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FINITE-DIFFERENCE SCHEMES FOR FRACTAL OSCILLATOR WITH A VARIABLE FRACTIONAL ORDER

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The paper considers explicit finite-difference schemes for a fractional oscillator. The questions of approximation, stability and convergence of these schemes are under investigation.

Key words: finite-difference scheme, convergence, stability

Introduction

Numerical methods are the important tools in mathematical simulation of different processes. They enable us to obtain approximate solutions for any model equations, and a numerical algorithm may be realized easily on a PC. For example, as a rule, we do not always succeed to obtain an analytical solution for a standard linear differential equation with constant coefficients, and it is impossible to obtain a family of theoretical curves for it.

In this paper we study one of numerical methods, finite-difference schemes, to solve a fractal or hereditary oscillator considered in the articles [1, 2]. Linear hereditary oscillators may be represented in the form of an oscillation equation with fractional differential Gerasimov-Caputo operators of the orders $1 < \beta < 2$ и $0 < \gamma < 1$ with friction coefficient λ and external influence $f(t)$:

$$\partial_{0t}^{\beta} x(\tau) + \partial_{0t}^{\gamma} x(\tau) + A(t)x(t) = f(t), x(0) = x_0, \dot{x}(0) = y_0, \quad (1)$$

where $\partial_{0t}^{\alpha} x(\tau) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t \frac{x^{(n)}(\tau)\,d\tau}{(x-\tau)^{\alpha-n+1}}, n < \alpha < n+1$ is the Gerasimov-Caputo operator; $A(t)$ is a some known function which determined the form of linear hereditary oscillator.

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The papers [3, 4] propose some finite-difference schemes to solve linear and nonlinear fractal oscillation equations. We may note, that construction of such difference schemes was based on the approximation of Rieman-Liouville fractional derivative, Gruenwald-Letnikov difference [5].

Another class of explicit finite-difference schemes may be constructed taking into consideration the fact that fractional control parameters, fractional derivative orders, which are included into the initial equation, are functions of time. In this case, equation (1) has the form:

$$\partial_{0t}^{\beta(t)} x(\tau) + \partial_{0t}^{\gamma(t)} x(\tau) + A(t) x(t) = f(t), x(0) = x_0, \dot{x}(0) = y_0, \tag{2}$$

Differential equations with variable-order fractional derivatives are described for different applications [6, 7, 8, 9].

Be specific, in equation (2) we will consider that $A(t) = \omega^{\beta(t)}$.

Statement of the problem and method of solution

A fractal oscillator with variable fractional orders is given by the following equation with initial conditions:

$$\partial_{0t}^{\beta(t)} x(\tau) + \partial_{0t}^{\gamma(t)} x(\tau) + \omega^{\beta(t)} x(t) = f(t), x(0) = x_0, \dot{x}(0) = y_0, \tag{3}$$

Equation (3) characterizes the fractal oscillator, the particular case of which is considered in the paper [10]. Variable-order fractional derivatives in equation (3) may be approximated as follows [11]:

$$\partial_{0t}^{\beta(t)} x(\eta) \approx \frac{\tau^{-\beta_j}}{\Gamma(3-\beta_j)} \sum_{k=0}^{j-1} \left[(k+1)^{2-\beta_j} - k^{2-\beta_j}\right] \left(x_{j-k+1} - 2x_{j-k} + x_{j-k-1}\right), \tag{4}$$

$$\partial_{0t}^{\gamma(t)} x(\eta) \approx \frac{\tau^{-\gamma_j}}{\Gamma(2-\gamma_j)} \sum_{k=0}^{j-1} \left[(k+1)^{1-\gamma_j} - k^{2-\gamma_j}\right] \left(x_{j-k+1} - x_{j-k}\right).$$

The following representation is also true [11]:

$$\partial_{0t}^{\beta(t)} x(\eta) \approx \sum_{k=0}^{j-1} \frac{\tau^{-\beta_k}}{\Gamma(3-\beta_k)} \left[(k+1)^{2-\beta_j} - k^{2-\beta_j}\right] \left(x_{j-k+1} - 2x_{j-k} + x_{j-k-1}\right), \tag{5}$$

$$\partial_{0t}^{\gamma(t)} x(\eta) \approx \sum_{k=0}^{j-1} \frac{\tau^{-\gamma_k}}{\Gamma(2-\gamma_k)} \left[(k+1)^{1-\gamma_j} - k^{1-\gamma_j}\right] \left(x_{j-k+1} - x_{j-k}\right).$$

