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PAINLEVÉ TEST FOR A MAGNETOHYDRODYNAMICS SYSTEM

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One of small size approximations of a magnetohydrodynamics equation, which describes cosmic object magnetic fields, is considered. The analytic properties of a nonlinear system are investigated by the Painlevé test. For a simplified magnetohydrodynamics system, we calculated the coefficients when the necessary condition of Painlevé property holds.

Key words: Painleve test, Kovalevski-Gambier method.

Introduction

Existance of large-scale magnetic fields of planets, stars and galacies is well explained by the dynamo mechanism [1]. Mathematical study of this mechanism is reduced to the solution of magnetodynamic equations. Nonlinearity and principle three-dimensionality (due to the known antidynamo theorems) of these equations makes it impossible to solve them analytically.

Large values of Reynolds number for cosmic dynamo-systems during direct numerical modeling require the space resolution and that means the number of difference mesh nodes which are invisible even for supercomputers now and in the foreseable future [2].

In the result, different simplified models which have qualitatively the main features of dynamo-mechaism are actively studied.

Besides the question on simulation of the main features of dynamo-systems in these models, the question on the analytical properties of the solutions is quite interesting. This paper investigates just one of such questions for a small size approximation of dynamo equations, the condition for this system to pass the Painlevé test.

First, we consider schematically the derivation of the system from dynamo equations.

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The process of generation of the magentic field by turbulent flows in a viscous incompressible fluid taking into accoutn the α-effect in a rotating coordinate system is described by the following magnetohydrodynamic equations:

$$
\begin{aligned}
&\partial_t \mathbf{v} + \mathrm{R}\,(\mathbf{v}\nabla)\,\mathbf{v} = \mathrm{Pm}\Delta\mathbf{v} - \nabla p - \mathrm{E}^{-1}\mathrm{Pm}\,(\mathbf{e}_z \times \mathbf{v}) + \mathrm{rot}\mathbf{B} \times \mathbf{B},\\
&\partial_t \mathbf{B} = \mathrm{R_m}\mathrm{rot}\,(\mathbf{v} \times \mathbf{B}) + \mathrm{R}_\alpha \mathrm{rot}\,(\alpha\mathbf{B}) + \triangle\mathbf{B},\\
&\nabla \cdot \mathbf{v} = 0,\\
&\nabla \cdot \mathbf{B} = 0.
\end{aligned}
\tag{1}
$$

Here $\mathbf{v}$ is the average (large-scale) velocity, $\mathbf{B}$ is the average (large-scale) magnetic field, p is pressure, $\mathbf{f}$ is the mass density of external forces, α is tensor of α-effect, R isсистема алгебраических уравнений является неразрешима the Reynold number, $\mathrm{R_m}$ is the Reynold magnetic number, E is the Ekman number, Pm Prandtl magnetic number, R_α is the amplitude of α-effect, $\mathbf{e}_z$ is ethe basis vector of rotation axis.

We will consider all the fields to be axially symmetric relatively the rotation axis.

Nondivergence of fields $\mathbf{v}$ and $\mathbf{B}$ makes it possible to represent them as a sum of toroidal and poloidal components.

We consider the case of detection of the largest-scale structures when toroidal and poloidal components of the velocity and the magnetic field may be represented as a product of stationary fields by amplitudes depending on time:

$$
\begin{aligned}
&\mathbf{v} = x_1(t)\mathbf{v}^T(\mathbf{r}) + x_2(t)\mathbf{v}^P(\mathbf{r}),\\
&\mathbf{B} = y_1(t)\mathbf{B}^T(\mathbf{r}) + y_2(t)\mathbf{B}^P(\mathbf{r}).
\end{aligned}
\tag{2}
$$

Substitution of expansions (2) into equations (1) and application of a standard Galerkin procedure result in an equation system for the amplitutes [3]:

$$
\begin{aligned}
&\frac{dx_1}{dt} = \mathrm{R}A_{112}x_1x_2 + \mathrm{E}^{-1}\mathrm{Pm}P_{12}x_2 + F_1 + L_{112}y_1y_2 - \mu_1x_1,\\
&\frac{dx_2}{dt} = \mathrm{R}A_{211}x_1^2 + \mathrm{E}^{-1}\mathrm{Pm}P_{21}x_1 + F_2 + L_{211}y_1^2 + L_{222}y_2^2 - \mu_2x_2,\\
&\frac{dy_1}{dt} = \mathrm{R_m}W_{112}x_1y_2 + \mathrm{R_m}W_{121}x_2y_1 + \mathrm{R}_\alpha W_{1\alpha2}y_2 - \eta_1y_1,\\
&\frac{dy_2}{dt} = \mathrm{R_m}W_{222}x_2y_2 + \mathrm{R}_\alpha W_{2\alpha1}y_1 - \eta_2y_2.
\end{aligned}
\tag{3}
$$

This system takes into accout the fact that in an axially symmetric case, vector lines of any poloidal field lie in the planes passing through a rotation axix, and the lines of any toroidal field are perpendecular to them. Capital letters with a suffix denote constant coefficients arising in the result of Galerkin method, μ_i and η_i denote dissipation rates of velocity and magnetic field modes from expansion (2). Besides, always $P_{12} = -P_{21}$.

