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MATHEMATICAL MODELING OF NONLOCAL OSCILLATORY DUFFING SYSTEM WITH FRACTAL FRICTION

R.I. Parovik[1, 2]

[1] Institute of Cosmophysical Researches and Radio Wave Propagation Far-Eastern Branch, Russian Academy of Sciences, 684034, Kamchatskiy Kray, Paratunka, Mirnaya st., 7, Russia

[2] Vitus Bering Kamchatka State University, 683031, Petropavlovsk-Kamchatsky, Pogranichnaya st., 4, Russia

E-mail: romanparovik@gmail.com

The paper considers a nonlinear fractal oscillatory Duffing system with friction. The numerical analysis of this system by a finite-difference scheme was carried out. Phase portraits and system solutions were constructed depending on fractional parameters.

Key words: Gerasimov-Caputo operator, phase portrait, Duffing oscillator, finite-difference scheme

Introduction

Investigation of nonlinear oscillatory system is of great practical importance [1]. With the development of the theory for modeling of fractal processes, the possibility to determine new properties of nonlinear fractal oscillatory systems appeared. Such oscillatory processes are described by differential equations with fractional derivatives [2]. Fractional orders of derivatives are associated with fractal dimension of a medium, and consideration of them in an oscillatory system as complementary degrees of freedom give prerequisites for new chaotic regimes which describe real processes and phenomena. For example, the paper [3] investigates the question on modeling of damped oscillation in a vehicle tire. The paper [4] studies viscoelastic properties of beams, plates, and cylindrical shells.

Investigation of nonlinear oscillatory system with friction (Duffing oscillator) is of interest. The papers [5, 6] consider modeling of Duffing oscillator with fractal friction. The present paper makes a generalization of the suggested earlier models for Duffing oscillator, when instead of a displacement second-order derivative, an operator of fractional differentiation is introduced into an initial equation. Regimes of an oscillatory system in the result of change of fractional parameters are under the investigation. Phase portraits are constructed.

Parovik Roman Ivanovich – Ph.D. (Phys. & Math.), Dean of the Faculty of Physics and Mathematics Vitus Bering Kamchatka State University, Senior Researcher of Lab. Modeling of Physical Processes, Institute of Cosmophysical Researches and Radio Wave Propagation FEB RAS.

Problem definition

Find a solution $x(t)$, where $t \in [0,T]$, satisfying the equation

$$\partial_{0t}^{\alpha} x(\eta) + a\partial_{0t}^{\beta} x(\eta) - x(t) + x^3(t) = \delta \cos(\omega t) \tag{1}$$

and the initial conditions

$$x(0) = x_0,\ \dot{x}(0) = y_0 \tag{2}$$

where $\partial_{0t}^{\alpha} x(\eta) = \frac{1}{\Gamma(2-\alpha)} \int\limits_0^t \frac{\ddot{x}(\eta)\,d\eta}{(t-\eta)^{\alpha-1}}$, $\partial_{0t}^{\beta} x(\eta) = \frac{1}{\Gamma(1-\beta)} \int\limits_0^t \frac{\dot{x}(\eta)\,d\eta}{(t-\eta)^{\beta}}$ are operators of fractional differentiation in the sense of Gerasimov-Caputo of the order α and β; $\dot{x}(t) = dx/dt$, $\ddot{x}(t) = d^2x/dt^2$; x_0, y_0, δ, ω, a, T are given parameters.

We should note that in the papers [2, 5, 6], differentiation operator of fractional order in the sense of Riman-Liuvill was used to describe friction. We applied Gerasimov-Kaputo operator, in this case local conditions (2) are true. In the case with Riman-Liuvill operator, it is necessary to specify nonlocal conditions [7].

