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CERTAIN PROPERTIES OF FRACTIONAL INTEGRO-DIFFERENTIATION OPERATOR OF FUNCTIONS IN OTHER FEATURES

A.S. Nishonboev

Fergana State University, 150100, Uzbekistan, Fergana c., Murabbiylar st., 19
E-mail: azizbek.nishoboev@mail.ru

Prove an important lemma for the operator of fractional integro-differentiation of functions of other functions.

Key words: Gaussian function, the function Meyer, Mellin transform

Assume the notations:

$$D^{l}_{a,g(x)}\phi(x)=\begin{cases}\frac{1}{\Gamma(-l)}\int\limits_a^x [g(x)-g(t)]^{-l-1}\phi(t)g'(t)\,dt, & l<0\\ \phi(x), & l=0\\ \frac{d^{n+1}}{d[g(x)]^{n+1}}D^{l-(n+1)}_{0g(x)}\phi(x), & l>0\end{cases} \tag{1}$$

$$F_{0,g(x)}\left[\begin{array}{c} a\ b\\ e\ x\end{array}\right]\phi(x)=\begin{cases}\frac{1}{\Gamma(-l)}\int\limits_a^x [g(x)-g(t)]^{-l-1}F(a,b,c;\frac{g(x)-g(t)}{g(x)})\phi(t)g(t)'\,dt, l<0\\ \phi(x), l=0\\ \left[g(x)]^{b}\right]\frac{d^{n+1}}{d[g(x)]^{n+1}}[g(x)]^{-b}F_{0,g(x)}\left[\begin{array}{cc} a+b+1, & b\\ l-n-1, & x\end{array}\right]\phi(x), l>0\end{cases} \tag{2}$$

were a, b and l are any real numbers, n is the integer part of l; $\Gamma(x)$ – gamma-function; $F[\ldots]$ is the Gaussian hypergeometric function [1].

Lemma 1. *Let* $\varphi(x)\in L(0,1)$, $l_1+l_3<0$, $l_3<0$. *Than almost everywhere at (0,1), the following relation holds*

$$D^{l_1}_{0x^2}D^{l_3}_{0x}\phi(x)=2^{l_3}\cdot x^{l_3+1}F_{0x^2}\left[\begin{array}{cc} -\frac{1+l_3}{2}, & -l_1-\frac{l_3}{2}\\ -l_1-l_3, & x\end{array}\right]\frac{\phi(x)}{x}. \tag{3}$$

Here l_1, l_3 *– are the given real numbers;*

Proof. Consider the case when $l_1<0$, $l_3<0$. Due to the definition (1) we have

$$D^{l_1}_{0x^2}D^{l_2}_{0x}\phi(x)=\frac{x^{-l_1}}{\Gamma(-l_1)\Gamma(-l_3)}\int\limits_0^x (x^2-t^2)^{-l_1-1}dt^2\int\limits_0^t (t-y)^{-l_3-1}\phi(y)dy. \tag{4}$$

Nishonboev Azizbek Solizhonovich – Lecturer, of Dep. Mathematical Analysis and Differential Equations, Fergana State University, Republic of Uzbekistan.

After some transformations from (4), we obtain

$$I=D_{0x^2}^{l_1}D_{0x}^{l_3}\phi\left(\sqrt{x}\right)=\frac{x^{-l_1}}{\Gamma(-l_1)\Gamma(-l_3)}\int\limits_0^x(x-t)^{-l_1-1}dt\int\limits_0^t(\sqrt{t}-\sqrt{y})^{-l_3-1}\phi\left(\sqrt{y}\right)d\sqrt{y}. \quad (5)$$

Changing the integration order, from (5) we find

$$I=\int\limits_0^x\phi(y)d\sqrt{y}\int\limits_y^x(x-t)^{-l_1-1}(\sqrt{t}-\sqrt{y})^{-l_3-1}dt, \quad (6)$$

