Bulletin KRASEC. Phys. & Math. Sci, 2015, V. 10, №. 1, pp. 4-10. ISSN 2313-0156

DOI: 10.18454/2313-0156-2015-10-1-4-10

MATHEMATICS

MSC 47F52+47F05

ON THE STABILITY OF THE BOUNDARY VALUE PROBLEM FOR EVEN ORDER EQUATION

A.V. Yuldasheva

National University of Uzbekistan by Mirzo Ulugbeka, 100174, Uzbekistan, Tashkent c., VUZ gorodok st.

E-mail: yuasv86@mail.ru

In this paper we consider ill-posed problem for one even-order equation. The stability of the problem is proved with the additional assumption.

Key words: partial differential equations, ill-posed problem, boundary value problem, algebraic numbers, the simple continued fraction

Introduction

We consider following problem for even order equation:

$$\frac{\partial^{2k}u}{\partial x^{2k}} - \frac{\partial^{2p}u}{\partial t^{2p}} = 0,\ k,p \in N,\ \ 0 < x < \pi,\ \ 0 < t < \alpha\pi,$$

$$\frac{\partial^{2m}u}{\partial x^{2m}}(0,t) = \frac{\partial^{2m}u}{\partial x^{2m}}(\pi,t) = 0, \quad m = 0,1,...,k-1, \quad 0 \le t \le \alpha\pi,$$

$$\frac{\partial^{j}u}{\partial t^{j}}(x,0) = \varphi_j(x),\ \ j = 0,1,...,p-1,\ 0 \le x \le \pi,$$

$$\frac{\partial^{j}u}{\partial t^{j}}(x,\alpha\pi) = \psi_j(x),\ \ j = 0,1,...,p-1,\ 0 \le x \le \pi,$$

where α is a positive constant.

If $k = p = 1$ we get The Dirichlet problem for the vibrating string equation, which is a classical ill-posed problem due to its irregular behavior. Its solution may neither exists, nor be uniquely determined, nor depend continuously on the data (see [1]-[3]).

In [2], the Dirichlet problem for the wave equation was studied with the additional assumption of an “a priori” bound for the gradient of the solution. Case when $p = 1$, $k \in N$ was studied in [6].

The present research leads to some problems of Diophantine approximation.

Let us note that formulate problem is ill-posed problem if $k - p$ is even number.

Therefore, the above problem cannot be suitably dealt with if α and $\varphi_j(x), \psi_j(x), j = 0,1,\ldots,p-1$ are known within a certain approximation.

Yuldasheva Asal Victorovna – Ph.D. (Phys. & Math.), Lecturer of the Dep. Differential Equations and Mathematical Physics, of the National University of Uzbekistan by M. Ulugbek, Tashkent.

Main results and comments

Let $\varphi_j(x)$, $\psi_j(x)$, $(j=0,1,...,p-1)$ be functions in $C^{2k}[0,\pi]$ such that $\varphi_j^{(2i)}(0)=\varphi_j^{(2i)}(\pi)=\psi_j^{(2i)}(0)=\psi_j^{(2i)}(\pi)=0,\ i=0,1,...,k-1,\ j=0,1,...,p-1$.

Let t, E, α, δ be positive constants. We consider solutions u in $C_{x,t}^{2k,2p}([0,\pi]\times(0,+\infty])$ of the following problem:

$$\frac{\partial^{2k}u}{\partial x^{2k}}-\frac{\partial^{2p}u}{\partial t^{2p}}=0,\ k,p\in N,\quad 0<x<\pi,\ \ t>0, \tag{1}$$

$$\frac{\partial^{2m}u}{\partial x^{2m}}(0,t)=\frac{\partial^{2m}u}{\partial x^{2m}}(\pi,t)=0,\quad m=0,1,...,k-1,\quad t\geq 0, \tag{2}$$

$$\left\|\frac{\partial^j u}{\partial t^j}(x,0)-\varphi_j(x)\right\|_{L_2[0,\pi]}\leq\delta\pi\sqrt{E},\ \ j=0,1,...,p-1, \tag{3}$$