Substituting approximations (4) and (5) into equation (3), we obtain two finite-difference schemes. The first scheme is:

$$x_1 = A_0 x_0 + f_0, j = 0,$$

$$x_{j+1} = A_j x_j - B_j x_{j-1} - B_j \sum_{k=1}^{j-1} p_k^j \left(x_{j-k+1} - 2x_{j-k} + x_{j-k-1}\right) - \tag{6}$$

$$-C_j\sum_{k=1}^{j-1} q_k^j\left(x_{j-k+1}-x_{j-k}\right)+f_{j+1},$$

$$A_j=\frac{2A_1^j+B_1^j-\omega^{\beta_j}}{A_1^j+B_1^j}, B_j=\frac{A_1^j}{A_1^j+B_1^j}, C_j=\frac{B_1^j}{A_1^j+B_1^j},$$

$$A_1^j=\frac{\tau^{-\beta_j}}{\Gamma\left(3-\beta_j\right)}, B_1^j=\frac{\lambda\tau^{-\gamma_j}}{\Gamma\left(2-\gamma_j\right)},$$

$$p_k^j=(k+1)^{2-\beta_j}-k^{2-\beta_j}, q_k^j=(k+1)^{1-\gamma_j}-k^{1-\gamma_j}, j=1,\ldots,N-1.$$

, and the second scheme is:

$$x_1=A_0x_0+f_0, j=0,$$

$$x_{j+1}=A_jx_j-Bx_{j-1}-C\sum_{k=1}^{j-1}\frac{\tau^{-\beta_k}}{\Gamma\left(3-\beta_k\right)}p_k^j\left(x_{j-k+1}-2x_{j-k}+x_{j-k-1}\right)- \tag{7}$$

$$-D\sum_{k=1}^{j-1}\frac{\lambda\tau^{-\gamma_k}}{\Gamma\left(2-\gamma_k\right)}q_k^j\left(x_{j-k+1}-x_{j-k}\right)+f_{j+1},$$

$$A_j=\frac{2A_0+B_0-\omega^{\beta_j}}{A_0+B_0}, B=\frac{A_0}{A_0+B_0}, C=D=\frac{1}{A_0+B_0},$$

$$A_0=\frac{\tau^{-\beta_0}}{\Gamma\left(3-\beta_0\right)}, B_0=\frac{\lambda\tau^{-\gamma_0}}{\Gamma\left(2-\gamma_0\right)},$$

$$p_k^j=(k+1)^{2-\beta_j}-k^{2-\beta_j}, q_k^j=(k+1)^{1-\gamma_j}-k^{1-\gamma_j}, j=1,\ldots,N-1.$$

We should note, that according to the second initial condition from (3): $x_1=\tau y_0+x_0$. Thus, we will take this condition into consideration during the numerical simulation.

Approximation, stability and convergence of explicit finite-difference schemes

In this section we consider the questions of approximation, stability and convergence of the explicit finite-difference schemes (6) and (7). First, we mention the properties of the coefficients from these schemes.

Lemma. *The coefficients for the explicit difference schemes (6) and (7) have the following properties:*

1) $0<B_j,C_j,B,C,D,p_k^j,q_k^j<1$;

2) $A,A_0,B_0,A_1,B_1,A_j,A_1^j,\omega^{\beta_j},B_1^j>0.$

We consider the structure of the explicit finite-difference scheme (6) which can be rewritten in a matrix form as follows:

$$M\cdot X=F, \tag{8}$$

$$M=\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \omega^{\beta_0}-2A_1^0-B_1^0 & A_1^0+B_1^0 & 0 & 0 & \dots \\ A_1^1p_0^1 & \omega^{\beta_1}-2A_1^1-B_1^1 & A_1^1+B_1^1 & 0 & \dots \\ A_1^2p_1^2 & A_1^2\left(1-2p_1^2\right)-B_1^2q_1^2 & \omega^{\beta_2}+A_1^2\left(p_1^2-2\right)+B_1^2\left(q_1^2-1\right) & A_1^2+B_1^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$X=\left[x_0,\tau y_0+x_0,\dots,x_N\right]^T, F=\left[f_0,f_1,\dots,f_N\right]^T.$$

Owing to the Lemma, the diagonal elements of the triangular matrix M are positive (the determinant is more than zero) and, consequently, it is invertible. Thus, the matrix equation (8) is solvable and the numerical solution for equation (3) may be obtained. Analogous arguments are also true for the explicit finite-difference scheme (7).