Method of solution

The problem (1), (2) is solved by numerical methods, explicit finite-difference scheme. Introduce τ, the sampling interval, and $t_j = j\tau$, $j = 1,2,..,N$, $N\tau = T$, $x(j\tau) = x_k$. Then fractional derivatives, entering equation (1), may be approximated as follows [8]:

$$\partial_{0t}^{\alpha} x(\eta) \approx \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{k=0}^{j-1} \left[(k+1)^{2-\alpha} - k^{2-\alpha}\right] \left(x_{j-k+1} - 2x_{j-k} + x_{j-k-1}\right) \tag{3}$$

$$\partial_{0t}^{\beta} x(\eta) \approx \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \sum_{k=0}^{j-1} \left[(k+1)^{1-\beta} - k^{1-\beta}\right] \left(x_{j-k+1} - x_{j-k}\right).$$

Substituting the relations (3) into equation (1), we obtain the following explicit finite-difference scheme:

$$x_1 = Ax_0 - Cx_0^3 + K,\ x_2 = Ax_1 - Bx_0 - Cx_1^3 + K\cos(\omega\tau),$$

$$x_{j+1} = Ax_j - Bx_{j-1} - Cx_i^3 - B\sum_{k=1}^{j-1} b_k \left(x_{j-k+1} - 2x_{j-k} + x_{j-k-1}\right) -$$

$$-M\sum_{k=1}^{j-1} c_k \left(x_{j-k+1} - x_{j-k}\right) + K\cos(\omega j\tau) \tag{4}$$

$$A = \left(\frac{2\tau^{-\alpha}}{\Gamma(3-\alpha)} + \frac{\tau^{-\beta}}{\Gamma(2-\beta)} + 1\right) \Bigg/ \left(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} + \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\right),$$

$$B = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \Bigg/ \left(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} + \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\right),\ K = \delta \Bigg/ \left(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} + \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\right),$$

$$C = 1 \Bigg/ \left(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} + \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\right), M = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \Bigg/ \left(\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} + \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\right),$$

$$b_k = (k+1)^{2-\alpha} - k^{2-\alpha}, c_k = (k+1)^{1-\beta} - k^{1-\beta}, j = 2, .., N-1.$$

The derivative $y(t) = \dot{x}(t) = dx/dt$ is approximated by a finite difference: $y_j = \dfrac{x_j - x_{j-1}}{\tau}$. Values x_0 and y_0 are determined from the initial conditions (2).

Modeling results

Numerical modeling was carried out taking into the account the following parameter values in solution of (4): $N = 4000$, $\tau = \pi/100$, $\omega = 1, \delta = 0.3, a = 0.15, x_0 = 0.2, y_0 = 0.3$. Phase portrait is drawn according to the points $(x(t), y(t))$ depending on α and β parameters.

To study the vibrational modes often use the Poincaré section. Poincaré section is a plane in the phase space, selected in such a way that all paths belonging to the attractor, crossed it under a non-zero angle.

Note, that the closed phase trajectories form a finite sequence of points in the Poincaré section (one point corresponds to the limit cycle with period T, two points correspond to the limit cycle with twice the period $2T$, non-recurring modes correspond to the infinite sequence of points in the Poincaré section. As a cross-section Poincaré choose the plane of constant phase of external influence $\omega t_n = 2\pi n$, which corresponds to the choice of the points of the phase trajectory exactly the period $T = 2\pi$ external force.

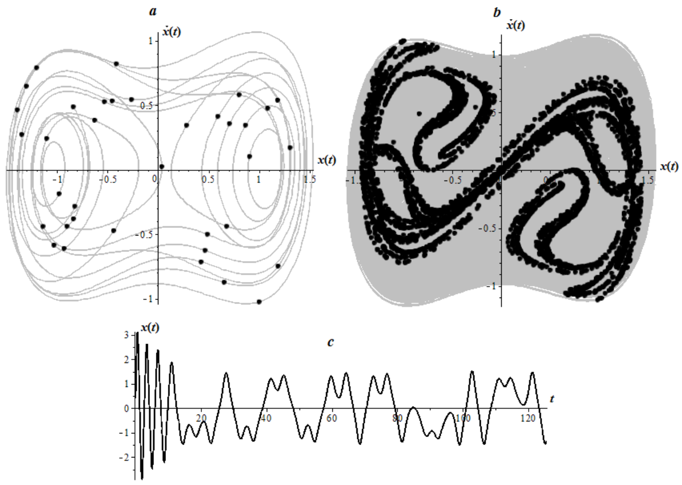

Fig. 1. Phase portrait and point Poincaré section (a), constructed in accordance with the numerical solution of (c), taking into account parameters: $N = 30000$, $\tau = \dfrac{\pi}{100}$, $\omega = 1$, $\delta = 0.3$, $a = 0.15$, $x_0 = -1.3311$, $y_0 = -0,1429$, $\alpha = 2$, $\beta = 1$; b) this Poincaré section at $N = 5 \cdot 10^5$ with the same parameter values