$$\left\|\frac{\partial^j u}{\partial t^j}\left(x,\tau_j\pi\right)-\psi_j(x)\right\|_{L_2[0,\pi]}\leq\delta\pi\sqrt{E},\ \ j=0,1,...,p-1,\ \left|\tau_j-\alpha\right|\leq\delta, \tag{4}$$

$$\int\limits_0^\pi\left(\left(\frac{\partial^k u}{\partial x^k}\right)^2+\left(\frac{\partial^p u}{\partial t^p}\right)^2\right)dx\leq E,\ \ t\geq 0. \tag{5}$$

for real numbers τ_j, $j=0,1,...,p-1$ depending on u and satisfying $\left|\tau_j-\alpha\right|\leq\delta$. The meaning $\left|\tau_j-\alpha\right|\leq\delta$ is that the final time $\alpha\pi$ is known up to a given error.

We denote by Γ_δ the set of all $C_{x,t}^{2k,2p}([0,\pi]\times(0,+\infty])$ solutions of (1-(5). We note that if $\delta=0$, then the problem (1-(5) is reduced to the classical boundary value problem with additional assumption (5). It was studied in [6] that this problem may have no solutions.

Let $Diam\Gamma_\delta=\sup\limits_{v,w\in\Gamma_\delta}\|v-w\|$. Let $v_1,\ v_2\in\Gamma_\delta$. Then there are τ_{ij} such that

$$\left\|\frac{\partial^j u}{\partial t^j}\left(x,\tau_j\pi\right)-\psi_j(x)\right\|_{L_2[0,\pi]}\leq\delta\pi\sqrt{E},\ \ i=0,1,\ j=0,1,...,p-1,\ \left|\tau_{ij}-\alpha\right|\leq\delta$$

and let

$$u(x,t)=v_1(x,t)-v_2(x,t),(x,t)\in[0,\pi]\times[0,+\infty) \tag{6}$$

Then $u\in C_{x,t}^{2k,2p}([0,\pi]\times[0,+\infty))$. Moreover, u satisfies equation (1), conditions (2) and following

$$\left\|\frac{\partial^j u}{\partial t^j}(x,0)\right\|_{L_2[0,\pi]}\leq 2\delta\pi\sqrt{E},\ \ j=0,1,...,p-1, \tag{7}$$

$$\left\|\frac{\partial^j u}{\partial t^j}(x,\alpha\pi)-\psi_j(x)\right\|_{L_2[0,\pi]}\leq 4\delta\pi\sqrt{E},\ \ j=0,1,...,p-1, \tag{8}$$

$$\int\limits_0^\pi\left(\left(\frac{\partial^k u}{\partial x^k}\right)^2+\left(\frac{\partial^p u}{\partial t^p}\right)^2\right)dx\leq 4E,\ \ t\geq 0. \tag{9}$$

It is easy to verify (1), (2), (7)-(9). We can write the function satisfying (1) and (2) in the following form

$$u(x,t)=\sum_{n\geq 1}\sin nx\left\{\frac{A_n\sin n^{\frac{k}{p}}(\alpha\pi-t)}{\sin n^{\frac{k}{p}}\alpha\pi}+B_n\sin n^{\frac{k}{p}}t\right\}.$$

Similarly, we can rewrite (7)-(9) as follows:

$$\sum_{n\geq 1}A_n^2\leq 8\delta^2\pi E, \tag{10}$$

$$\sum_{n\geq 1}B_n\sin^2 n^{\frac{k}{p}}\alpha\pi\leq 32\delta^2 E, \tag{11}$$

$$\left\|\frac{\partial^k u}{\partial x^k}(\cdot,t)\right\|^2_{L_2[0,\pi]}+\left\|\frac{\partial^p u}{\partial t^p}(\cdot,t)\right\|^2_{L_2[0,\pi]}\leq 4E,\ t\geq 0. \tag{12}$$