Then we rewrite scheme (6) as follows:

$$x_{j+1}=\left(AA_j-BB_j-CC_j\right)x_j-D_jx_{j-1}- \tag{9}$$

$$-B_j\sum_{k=2}^{j-1}p_k^j\left(x_{j-k+1}-2x_{j-k}+x_{j-k-1}\right)-C_j\sum_{k=2}^{j-1}q_k^j\left(x_{j-k+1}-x_{j-k}\right)+f_j,$$

where $AA_j=A_j-B_jp_1^j-C_jq_1^j$, $BB_j=p_1^jB_j$, $CC_j=q_1^jC_j$, $D_j=B_j-2B_jp_1^j-C_jq_1^j$.

Following the method described in the paper [12], de derive the error as a difference: $\varepsilon_{j+1}=x_{j+1}-\bar{x}_{j+1}$, where $\bar{x}_{j+1}$ is the exact value of solution of $x(t)$ at the point $t=t_{j+1}$, which we substitute into equation (9). The error will satisfy the following inequality:

$$|\varepsilon_{j+1}|\leq|AA_j-BB_j-CC_j||\varepsilon_j|. \tag{10}$$

We define the following theorem.

Theorem 1. *The explicit finite-difference scheme (6) is stable if the following condition if fulfilled:*

$$A_1^j\leq\omega^{\beta_j}\leq A_1^j\left(3-p_1^j\right)+B_1^j\left(2-q_1^j\right). \tag{11}$$

Proof. We rewrite the condition (10) as follows:

$$|\varepsilon_{j+1}|\leq\left|\frac{2A_1^j+B_1^j-\omega^{\beta_j}-A_1^jp_1^j-B_1^jq_1^j}{A_1^j+B_1^j}\right||\varepsilon_j|. \tag{12}$$

It is obvious that in the case of $A_1^j-\omega^{\beta_j}\leq 0$ and taking into account the Lemma, the following condition holds:

$$\frac{2A_1^j+B_1^j-\omega^{\beta_j}-A_1^jp_1^j-B_1^jq_1^j}{A_1^j+B_1^j}\leq 1.$$

On the other hand, the condition

$$\frac{2A_1^j+B_1^j-\omega^{\beta_j}-A_1^jp_1^j-B_1^jq_1^j}{A_1^j+B_1^j}\geq -1$$

can be fulfilled if:

$$\omega^{\beta_j} \leq A_1^j\left(3-p_1^j\right)+B_1^j\left(2-q_1^j\right).$$

Thus, according to (11) we have:

$$\left|\frac{2A_1^j+B_1^j-\omega^{\beta_j}-A_1^jp_1^j-B_1^jq_1^j}{A_1^j+B_1^j}\right| \leq 1$$

Consequently, the error is limited, and as the interval τ is reduced, the following condition is fulfilled: $\varepsilon_{j+1} \to 0$ at $j \to \infty$.

□

In much the same way, we may define a theorem for the explicit finite-difference (7).

Theorem 2. *The explicit finite-difference scheme (7) is stable if the following condition holds:*

$$A_0 \leq \omega^{\beta_j} \leq 3A_0+2B_0-A_1p_1^j-B_1q_1^j, \tag{13}$$

where $A_1=\dfrac{\tau^{-\beta_1}}{\Gamma(3-\beta_1)}, B_1=\dfrac{\lambda\tau^{-\gamma_1}}{\Gamma(2-\gamma_1)}$.

This theorem is proved just in the same way taking into account the Lemma.

The convergence of the explicit finite-difference schemes is associated with the approximation of fractional differentiation Gerasimov-Caputo operators from which it follows that approximation in the inner nodes has the second order, but in general the scheme order decreases to the first one due to approximation at the frontier points. Thus, we may conclude that numerical solution approximates the exact solution with first order approximation.

The convergence of the explicit schemes (6) and (7) taking into account their stability based on the theorems 1 and 2 follows from the Lax-Richtmyer theorem [13], according to which there is first order convergence of the numerical solution to the exact one. We can show that the inequality $|x(t_j)-x_j| \leq C\tau$ is fulfilled, where C is an arbitrary constant which does not depend on the interval τ.