Fig. 1 a case $\alpha = 2$, $\beta = 1$, corresponding to the classical Duffing oscillator with friction. In this case, the memory effect in the vibrating system disappears. The solution is not periodic, and it has a chaotic character (Fig. 1 a). Confirmation of the chaotic regime for forced oscillations of fractal Duffing oscillator can be seen in Fig. 1 b, which shows the Poincare section, built with a large number of points of $N = 5 \cdot 10^5$, and the shift function $x(t)$, which is shown in Fig. 1c. Based on the points of the Poincare section

Fig. 1 b, we can conclude that the classic is a bistable Duffing oscillator oscillating system [9], which has a chaotic attractor, characteristic of deterministic chaos [10].

Fig. 2 shows the phase portrait (Fig. 2 a) and shift function (Fig. 2 b) obtained by the numerical scheme (4) in the case of: $\alpha = 2,\ \beta = 0.6$.

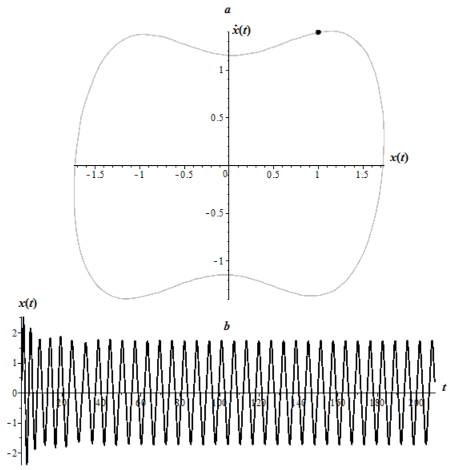

Fig. 2. Phase portrait and point Poincaré section (a), constructed in accordance with the numerical solution of (b), taking into account parameters: $N = 4000, \tau = \frac{\pi}{100}, \omega = 1, \delta = 0.3, a = 0.15, x_0 = 1.0052, y_0 = 1.3901, \alpha = 2, \beta = 0.6$

It may be noted that the solution in this mode is periodic, and the phase trajectory is a limit cycle. Poincare section consists of a single point, as shown in Fig. 2b, and this point is the same as the initial point of (x_0, y_0). Similar results were presented in the work [5]. You can also note that the cubic nonlinearity in the equation (1) leads to an increase in the frequency of oscillations (Fig. 2 b).

Fig. 3 presents the calculated curve constructed by the formula (4). The calculation parameters: the number of points of $N = 1000$, sample rate $\tau = 0.16$, $\xi = 4, \alpha = 2, \beta = 0.8, (x(0), \dot{x}(0)) = (-2.623, -4.0705)$.

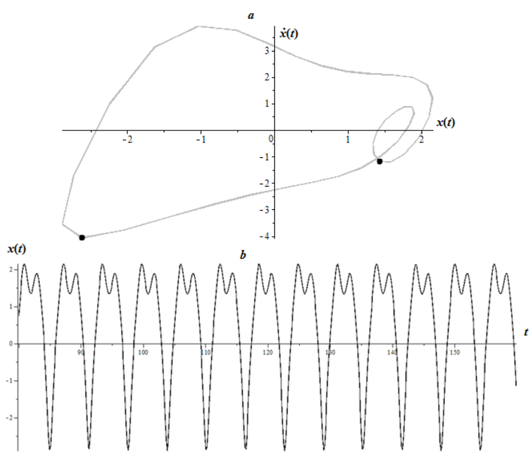

Fig. 3. The limit cycle with the points of the Poincare sections (a) and numerical two periodic solution (b) obtained by the formula (4) within the parameters: $\alpha = 2, \beta = 0.8, \tau = 0.16, a = 0.15, \delta = 4, (x(0), \dot{x}(0)) = (-2.623, -4.0705)$

Fig. 3 and and rice. 3 b that the decision has pridelnyh cycle loop, the Poincare section contains two points. Therefore, the solution is two-periodic. Have loop leads to a bifurcation of the vibration amplitude (Fig. 3 a). Similar patterns were obtained by the authors of [5].