Defining:

$$\sigma_n=\sqrt{\frac{\pi}{2}}\left(\frac{A_n\sin n^{\frac{k}{p}}(\alpha\pi-t)}{\sin n^{\frac{k}{p}}\alpha\pi}+B_n\sin n^{\frac{k}{p}}t\right),$$

we obtain from (12)

$$\left\|\frac{\partial^k u}{\partial x^k}(\cdot,t)\right\|^2_{L_2[0,\pi]}\leq\sum_{n\geq 1}n^{2k}\sigma_n^2\leq 4E,$$

whence

$$\|u(\cdot,t)\|^2_{L_2[0,\pi]}=\sum_{n=1}^{N}\sigma_n^2+\sum_{n=N+1}^{\infty}\sigma_n^2<\sum_{n=1}^{N}\sigma_n^2+\frac{4E}{N^{2k}}.$$

We now have following bound:

$$\|u(\cdot,t)\|^2_{L_2[0,\pi]}<\tfrac{\pi}{2}\max_{n=1,N}\left(\sin n^{\frac{k}{p}}\alpha\pi\right)^{-2}\sum_{n=1}^{N}\Big[A_n^2\sin^2 n^{\frac{k}{p}}(\alpha\pi-t)+B_n^2\sin^2 n^{\frac{k}{p}}\alpha\pi\cdot\sin^2 n^{\frac{k}{p}}t+$$
$$+2\left|A_n\right|\left|B_n\right|\left|\sin n^{\frac{k}{p}}(\alpha\pi-t)\right|\left|\sin n^{\frac{k}{p}}t\right|\left|\sin n^{\frac{k}{p}}\alpha\pi\right|\Big]+\tfrac{4E}{N^{2k}}.$$

Therefore, from (10) and (11) it follows that

$$\max_{t\in[0,\alpha\pi]}\|u(\cdot,t)\|^2_{L_2[0,\pi]}=\|u\|^2<40\delta^2\pi^2E\max_{n=1,N}\left(\sin n^{\frac{k}{p}}\alpha\pi\right)^{-2}+\frac{4E}{N^{2k}},N=1,2,\ldots.$$

Let

$$\alpha=\cfrac{1}{a_1+\cfrac{1}{a_2+\ldots}}$$

be the simple continued fraction for α, where the partial quotients a_n are integers such that, $a_n\geq 1$.

We consider the set of irrational numbers with bounded partial quotients, i.e. the numbers α, for which there exists a constant A_α satisfying $a_n\leq A_\alpha$ for all n . We note that if α is a quadratic irrational, then the expansion of α as a simple continued fraction is ultimately periodic, which implies that a_n has bounded partial quotients.

Then from theory of continued fractions (see [5] p.37) we easily obtain

$$\max_{n=1,N}\left(\sin n^{\frac{k}{p}}\alpha\pi\right)^{-2}<\left(\sin\frac{\pi}{(A_\alpha+2)N^{\frac{k}{p}}}\right)^{-2},N=1,2,\ldots.$$

Since $\sin x\geq\frac{3\sqrt{3}}{2\pi}x$ for $x\in\left[0,{}^{\pi}/_{3}\right]$, we have for every N

$$\|u\|^2<\frac{160}{27}\delta^2\pi^2E(A_\alpha+2)^2N^{\frac{2k}{p}}+\frac{4E}{N^{2k}},\ N=1,2,\ldots. \tag{13}$$

Now let

$$g(t)=\frac{160}{27}\delta^2\pi^2E(A_\alpha+2)^2t^{\frac{2k}{p}}+4Et^{-2k}.$$

The minimum value of g for $t>0$ is attained at

$$\bar{t}=\left(\frac{27p}{40}\right)^{\frac{p}{2k(p+1)}}(\delta\pi(A_\alpha+2))^{-\frac{p}{k(p+1)}}$$