Results of the simulation

In order to solve the problems of approximation, stability and convergence of the explicit schemes (6) and (7), we consider the following Cauchy problem:

$$\partial_{0t}^{\beta(t)}x(\tau)+\lambda\partial_{0t}^{\gamma(t)}x(\tau)+\omega^{\beta(t)}x(t)=\frac{2t^{2-\beta(t)}}{\Gamma(3-\beta(t))}+\frac{2\lambda t^{2-\gamma(t)}}{\Gamma(3-\gamma(t))}+\omega^{\beta(t)}t^2, \tag{14}$$

with homogeneous initial conditions $x(0)=\dot{x}(0)=0$ and the right part:

$$f(t)=\frac{2t^{2-\beta(t)}}{\Gamma(3-\beta(t))}+\frac{2\lambda t^{2-\gamma(t)}}{\Gamma(3-\gamma(t))}+\omega^{\beta(t)}t^2$$

Problem (14) has the exact solution:

$$x(t)=t^2. \tag{15}$$

We consider a numerical solution of problem (14) obtained by the explicit finite-difference schemes (6) and (7) taking into account the following control parameters:

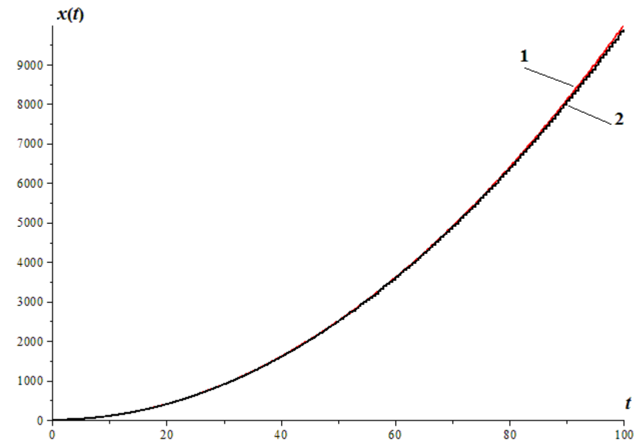

Fig. 1. Computational explicit finite-differential schemes: 1 - scheme (6); 2 - scheme (7)

Table 1

Convergence of the explicit schemes (6) and (7)

t	Exact solution	Error for (6)	Error for (7)
0.05	0.0025	0.0425	0.0425
0.1	0.01	0.084	0.084
0.15	0.0225	0.119	0.12
0.2	0.04	0.15	0.152
0.25	0.0625	0.175	0.18
0.3	0.09	0.19	0.2

$\beta(t) = 1.8 - \frac{0.5t}{T}, \gamma(t) = 0.8 - \frac{0.3t}{T}, t \in (0,T), T = 1000, N = 2000, \tau = 0.5, \lambda = 0.02, \omega = 1, t_j = j\tau, j = 0,\cdots,N-1.$

Results of the simulation are shown in Fig. 1 and in Table 1.

It follows from Table 1 that the numerical solutions obtained by the explicit schemes (6) and (7) approximate quite well the exact solution (15). We may also note that the first explicit finite-differential scheme (6) approximates better the exact solution than the second explicit scheme (7) which is shown in Fig. 1. So, we note that the calculations by formulas (6) and (7) may be considered to be similar.

We may also mention that as the sampling interval τ is reduced, absolute errors for schemes (6) and (7) vanish. Results of the simulation are illustrated in Table (2).

Table 2

Convergence of the explicit schemes (6) and (7)

τ	Exact solution	Error for (6)	Error for (7)
0.05	0.0025	0.0425	0.0425
0.025	0.000625	0.0218	0.0218
0.0125	0.00015	0.011	0.011
0.00625	0.00007	0.0074	0.0074

We should note that in this example the approximation has the first order. According to the definition, approximation of p-th order is really characterized by the inequality $|\varepsilon_j| = |x(t_j) - x_j| \leq C\tau^p$, where C is an arbitrary constant which does not depend on the interval τ. Having found the algorithm of this inequality, we obtain the ratio: $\ln|\varepsilon_j| \leq \ln C + p\ln\tau$. Having plotted a graph of the straight line in a double logarithmic scale, we determine the approximation order p as the angle of its inclination (Fig. 2).