Moreover, we assume that the foloing ratios are fulfilled $A_{112} = -A_{211}$, $L_{112} = -W_{112}$, $L_{211} = -W_{121}$, $L_{222} = -W_{222}$. These ratios arise in the result of the following considerations. It is known that in a dissipationless limit, three-dimensional magnetohydrodynamic equations satisfy tree conservation laws: conservation of total energy, conservation of cross helicity, conservation of magnetic helicity [2]. If we demand the realization of an analogue of the total energy conservation law in our system in the form $x_1^2 + x_2^2 + y_1^2 + y_2^2 =$

const, then there is a need in such relations between the coefficient. For the sake of simplisity we also assmume that dissipation rates of toroidal and poloidal modes coinside, from which $\mu_1 = \mu_2$, $\eta_1 = \eta_2$.

We also assume that $F_1 = 0$. Physically that means that there is no source of toroidal motion in the dynamo-system. Toroidal component of velocity arises only due to the Coriolis deflection of the velocity poloidal component having an external source F_2.

System (3) is one of the possible simpliest models of non-kinematic dynamo. Further in the paper we investigate some analytical properties of it.

Simplified dynamic system

We simplify the dynamic system (3), imposing (just formally and not from physical considerations) additional restrictions accepting the following equalities and introducing the change of notations: $L_{222} = L_{211} = W_{121} = W_{222} = F_1 = 0$, $F_2 = M$, $R_\alpha W_{2\alpha 1} = R_\alpha W_{1\alpha 2} = \alpha$, $P_{12} = K$, $P_{21} = -K$, $x_1 = u_2$, $x_2 = u_1$, $y_1 = u_3$, $y_2 = u_4$.

For the simplified system we find the parameter values when it passes the Painlevé test. In this case, solution of the system can be represented in the form of Laurent series with free arbitrary coefficients. Now we realize the Kovalevskaya-Gambier method [4].

The main stages of the method are:

1) We substitute $u(x) = u_0 x^p$. Then we determine the value p of the highest power to equal the corresponding summands.

2) After that, for each index j starting from $j = 0$, we calculated the expansion coefficients $u(x) = u_j x^{p+j}$. For each index j we obtain a system of linear algebraic equations. If we succeed to solve the systems for each index j uniquely, we obtain a unique representation of the solution in the form of Laurent series. To determine the indexes j, when the sistem is overdetermined system, it is necessary to calculate the Fuchs indexes.

3) If the system of algebraic euations is unsolvable for at least one index j, the system of differential equations does not pass the Painlevé test. If all the sytems of algebraic equations determined by the Fuchs inexes are consistent, the sytem passes the Painlevé test.

A simplified dynamic system has the following form:

$$\begin{cases} \dfrac{du_1}{dx} = -\lambda\, u_1(x) + K u_2(x) - L u_3(x)\, u_4(x) \\ \dfrac{du_2}{dx} = -\lambda\, u_2(x) - K u_1(x) + M \\ \dfrac{du_3}{dx} = L u_1(x)\, u_4(x) + \alpha\, u_4(x) - u_3(x) \\ \dfrac{du_4}{dx} = -\alpha\, u_3(x) - u_4(x) \end{cases} \tag{4}$$

where M, L, K, λ are the system parameters.

1 stage. We substitute the expansion $u_i(x) = u_{i,0}x^{p_i}, i = 1..4$ into the sytem of equations (4). The equation system for the leading summands from the expansion result in the following ratios:

$$\begin{cases} p_1 - 1 = p_3 + p_4 \\ p_2 - 1 = p_1 \\ p_3 - 1 = p_1 + p_4 \\ p_4 - 1 = p_3 \end{cases} \tag{5}$$

Solution of the linear sytem (5) give only integer coefficients for the veriables p_i

$$p_1 = -2, p_2 = -1, p_3 = -2, p_4 = -1 \tag{6}$$

We find the coefficients $u_{i,0}$ from the system at the expansion highest powers for the given values p_i, $i = 1..4$:

$p_1x_1 = -Lx_3x_4, p_2x_2 = -Kx_1, p_3x_3 = Lx_1x_4, p_4x_4 = -\alpha x_3$

Thus, we have several solution sets, they are: one trivial solutioan and four nonnull solutions.

$$u_{1,0} = \frac{-2}{L\alpha}, u_{2,0} = \frac{-2K}{L\alpha}, u_{3,0} = \pm\frac{2I}{L\alpha}, u_{4,0} = \pm\frac{\pm 2I}{L} \tag{7}$$

where I is an imaginary unit.