Since g is an increasing function on the interval $[\bar{t},+\infty)$, we have

$$g([\bar{t}+1])<g(\bar{t}+1).$$

We obtain

$$\|u\|^2\leq\frac{160E}{27}(\delta\pi(A_\alpha+2))^{\frac{2p}{p+1}}\left[1+\left(\left(\frac{27p}{40}\right)^{\frac{p}{(p+1)2k}}+(\delta\pi(A_\alpha+2))\right)^{\frac{p}{(p+1)k}}\right]^{\frac{2k}{p}}. \tag{14}$$

So we proved following

Theorem 1. *Let α is an irrational number and has the simple continued fraction with bounded partial quotients. Then for $(Diam\Gamma_\delta)^2$ (14) is valid.*

Now we use some results obtained in [4]. By corollary 6 of [7], since α has a type $\Omega<\infty$, there exist $K=K(\theta,\alpha)>0$ and, for any $\delta>0$, a number $\xi\in R\backslash Q$ such that

$$|\xi-\alpha|<\delta, \tag{15}$$

and

$$\max_{n=1,N}(\sin n\pi\xi)^{-2}\leq\left(\sin\left(\frac{\pi\left(3-\sqrt{5}\right)}{2N}\right)\right)^{-2} \tag{16}$$

for all $N\geq K\delta^{-\theta}$. From (15) it follows that $\left|\tau_j-\alpha\right|\leq\delta$, for every τ_j satisfying $\left|\xi-\tau_j\right|\leq 2\delta$.

If u is defined by (6), we obtain from (8)

$$\left\|\frac{\partial^j u}{\partial t^j}(x,\xi\pi)\right\|_{L_2[0,\pi]}\leq 4\delta\pi\sqrt{E},\ \ j=0,1,...,p-1. \tag{17}$$

Therefore, u satisfies conditions (1), (2), (7), (9) and (17). The solutions of the problem (1), (2), (7), (9) and (17) $u \in C^{2k,2p}_{x,t}([0,\pi]\times[0,+\infty))$ of the form

$$u(x,t)=\sum_{n\geq 1}\sin nx\left\{\frac{A_n \sin n^{\frac{k}{p}}(\xi\pi - t)}{\sin n^{\frac{k}{p}}\xi\pi}+B_n \sin n^{\frac{k}{p}} t\right\}.$$

which satisfies (10), (12) and

$$\sum_{n\geq 1} B_n \sin^2 n^{\frac{k}{p}}\xi\pi \leq 72\delta^2 E. \tag{18}$$

As in proof of theorem 1 we obtain

$$\|u\|^2 < 80\delta^2\pi^2 E \max_{n=1,N}\left(\sin n^{\frac{k}{p}}\xi\pi\right)^{-2}+\frac{4E}{N^{2k}},\ N=1,2,\ldots.$$

Using (16) and $\sin x \geq \frac{2}{\pi}x$ for all $x\in\left[0,\frac{\pi}{2}\right]$, we obtain

$$\|u\|^2 < \frac{80}{(3-\sqrt{5})^2}\delta^2\pi^2 E N^{\frac{2k}{p}}+\frac{4E}{N^{2k}},\ \ N\geq K\delta^{-\theta}. \tag{19}$$

Let

$$g(t)=\frac{80}{(3-\sqrt{5})^2}\delta^2\pi^2 E t^{\frac{2k}{p}}+\frac{4E}{t^{2k}},\ \ t>0. \tag{20}$$

The minimum of g for $t>0$ is attained at

$$\bar{t}=\left(\frac{p}{20}\right)^{\frac{p}{2k(p+1)}}\left(\frac{3-\sqrt{5}}{\pi\delta}\right)^{\frac{p}{k(p+1)}}.$$

We choose δ as

$$0<\delta<\left\{K\left(\frac{p}{20}\right)^{\frac{p}{2k(p+1)}}\left(\frac{3-\sqrt{5}}{\pi\delta}\right)^{\frac{p}{k(p+1)}}\right\}^{\frac{k(p+1)}{k\theta(p+1)-p}}. \tag{21}$$