Changing $t=xz$ from (6) we obtain

$$I=\frac{x^{-l_1}}{\Gamma(-l_1)\Gamma(-l_3)}\int\limits_0^x(\sqrt{y})^{-l_3-1}\phi(y)K(\sigma)d\sqrt{y}, \quad (7)$$

where

$$K(\sigma)=\int\limits_0^\infty K_1\left(\sqrt{\sigma z}\right)K_2(z)\,dz, \quad (8)$$

$$K_1(z)=(z-1)_+^{-l_3-1},\quad K_2(z)=(1-z)_+^{-l_1-1}, \quad (9)$$

$$\sigma=\frac{x}{y},\quad z_+^l=\begin{cases}0, & z\le 0\\ z^l, & z>0\end{cases}.$$

To calculate the integral (8), Mellin transform is used a [1]

$$f*(s)=M\{f(x);s\}=\int\limits_0^\infty f(x)x^{s-1}dx. \quad (10)$$

Then, considering the relation [3]

$$M\left\{x^{\delta_1}\int\limits_0^\infty\xi^{\delta_2}g_1(x\xi)\,g_2(\xi)\,d\xi;s\right\}=g_1^*(s+\delta_1)\,g_2^*(1-\delta_1+\delta_2-s), \quad (11)$$

we obtain

$$M\{K(\sigma);s\}=K^*(s)K_1^*(s)\,K_2^*(1-s). \quad (12)$$

Further, in virtue of the formula [3]

$$M\left\{(x-1)_+^{c-1};s\right\}=\Gamma(c)\Gamma\left[\begin{array}{c}1-c-s\\1-s\end{array}\right],\quad \mathrm{Re}c>0,\quad \mathrm{Re}s<1-\mathrm{Re}c. \quad (13)$$

$$M\{f(x^p)\}=\frac{1}{p}f*\left(\frac{s}{p}\right),\quad p\neq 0, \quad (14)$$

$$M\left\{(1-x)_+^{c-1};s\right\}=\Gamma(c)\Gamma\left[\begin{array}{c}s\\s+c\end{array}\right],\quad \mathrm{Re}c>0,\quad \mathrm{Re}s>0 \quad (15)$$

find

$$K_1*(s)=2\Gamma(-l_3)\Gamma\left[\begin{array}{c}1+l_3-2s\\ 1-2s\end{array}\right],\quad \mathrm{Re}l_3<0,\quad 2\mathrm{Re}s<1+l_3, \tag{16}$$

and

$$K_2*(s)=\Gamma(-l_1)\Gamma\left[\begin{array}{c}s\\ s-l_1\end{array}\right],\quad \mathrm{Re}l_1<0,\quad \mathrm{Re}s>0, \tag{17}$$

Substituting (16), (17) into (12) we obtain

$$K*(s)=2\Gamma(-l_1)\Gamma(-l_3)\Gamma\left[\begin{array}{cc}1+l_3-2s & 1-s\\ 1-2s & 1-l_1-s\end{array}\right], \tag{18}$$

$$\mathrm{Re}l_1<0,\quad \mathrm{Re}l_3<0,\quad 2\mathrm{Re}s<1+l_3,\quad \mathrm{Re}s<1.$$

Hence, applying the formula [2]

$$\Gamma(2a)=2^{2a-1}\sqrt{\pi}\Gamma(a)\Gamma(a+1/2), \tag{19}$$

from (18) we have

$$K*(s)=2^{l_3+1}\Gamma(-l_1)\Gamma(-l_3)\Gamma\left[\begin{array}{cc}\frac{1+l_3}{2}-s & \frac{2+l_3}{2}-s\\ 1-l_1-s & \frac{1}{2}-s\end{array}\right], \tag{20}$$

$$\mathrm{Re}l_1<0,\quad \mathrm{Re}l_3<0,\quad 2\mathrm{Re}s<1+l_3,\quad 2\mathrm{Re}s<2+l_3.$$

