_n(t) = \frac{1}{\sqrt{n^k}} \sin n^k t,$$

then

$$u_n(x,t) = \left(\sqrt{n^k} \sin n\alpha\pi \right)^{-1} \sin n^k t \sin nx.$$

It is known that [2] there is a sequence of such integers p_n, q_n that

$$\lim_{n\to\infty} q_n = +\infty, \quad \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$$

and then the theorem statement appears from the following estimation

$$|\sin q_n\alpha\pi| = |\sin(q_n\alpha - p_n)\pi| < \frac{\pi}{q_n}.$$

Note, that for any given integer s there is such an irrational number α (for example, of the order $s+2$) that the solution

$$u_n(x,t) = n^{-1-s}(\sin n\alpha\pi)^{-1}\sin n^k t\cdot\sin nx$$

of the problem (1), (2) and $u_n(\alpha\pi,t) = n^{-1-s}\sin n^k t$ satisfies the following estimation

$$\lim_{n\to\infty}\|u_n\|_{L_2(D)} = +\infty \quad \lim_{n\to\infty}\|f_n\|_{C^s(D)} = 0,$$

from which it is clear that the problem is ill-posed. □

Further we shall show that the problem is also unstable relative to α.

Theorem 3. *Let $p,q\in N, p<q$ and $\{\alpha_n\}$ be a sequence of irrational numbers converging to $\frac{p}{q}$. And let $u_n\in C^{2k,2}_{x,t}(D)$ be the solution of the problem (1), (2) and $u_n(\alpha\pi,t) = \sin q^k t$, then*

$$\lim_{\to\infty}\|u_n\|_{L_2(D)} = +\infty.$$

The solution is written in the form $u_n(x,t) = \frac{\sin q^k t\cdot\sin qx}{\sin q\alpha_n\pi}$, from which the theorem statement is obvious.

Thus, a side condition is required.

$$\int_0^\pi\int_0^{2\pi}\left(\frac{\partial^k u}{\partial x^k}\right)^2 dtdx \le E^2.$$

Problem with a bounded solution

Let α,ε,E be some positive constants, and $\alpha\in(0,1)$.

Let $f\in L_2(0,2\pi)$. The function class $u\in C^{2k,2}_{x,t}(D)$ satisfying (1), (2) and

$$\|u(\alpha\pi,\cdot) - f\|_{L_2(0,2\pi)} \le \varepsilon, \tag{9}$$

$$\left\|\frac{\partial^k u}{\partial x^k}\right\|_{L_2(D)} \le E. \tag{10}$$

is indicated by $\Gamma(\varepsilon,E)$. The condition (9) substitutes the condition (3), and the priori estimate (10) is required for the problem to be well-posed.

We introduce the following notations

$$\|u\|^2 = \sup_{x\in[0,\pi]}\int_0^{2\pi} u^2(x,t)dt, \tag{11}$$

$$\Delta(\varepsilon,E) = \sup_{v,w\in\Gamma(\varepsilon,E)}\|v-w\|. \tag{12}$$

Theorem 4. *Let* α *be a rational number,* $\alpha = \frac{p}{q}, (p,q) = 1$ *and*

$$q^2 \le 2\frac{E}{\varepsilon} \quad . \tag{13}$$

Then

$$\Delta(\varepsilon, E) \le 3\frac{E}{q^k}. \tag{14}$$

If $\quad q = \left[(\frac{2E}{\varepsilon})^{\frac{1}{k}} \right]$, *then*

$$\Delta(\varepsilon, E) \le 3\sqrt{E}\sqrt{\varepsilon}.$$

Proof. Let $v, w \in \Gamma(\varepsilon, E)$, then $u = v - w \in C_{x,t}^{2k,2}(D)$ satisfies (1), (2) and

$$\|u(\alpha\pi, \cdot)\|_{L_2(0,2\pi)} \le 2\varepsilon, \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L_2(D)} \le 2E. \tag{15}$$

Since u is represented as (5), the conditions (15) are rearranged as

$$\sum_{n=1}^{+\infty} (a_n^2 + b_n^2) \sin^2 n\frac{p}{q}\pi \le \frac{4\varepsilon^2}{\pi^2}, \tag{16}$$

$$\sum_{n=1}^{+\infty} n^{2k}(a_n^2 + b_n^2) \le \frac{8E^2}{\pi^2} \quad , \tag{17}$$

whence it follows

$$\sum_{n=1}^{+\infty} \left(\sin^2 n\frac{p}{q}\pi + n^{2k}\frac{\varepsilon^2}{E^2} \right) (a_n^2 + b_n^2) \le \frac{8\varepsilon^2}{\pi}. \tag{18}$$

From (5) we have

$$\|u\|^2 = \pi \max_{x\in[0,\pi]} \sum_{n=1}^{+\infty} (a_n^2 + b_n^2) \sin^2 nx \le \pi \sum_{n=1}^{+\infty} (a_n^2 + b_n^2).$$

It follows from (12)

$$\Delta^2(\varepsilon, E) \le \pi \sup \left\{ \sum_{n=1}^{+\infty} (a_n^2 + b_n^2) : a_n, b_n \right\},$$

satisfying (18).

According to the Lagrange multiplier role we find

$$\Delta^2(\varepsilon, E) \le 8\varepsilon^2 \min(\sin^2 r\frac{p}{q}\pi + r^{2k}\frac{\varepsilon^2}{E^2})^{-1}, \quad r \in N. \tag{19}$$

From (13) we obtain

$$\sin^2 r\frac{p}{q}\pi + r^{2k}\frac{\varepsilon^2}{E^2} \ge \frac{\varepsilon^2}{E^2}q^{2k}, \quad 1 \le r \le q.$$

Substituting this estimation into (19) we obtain (14). The theorem has been proved. □

Assume that α is an irrational number expanded into a continued fraction

$$\alpha = \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \ldots}} \quad .$$

Theorem 5. *Let* $\alpha \in (0,1)$ *be an irrational number and* $\alpha_i \leq K_\alpha$*, then*

$$\Delta(\varepsilon, E) \leq 3 \left(\frac{K_\alpha + 2}{2} \varepsilon E \right)^{\frac{1}{2}}. \tag{20}$$

Proof. To make sure that this estimation is valid, note that it follows from the theorem 4 that the estimation (19) is true for $\Delta(\varepsilon, E)$ then from the condition $\alpha_i \leq K_\alpha$ [3] we obtain

$$\sin^2 r\alpha\pi \geq \frac{27}{4\left(K_\alpha + 2\right)^2 r^2}, \ r \geq 1.$$

Then

$$\min_{r \in N} \left\{ \frac{27}{4\left(K_\alpha + 2\right)^2 r^2} + r^{2k} \frac{\varepsilon^2}{E^2} \right\} \geq \frac{\varepsilon}{E} \frac{\sqrt{27}}{\left(K_\alpha + 2\right)}.$$

Thus, the required estimation follows from the above

$$\Delta^2(\varepsilon, E) \leq 9\varepsilon E \left(\frac{K_\alpha + 2}{2} \right).$$

□

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Original article submitted: 23.10.2014