It follows from (21) that $\bar{t}<K\delta^{-\theta}$. Let $\overline{N}$ be the integer $\geq K\delta^{-\theta}$ for which the right side of (20) is minimum. Since g is increasing on the interval $[\bar{t},+\infty)$, $\overline{N}$ satisfies $K\delta^{-\theta}\leq\overline{N}<K\delta^{-\theta}+1$. Hence

$$\|u\|^2\leq g\left(K\delta^{-\theta}+1\right),$$

and finally

$$\|u\|^2\leq\frac{80\pi^2 E}{(3-\sqrt{5})^2}\left[K\delta^{\frac{k}{p}-\theta}+\delta^{\frac{k}{p}}\right]^{\frac{2k}{p}}+\frac{4E\delta^{2k\theta}}{K^{2k}} \tag{22}$$

which proofs following:

Theorem 2. *Let α be an irrational number and has a type $\Omega < \infty$. Then for any fixed θ, $\frac{\Omega}{\Omega+1} < \theta < 1$, there is constant $K = K(\theta, \alpha) > 0$ such that*

$$\|u\|^2 \le \frac{80\pi^2 E}{(3-\sqrt{5})^2}\left[K\delta^{\frac{k}{p}-\theta} + \delta^{\frac{k}{p}}\right]^{\frac{2k}{p}} + \frac{4E\delta^{2k\theta}}{K^{2k}}$$

for any $0 < \delta < \left\{K\left(\frac{p}{20}\right)^{\frac{p}{2k(p+1)}}\left(\frac{3-\sqrt{5}}{\pi\delta}\right)^{\frac{p}{k(p+1)}}\right\}^{\frac{k(p+1)}{k\theta(p+1)-p}}$.

We conclude with the proof of the following:

Theorem 3. *The problem (1)-(5) is stable if and only if α is irrational. Moreover, if α is irrational then $\lim_{\delta\to 0}(Diam\Gamma_\delta) = 0$ uniformly in $\varphi_j(x)$, $\psi_j(x)$, $(j = 0, 1, ..., p-1)$*

Proof. Let $\alpha \notin Q$. By corollary 9 of [4], there exist a function $f(\delta)$ such that

$$\lim_{\delta\to 0} f(\delta) = \infty, \lim_{\delta\to 0} \delta f(\delta) = 0, \tag{23}$$

and, for any sufficiently small δ, a number $\xi \notin Q$, satisfying (15) and (16) for all $N \ge f(\delta)$. The same argument given in the proof of theorem 2 shows that

$$\|u\|^2 \le g(f(\delta)+1),$$

where g is defined by (20), i.e.

$$\|u\|^2 \le \frac{80\pi^2 E}{(3-\sqrt{5})^2}\left[f(\delta)\,\delta^{\frac{k}{p}} + \delta^{\frac{k}{p}}\right]^{\frac{2k}{p}} + \frac{4E}{f(\delta)^{2k}}.$$

By (23), this yields

$$\lim_{\delta\to 0}(Diam\Gamma_\delta) = 0.$$

□

References

1. Bourgin-R. Duffin. D.G. The Dirichlet problem for the vibrating string equation. Bull. Amer. Math. Soc., 45(1939), 851-858.
2. Fox-C. Pucci. D. The Dirichlet problem for the wave equation. Ann. Mat. Pura Appl. (IV), vol. XLVI (1958),pp. 155-182.
3. John F. The Dirichlet problem for a hyperbolic equation. Amer. J. Math., 63(1941), pp. 141-154.
4. Viola C. Diophantine approximation in short intervals, Ann. Scuola Norm. Sup. Pisa, 6(1979), pp. 703-717.
5. Khinchin A. Ya. Continued fractions. The Universiry of Chicago Press, 1964, 112 p.
6. Yuldasheva A.V. On one problem for high-order. Reports of the Academy of Sciences of Uzbekistan, Tashkent, 2012, №5, pp. 11-14.

Original article submitted: 23.03.2015