In virtue of the formula [3]

$$M\left\{(x-1)_+^{c-1}F(a,b,c;1-x);s\right\}=\Gamma(c)\Gamma\left[\begin{array}{cc}1+a-c-\gamma-s, & 1+b-c-s\\ 1-s, & 1+a+b-c-s\end{array}\right], \tag{21}$$

$$\mathrm{Re}c>0,\ \mathrm{Re}s<1+\mathrm{Re}(a-c),1+\mathrm{Re}(b-c).$$

$$M\{x^p f(x);s\}=f*(s+p), \tag{22}$$

$$F(a,b,c;z)=(1-z)^{-a}F(a,c-b,c;\frac{z}{z-1}), \tag{23}$$

from (20) we obtain

$$K(\sigma)=\frac{2^{l_3+1}\Gamma(-l_1)\Gamma(-l_3)}{\Gamma(-l_1-l_3)}x^{l_1+\frac{1+l_3}{2}}y^{\frac{l_3+1}{2}}(x-y)_+^{-l_1-l_3-1}\times \tag{24}$$

$$\times F(-\frac{1+l_3}{2},-l_1-\frac{l_3}{2},-l_1-l_3;\frac{x-y}{x}).$$

Substituting (24) into (7) and after some transforms we obtain the equality (3).

The similar proof is for the case when $0<l_1<-l_3$. □

Lemma 2. *If the conditions:*

$$1)\varphi(x)\in L(0,1),2)l_1+l_3<0,l_3<0,l_2>-l_1,$$

hold, almost everywhere at (0.1) the following formula is valid

$$D_{0x^2}^{l_1}\left(x^2\right)^{l_2}D_{0x}^{l_3}\phi(x)=\frac{\left(x^2\right)^{l_2-l_1}}{2^{l_3}}\int\limits_0^x y^{-l_3-1}G_{33}^{30}\left(\frac{y^2}{x^2}\left|\begin{array}{c}1+l_2-l_1,\ 1,\ \frac{1}{2}\\ \frac{3+l_3}{2},\ \frac{2+l_3}{2},\ 1+l_2\end{array}\right.\right)\phi(y)\,dy. \tag{25}$$

Proof. To prove it, consider the case when $l_1 < 0$, $l_3 < 0$.
Then due to the definition (1) we have

$$D_{0x^2}^{l_1}\left(x^2\right)^{l_2} D_{0x}^{l_3}\phi\left(x\right) = \frac{1}{\Gamma(-l_1)\Gamma(-l_3)}\int\limits_0^x (x^2-t^2)^{-l_1-1}(t^2)^{l_2}dt^2\int\limits_0^t (t-y)^{-l_3-1}\phi\left(y\right)dy. \quad (26)$$

After some transforms from (26) we find

$$I = D_{0x}^{l_1}x^{l_2}D_{0\sqrt{x}}^{l_3}\phi(\sqrt{x}) = \quad (27)$$

$$= \frac{1}{\Gamma(-l_1)\Gamma(-l_3)}\int\limits_0^x (x-t)^{-l_1-1}t^{l_2}dt\int\limits_0^t (\sqrt{t}-\sqrt{y})^{-l_3-1}\phi\left(\sqrt{y}\right)d\sqrt{y}.$$

Changing the integration order, we obtain

$$I = \frac{1}{\Gamma(-l_1)\Gamma(-l_3)}\int\limits_0^x \phi\left(\sqrt{y}\right)d\sqrt{y}\int\limits_y^t (x-t)^{-l_1-1}t^{l_2}(\sqrt{t}-\sqrt{y})^{-l_3-1}dt. \quad (28)$$