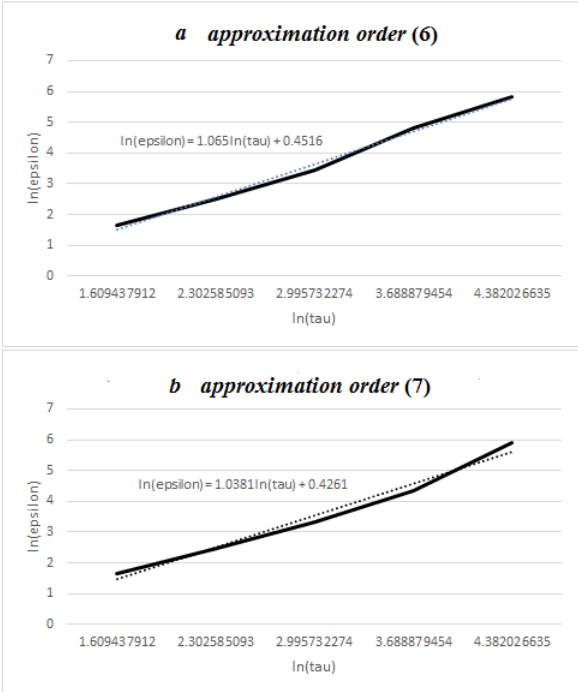

Fig. 2. Approximation order of explicit finite-difference schemes: a) – scheme (6); b)– scheme (7)

It is clear from Fig.2 that approximation order for the first scheme (6) is $p = 1.065$, and for the second scheme (7) – $p = 1.038$.

Conclusions

In this paper, two explicit finite-differential schemes for a fractal oscillator were considered. They are stable, approximate and converge to the exact solution with the first order. Approximation order may be increased to the second one if initial conditions are properly approximated.

On the example, the results of the simulation showed that calculations by the formulas (6) and (7) may be considered to be almost similar and, consequently, they may be applied in the simulation of oscillatory systems.

References

1. Parovik R.I. Numerical analysis of some oscillation equations with fractional order derivatives. Bulletin KRASEC. Physical and Mathematical Sciences, 2014, vol. 9, no. 2, pp. 34–38.

2. Parovik R.I. Matematicheskoe modelirovanie ereditarnogo ostsillyatora [Mathematical simulation of heredity oscillator]. Komp'yuternye issledovaniya i modelirovaniya – Computer Research and Modeling, 2015, V. 7, No. 5, pp. 1002–1023.

3. Momani S., Odibat Z. Numerical comparison of methods for solving linear differential equations of fractional order. Chaos, Solitons & Fractals. 2007. Vol. 31, no. 5. P. 1248–1255.

4. Petras I. Fractional-Order Nonlinear Systems. Modeling, Analysis and Simulation. Beijing and Springer Verlag Berlin Heidelberg: Springer, 2011. 218 p.

5. Oldham K. B., Spanier J. The fractional calculus. Theory and applications of differentiation and integration to arbitrary order. London: Academic Press, 1974. 240 p.

6. Mainardi F. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. Chaos, Solitons & Fractals. 1996. Vol. 7, no. 9. P. 1461–1477.

7. Coimbra C.F.M. Mechanics with variable-order differential operators. Annalen der Physik. 2003. Vol. 12, no. 11-12. P. 692–703.

8. Sheng H., Sun H.G., Coopmans C. et al. A physical experimental study of variable-order fractional integrator and differentiator. The European Physical Journal Special Topics. 2011. Vol. 193, no. 1. P. 93–104.

9. Shichang Ma, Yufeng Xu, Wei Yue. Numerical solutions of a variable-order fractional financial system. Journal of Applied Mathematics. 2012.

10. Meilanov R. P. K teorii fil'tratsii v poristykh sredakh s fraktal'noi strukturoi [To the theory of filtration in porous mediums with fractal structure]. Pis'ma v ZhTF – Technical Physics Letters, 1996, V. 22, No. 23. pp. 40–43.

11. Sun H. G., Chen W., Wei H., Chen Y. Q. A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems. The European Physical Journal Special Topics. 2011. Vol. 193, no. 1. P. 185–192.

12. Yufeng Xu, Vedat Suat Erturk. A finite difference technique for solving variable-order fractional integro-differential equations. Bulletin of the Iranian Mathematical Society. 2014. Vol. 40, no. 3. P. 699–712.

13. Samarskii A.A. Teoriya raznostnykh skhem [Theory of difference schemes], Moscow, Nauka, 1977, 656 p.

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