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NECESSARY AND SUFFICIENT CONDITIONS FOR THE UNIQUENESS OF DIRICHLET PROBLEM SOLUTION FOR NONLOCAL WAVE EQUATION

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In this paper we find the necessary and sufficient conditions for the uniqueness of Dirichlet problem solution for a wave equation.

Key words: Dirichlet problem, Mittag-Leffler function, wave equation

Introduction

In the domain $\Omega = \{(x,y) : 0 < x < a, 0 < y < b\}$ we consider the equation:

$$\frac{\partial^2 u}{\partial x^2} - \partial_{0y}^{\alpha} u(x,\eta) = 0, \quad 1 < \alpha < 2, \tag{1}$$

where $\partial_{0y}^{\alpha} u(x,\eta) = D_{0y}^{\alpha-2} u_{\eta\eta} = \frac{1}{\Gamma(2-\alpha)} \int\limits_0^y (y-\eta)^{1-\alpha} u_{\eta\eta}(x,\eta) d\eta$ – is a regularized partial fractional derivative of the order α with respect to a variable y [1, p. 11].

The paper [2] proves the unique solvability of Dirichlet problem for a wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

in a rectangular domain Ω with irrational side ratio:

$$\frac{b}{a} \notin \mathbb{Q},$$

in the class of functions, which are continuously differentiable and have in Ω second derivatives Lebesgue integrable.

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The papers [3], [4, p. 374] are devoted to the investigation of Cauchy problem for equation (1). Solution of the first boundary value problem for equation (1) can be found in the paper [5].

The paper [6] proves that Dirichlet problem for equation (1) in the domain Ω has a unique solution only when

$$\sin_\alpha(\lambda_n^{1/\alpha} b) \neq 0,$$

where $\sin_\alpha(z)$ is a generalized trigonometric sine [1, p. 238]:

$$\sin_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{\alpha k+1}}{\Gamma(\alpha k+2)}.$$

In this paper, the necessary and the sufficient condition of the unique solvability of Dirichlet problem for equation (1) in a rectangular domain Ω was found. It agrees with the condition of the uniqueness of Dirichlet problem solution for a wave equation.

We consider the Mittag-Leffler function [7, p. 117]:

$$E_{1/\alpha}(z;\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\mu)},\ \alpha > 0. \tag{2}$$

It is known that at $0 < \alpha < 2$, the Mittag-Leffler function (2) has only a finite number of real zeros [7, p. 142]; at $\alpha \leq \frac{4}{3}$, $\mu = 2$ the function does not have zeros [1, p. 129], [8]; at $\alpha \in \left[\frac{5}{3}, 2\right)$, $\mu = 2$ function (2) has not less than two zeros [9].

Denote by $\mathbb{Q}^\alpha$ the subset of real numbers of the the type

$$\frac{\lambda^{1/\alpha}}{(\pi n)^{2/\alpha}},$$

where $n \in \mathbb{N}$, λ are all the real roots of the function $E_{1/\alpha}(z;2)$. It is obvious that the set $\mathbb{Q}^\alpha$ is limited, point 0 is the accumulation point.

We call a regular solution of equation (1) in the domain Ω a function $u = u(x,y)$ from the class $C(\bar{\Omega})$, having the derivatives $u_{xx}, \partial_{0y}^\alpha u \in C(\Omega)$, and satisfying equation (1) at all the points of the domain Ω.

Statement of the problem

Task. *Find in the domain Ω a regular solution of equation (1) satisfying the conditions*

$$u(0,y) = u(a,y) = 0, \tag{3}$$

$$u(x,0) = u(x,b) = 0. \tag{4}$$

Theorem. *For the task (1), (3),(4) to have only a trivial solution, it is necessary and enough that*

$$\frac{b}{a^{\frac{\alpha}{2}}} \notin \mathbb{Q}^\alpha.$$

Proof. 1. Sufficiency. We denote

$$v(x,y)=\sin\sqrt{\lambda_n}x\cdot(b-y)^{\alpha-1}E_{1/\alpha}\left(-\lambda_n(b-y)^{\alpha};\alpha\right),\quad \lambda_n=\left(\frac{\pi n}{a}\right)^2,\ n=1,2,...$$

Applying the formula of fractional integration by parts

$$\int\limits_c^d f(t)D_{ct}^{\gamma}g(\eta)dt=\int\limits_c^d D_{dt}^{\gamma}f(\eta)g(t)dt,\ \gamma\leq 0,$$

for the operator L we obtain the following Green formula

$$\int\limits_{\Omega}(vLu-uL^*v)dxdy=\int\limits_{\partial\Omega}(vu_x-uv_x)dy+(u_yD_{by}^{\alpha-2}v+uD_{by}^{\alpha-1}v)dx, \tag{5}$$

where

$$L^*v\equiv v_{xx}-D_{by}^{\alpha}v,$$

D_{by}^{α} is the operator of Riemann-Liouville fractional differentiation of the order α with the origin at the point b [1, p. 9].