Assuming $t = xz$, from (28) we find

$$I = \frac{1}{\Gamma(-l_1)\Gamma(-l_3)}\int\limits_0^x y^{-l_1-l_2-\frac{l_3+1}{2}}Q(\sigma)d\sqrt{y}. \quad (29)$$

where

$$Q(\sigma) = \sigma^{l_2-l_3}\int\limits_0^\infty z^{l_2}f_1(\sqrt{\sigma z})f_2(z)dz, \quad (30)$$

$$f_1(z) = (z-1)_+^{-l_3-1}; \quad f_2(z) = (1-z)_+^{-l_1-1}, \quad (31)$$

$$\sigma = \frac{x}{y}; \quad z_+^l = \begin{cases} 0, \ z \le 0 \\ z^l, \ z > 0 \end{cases}.$$

Applying the formula (11) from (30) we obtain

$$M\{Q(\sigma);s\} = f_1*(s)f_2*(1+l_2-s), \quad (32)$$

Further, in virtue of the formula (13), (14), (15) we find

$$f_1*(s) = 2\Gamma(-l_3)\Gamma\left[\begin{array}{c} 1+l_3-2s \\ 1-2s \end{array}\right], \quad \mathrm{Re} l_3 < 0, \quad 2\mathrm{Re} s < 1+l_3, \quad (33)$$

$$f_2*(s) = \Gamma(-l_1)\Gamma\left[\begin{array}{c} s \\ s-l_1 \end{array}\right], \quad \mathrm{Re} l_1 < 0, \quad \mathrm{Re} s > 0, \quad (34)$$

Substituting (33), (34) into (32), we obtain

$$Q*(s) = 2\Gamma(-l_1)\Gamma(-l_3)\Gamma\left[\begin{array}{cc} 1+l_3-2s & 1+l_2-s \\ 1-2s & 1+l_2-l_1-s \end{array}\right], \quad (35)$$

$$\mathrm{Re}l_1 < 0, \quad \mathrm{Re}l_3 < 0, \quad 2\mathrm{Re}s < 1 + l_3, \quad \mathrm{Re}s < 1 + l_2.$$

On the basis of the formula (19) (35), we have

$$Q*(s) = 2^{l_3+1}\Gamma(-l_1)\Gamma(-l_3)\Gamma\left[\begin{array}{ccc} \frac{1+l_3}{2}-s & \frac{2+l_3}{2}-s & 1+l_3-s \\ \frac{1}{2}-s & 1-s & 1+l_3-l_1-s \end{array}\right], \tag{36}$$

$$\mathrm{Re}l_1 < 0, \quad \mathrm{Re}l_3 < 0, \quad 2\mathrm{Re}s < 1 + l_3, \quad 2\mathrm{Re}s < 2 + l_3, \quad \mathrm{Re}s < 1 + l_2.$$

In virtue of the formula [2]

$$M\left\{G^{m'n'}_{p'q'}\left(z\left|\begin{array}{c} a_1, a_2, ...a_{p'} \\ b_1, b_2, ...b_{q'} \end{array}\right.\right); s\right\} =$$

$$= \Gamma\left[\begin{array}{c} s+b_1,\, s+b_2, ...s+b_{m'},\, 1-a_1-s,\, 1-a_2-s, ...1-a_{n'}-s \\ s+a_{n'+1},\, s+a_{n'+2}, ...s+a_{p'},\, 1-b_{m'+1}-s,\, 1-b_{m'2}-s, ...1-b_{q'}-s \end{array}\right] \tag{37}$$

$$-\min_{1\le k'\le m'} Reb_{k'} < Res < 1 - \max_{1\le j\le n'} Rea_j, \quad p' = q' \ge 1, \quad m' + n' = p', \quad \sum_{j=1}^{n'}(a_j - b_j) > 0$$

from (35) we obtain

$$Q(\sigma) = 2^{l_3}\frac{1}{\Gamma(-l_1)\Gamma(-l_3)}G^{30}_{33}\left(\frac{y}{x}\left|\begin{array}{ccc} -\frac{1+l_3}{2}, & -\frac{l_1}{2}, & -l_2 \\ l_1-l_2, & 0, & -\frac{1}{2} \end{array}\right.\right). \tag{38}$$

Substituting (35) into (29) and after some transforms, we obtain the equality (25). □

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