We consider the first set of coefficients $u_{3,0} = \frac{2I}{L\alpha}$, $u_{4,0} = -\frac{2I}{L}$. The coefficients $u_{3,0}$, $u_{4,0}$ have different signs. We calculate the Fuchs indexes from the following matrix determinant:

$$\begin{bmatrix} \alpha(j-2) & 0 & -2I\alpha & 2I \\ K & j-1 & 0 & 0 \\ 2I\alpha & 0 & \alpha(j-2) & 2 \\ 0 & 0 & \alpha & j-1 \end{bmatrix} \tag{8}$$

We obtain irrational values of Fuchs indexes

$$0, 1, 5/2 - 1/2\sqrt{17}, 5/2 + 1/2\sqrt{17} \tag{9}$$

In this context, we need to choose $u_{3,0}$, $u_{4,0}$ only with the same signs to contine the analysis of the system. If $u_{3,0}$ and $u_{4,0}$ have the same signes, the matric to calculate Fuchs indexes has the following form:

$$\begin{bmatrix} \alpha j - 2\alpha & 0 & 2I\alpha & 2I \\ K & j-1 & 0 & 0 \\ -2I\alpha & 0 & \alpha j - 2\alpha & 2 \\ 0 & 0 & \alpha & j-1 \end{bmatrix} \tag{10}$$

Equating the determinant of matrix (10) to zero, we obtain Fuchs indexes $-1, 1, 2, 4$. For each Fuchs index, we calculate the invariants of the dynamic system:

$$Q_1 = -\frac{8I(\lambda - 1)}{3L}$$

$$Q_2 = \frac{IK^2}{L} + \frac{I\left(17\lambda^2 - 4\lambda - 4\right)}{9L}$$

$$Q_4 = \frac{(\lambda-1)}{3}\left(-\frac{33\alpha^2\lambda + 12\lambda^3 + 27K^2 - 15\alpha^2 + 43\lambda^2 - 56\lambda + 16}{3L\alpha} + \right.$$
$$\left. +2\,\frac{(7\lambda-4)(u_{2,2}-\Delta) + \left(3K^2 + 7\lambda^2 - 4\lambda\right)u_{2,1}}{K} + \right) - \frac{\alpha K^2}{L} - \frac{\alpha}{L} - KM$$

Assuming that $Q_1 = Q_2 = Q_4 = 0$, we obtain the following values for the system parameters $\lambda = 1$, $K = \pm I$, $M = 0$. The system of differential equations changes to:

$$\frac{d}{dx}u_1(x) = -u_1(x) + Iu_2(x) - Lu_3(x)u_4(x) \tag{11}$$

$$\frac{d}{dx}u_2(x) = -u_2(x) - Iu_1(x) \tag{12}$$

$$\frac{d}{dx}u_3(x) = Lu_1(x)u_4(x) + \alpha u_4(x) - u_3(x) \tag{13}$$

$$\frac{d}{dx}u_4(x) = -\alpha u_3(x) - u_4(x) \tag{14}$$

Thus, we obtain a unique representation of the solution in the form of Laurent series. Consequently, for such parameters λ, K, M we write the explicit expansion into a series:

$$u_1 = 2\frac{\left(x^{-1} - x^{-2}\right)}{L\alpha} + I(1-2x)(u_{2,1} + u_{2,2}) - \frac{2\left(\alpha^2+1\right)}{L\alpha}x + u_{1,4}x^2 + \ldots \tag{15}$$

$$u_2 = \frac{-2I}{L\alpha x} + \left(1 - x^2\right)u_{2,1} + \left(x - 3/2x^2\right)u_{2,2} + \frac{I\left(\alpha^2+1\right)}{L\alpha}x^2 + u_{2,4}x^3 + \ldots \tag{16}$$

$$u_3 = \frac{2I\left(x^{-2} - x^{-1}\right)}{L\alpha} + (1-2x)u_{2,2} + \left(\frac{I\left(\alpha^2+1\right)}{L\alpha} - u_{2,1}\right)(x-1) + u_{3,4}x^2 + \ldots \tag{17}$$

$$u_4 = \frac{2I}{Lx} + \alpha\left(x^2 - x\right)u_{2,1} + \alpha\left(3/2x^2 - x\right)u_{2,2} + \frac{I\left(\alpha^2+1\right)\left(x - x^2\right)}{L} + u_{4,4}x^3 + \ldots \tag{18}$$

Conclusions

In the simplified representation, the dynamic system passes the Painlevé test with the determed set of values for the coefficients. To pass the Painlevé test is not the suffisient condition to have Painlevé properties. The obtained conditions for the coefficients of a simplified system may be applied for comparison with analytical and numerical studies of the system solutions.

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