Noticing that

$$L^*v=0,$$
$$v(0,y)=v(r,y)=0,\ \lim_{y\to b}D_{by}^{\alpha-2}v(x,\eta)=0,$$

from (5) at homogeneous boundary conditions we pass on to the formula

$$\lambda_n^{-1/\alpha}\sin_{\alpha}(\lambda_n^{1/\alpha}b)\int\limits_0^a u_y(x,0)\sin\sqrt{\lambda_n}xdx=0,\ n=1,2,...$$

The theorem hypotheses allow us to write

$$\int\limits_0^r u_y(x,0)\sin\sqrt{\lambda_n}xdx=0,\ n=1,2,...$$

It is known that the function system $\{\sin\sqrt{\lambda_n}x\}$ is complete in the space $L^2(0,r)$. Thus, from the Lagrange lemma we obtain that

$$u_y(x,0)=0,\ x\in(0,r). \tag{6}$$

Consequently,

$$\partial_{0y}^{\alpha}u(x,\eta)=D_{0y}^{\alpha}u(x,\eta)-\frac{y^{1-\alpha}}{\Gamma(2-\alpha)}u_y(x,0)-\frac{y^{-\alpha}}{\Gamma(1-\alpha)}u(x,0)=D_{0y}^{\alpha}u(x,\eta).$$

Owing to equality (6), there is a formula

$$D_{0y}^{\alpha-1}u(x,\eta)=D_{0y}^{\alpha-2}u_{\eta}(x,\eta).$$

Therefore,

$$Lu\equiv u_{xx}(x,y)-D_{0y}^{\alpha}u(x,\eta)=0, \tag{7}$$

$$\lim_{y \to 0} D_{0y}^{\alpha-1} u(x,\eta) = 0,$$
$$\lim_{y \to 0} D_{0y}^{\alpha-2} u(x,\eta) = 0, \quad (8)$$

Thus, the problem (1), (3), (4) may be reduced to the problem (7), (3), (8). It is known (see [10, p. 123]) that this problem has only a null solution. Consequently, $u(x,y) \equiv 0$ in the domain Ω.

2. Necessity. Assume that $\frac{b}{a^{\frac{\alpha}{2}}} \in Q^{\alpha}$, i.e. at constant n and λ

$$\frac{b}{a^{\alpha/2}} = \frac{\lambda^{1/\alpha}}{(\pi n)^{2/\alpha}}.$$

Then it is not difficult to show that the function

$$u(x,y) = yE_{1/\alpha}\left[-\left(\frac{\pi n}{a}\right)^2 y^{\alpha}; 2\right] \sin \frac{\pi n}{a} x,$$

satisfies the equation (1) and the conditions (3), (4). □

References

1. Nakhushev A. M. Drobnoe ischislenie i ego primenenie [Fractional calculus and its application]. Moscow, Fizmatlit, 2003.

2. Bourgin D. G., Duffin R. The Dirichlet problem for the vibrating string equation. Bull. Amer. Math. Soc. 45, 851-858, 1939.

3. Pskhu A.V. Fundamental'noe reshenie diffuzionno-volnovogo uravneniya drobnogo poryadka [Fundamental solution of diffusion-wave equation of the second order].Izvestiya RAN - Bulletin of the Russian Academy of Sciences, 2009, V. 73, № 2. pp. 141-182.

4. Kilbas A.A., Srivastava H.M., Trujillo J.J. Theory and applications of Fractional Differential Equations. Amsterdam: Elsevier, 2006.

5. Pskhu A.V. Pervaya kraevaya zadacha dlya diffuzionno-volnovogo uravneniya drobnogo poryadka [First boundary value problem for diffusive-wave equation of fractional order].Issledovaniya po differentsial'nym uravneniyam i matematicheskomu modelirovaniyu - Investigations on differential equations and mathematical modelling, Vladikavkaz, VNTs RAN, 2008. pp. 235-242.

6. Masaeva O.Kh. Zadacha Dirikhle dlya nelokal'nogo volnovogo uravneniya [Dirichlet problem for nonlocal wave equation]. Differents. uravneniya - Differential equations, 2013, V. 49, No.12, pp. 1554-1559.

7. Dzhrbashyan M.M. Integral'nye preobrazovaniya i predstavleniya funktsii v kompleksnoi oblasti [Integral transformations and presentation of functions in a complex domain]. Moscow, Nauka, 1966.

8. Pskhu A.V. O veshchestvennykh nulyakh funktsii tipa Mittag-Lefflera [Real zeros of Mittag-Leffler function]. Matematicheskie zametki - Mathematical Notes, 2005, V. 77, I. 4. pp. 592-599.

9. Popov A.Yu. O kolichestve veshchestvennykh sobstvennykh znachenii odnoi kraevoi zadachi dlya uravneniya vtorogo poryadka s drobnoi proizvodnoi [On the number of real eigen values of a boundary value problem for an equation of the second order with fractional derivative]. Fundamental'naya i prikladnaya matematika - Fundamental and Applied Mathematics, 2006, V. 12, No. 6. pp. 137-155.

10. Pskhu A.V. Uravneniya v chastnykh proizvodnykh drobnogo poryadka [Partial fractional equations]. Moscow, Nauka, 2005.

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