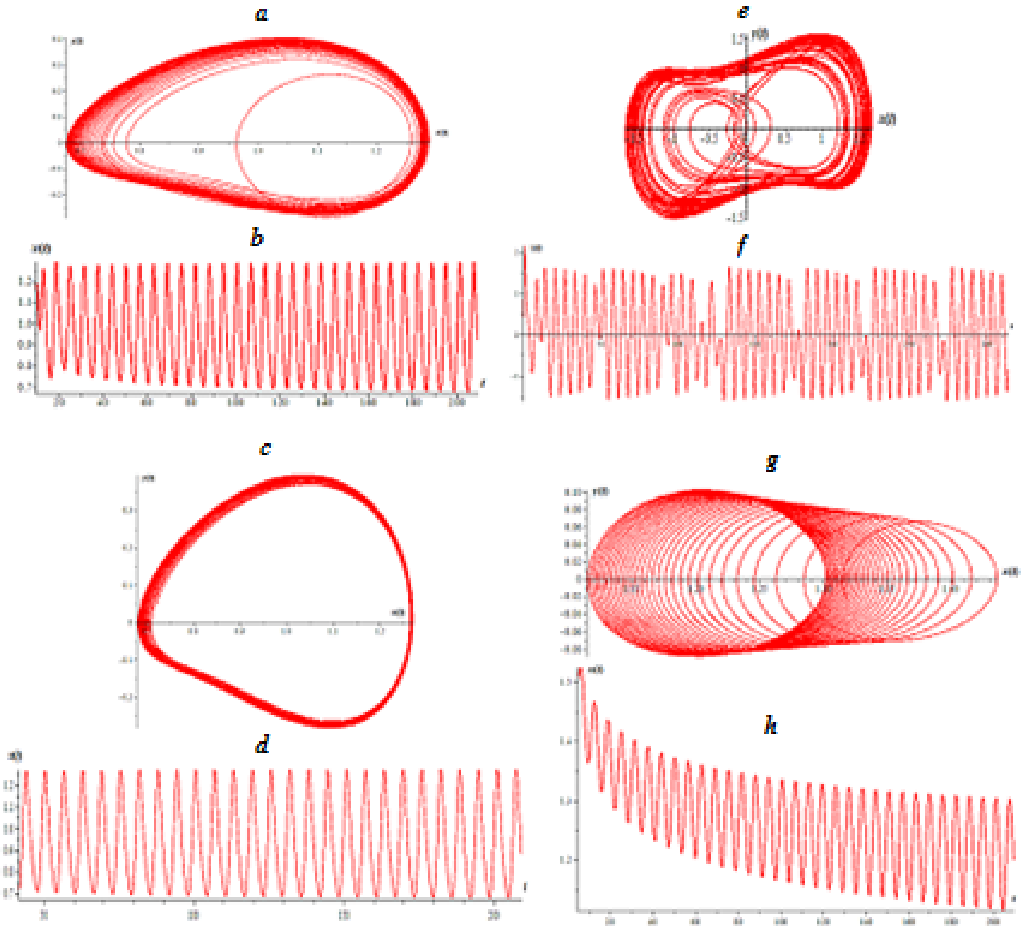

Fig. 4. Phase portrait and numerical solution (4) with regard to the parameters: (a,b) $\alpha = 1.7, \beta = 1, \tau = \frac{\pi}{60}$; (c,d) $\alpha = 1.8, \beta = 1, \frac{\pi}{40}$; (e,f) $\alpha = 1.8, \beta = 0.2, \tau = \frac{\pi}{60}$; (g,h) $\alpha = 1.3, \beta = 0.2, \tau = \frac{\pi}{60}$

Fig. 4 illustrates solution evolution and phase portraits for different parameters α, β and τ. In Fig. 4, mainly phase trajectories reach the boundary cycle. In Fig. 4c, chaotic regime is observed.

It can be concluded that the emergence of new parameters (fractional exponents) in hereditarity equation (1), widens properties Duffing oscillator and anticipates the emergence of new modes and effects in a nonlinear oscillatory systems. Orders fractional derivatives act as control parameters that define the fractal vibrational modes of the system that need to be considered when modeling.

Conclusions

The paper presents a model of fractal Duffing oscillator with friction. Numerical solutions were obtained depending on fractional parameters α and β. Phase trajectories were drawn. Solution analysis showed that there are both periodic solutions and chaotic regimes. For a more qualitative analysis in future, bifurcation diagrams will be drawn, and a test to determine the conditions for periodic solutions will be made.

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