Bulletin KRASEC. Phys. & Math. Sci, 2015, V. 11, №. 2, pp. 15-18. ISSN 2313-0156

DOI: 10.18454/2313-0156-2015-11-2-15-18

MSC 35K57

PROBLEM WITH SAMARSKII CONDITION FOR FRACTIONAL DIFFUSION EQUATION IN THE HALF-STRING

F.M. Losanova

Institute of Applied Mathematics and Automation, 360000, Republic of Kabardino-Balkariya, Nalchik, st. Shortanova, 89a

E-mail: losanovaf@gmail.com

In this paper, we solve a nonlocal boundary value problem with Samarskii condition for a fractional diffusion equation in the half-string.

Key words: fractional diffusion equation, nonlocal problem, Samarskii condition, Wright type function

Introduction

Fractional equations have been studied quite intensively in recent times. It is due to the multiple applications of fractional calculus in Physics, Mechanics, Biology and so on.

Fractional equations are the basis of mathematical models of different physical processes in fractal mediums [1], [2].

Statement of the problem

In the domain $\Omega = \{(x,y): 0 < x < \infty,\ 0 < y < T\}$ we consider an equation

$$u_{xx}(x,y) - D^{\alpha}_{0y}u(x,\eta) = f(x,y), \tag{1}$$

where

$$D^{\alpha}_{0y}u(x,\eta) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^y \frac{u(x,\eta)d\eta}{(y-\eta)^{\alpha+1}}, & \alpha < 0, \\ u(x,y), & \alpha = 0, \\ \frac{\partial^n}{\partial y^n} D^{\alpha-n}_{0y}u(x,\eta), & n-1 < \alpha \le n, \quad n \in \mathbb{N} \end{cases}$$

is the fractional integro-differentiation operator (in the sense of Reimann-Liouville) of the order α [1, p. 9], $\Gamma(\alpha)$ is Euler gamma function, $0 < \alpha \le 1$.

Boundary value problems with integral conditions for parabolic equations, including those with fractional derivative, were investigated in the papers [3]-[6]. Equations of such a type as (1) were studied in the articles [7], [8].

Losanova Fatima Muhamedovna – Researcher of Dep. Equations of mixed type, Institute of Applied Mathematics and Automation, Kabardino-Balkaria, Nalchik.

This paper solves a nonlocal boundary value problem with Samarskii condition for equation (1).

Solution $u(x,y)$ of equation (1) is called *regular* in the domain Ω, in $y^{1-\alpha}u(x,y)\in C(\overline{\Omega})$, $u_{xx}(x,y), D^{\alpha}_{0y}u(x,\eta)\in C(\Omega)$.

Task. *Find a regular solution $u(x,y)$ of equation (1) in the domain Ω, satisfying the following conditions:*

$$\lim_{y\to 0} D^{\alpha-1}_{0y}u(x,\eta)=\tau(x), \qquad 0\le x<\infty, \tag{2}$$

$$wesatisfy \int\limits_0^l u(x,y)dx=\psi(y), \qquad 0<y<T, \tag{3}$$

where $\tau(x),\psi(y)$ are specified continuous functions.

To find the solutions of problem (1)-(3) we use the solution of the problem with conditons (2), $u(0,y)=\varphi(y)$ for equation (1) [4]

$$u(x,y)=\int\limits_0^y G_{\xi}(x,y,0,\eta)\varphi(\eta)d\eta+\int\limits_0^{\infty} G(x,y,\xi,0)\tau(\xi)d\xi-\int\limits_0^y\int\limits_0^{\infty} f(\xi,\eta)G(x,y,\xi,\eta)d\xi d\eta, \tag{4}$$

where $G(x,y,\xi,\eta)=\frac{(y-\eta)^{\delta-1}}{2}\left[e^{1,\delta}_{1,\delta}\left(-\frac{|x-\xi|}{(y-\eta)^{\delta}}\right)-e^{1,\delta}_{1,\delta}\left(-\frac{|x+\xi|}{(y-\eta)^{\delta}}\right)\right], e^{\mu,\delta}_{\alpha,\beta}(z)$ is Wright type function [7, c. 22].

To determine the unknown $\varphi(y)$, function (4) should satisfy the condition (3). Then after simple transformations we have

$$\int\limits_0^y \varphi(\eta)\int\limits_0^l G_{\xi}(x,y,0,\eta)dxd\eta+\int\limits_0^{\infty}\tau(\xi)\int\limits_0^l G(x,y,\xi,0)dxd\xi-$$

$$-\int\limits_0^y\int\limits_0^{\infty} f(\xi,\eta)\int\limits_0^l G(x,y,\xi,\eta)dxd\xi d\eta=\psi(y). \tag{5}$$

Applying the Wright type function differentiation formula we obtain that [8, p. 26] $\frac{d^n}{dx^n}x^{\mu-1}e^{\mu,\delta}_{\alpha,\beta}(cx^{\alpha})=x^{\mu-n-1}e^{\mu-n,\delta}_{\alpha,\beta}(cx^{\alpha})$ for any $\mu\in\mathbb{R}$

$$G_{\xi}(x,y,\xi,\eta)=\frac{(y-\eta)^{\delta-1}}{2}\left[\frac{sign(x-\xi)}{|x-\xi|}e^{0,\delta}_{1,\delta}\left(-\frac{|x-\xi|}{(y-\eta)^{\delta}}\right)+\frac{sign(x+\xi)}{|x+\xi|}e^{0,\delta}_{1,\delta}\left(-\frac{|x+\xi|}{(y-\eta)^{\delta}}\right)\right].$$

From which at $\xi=0$ taking into account the autotransformation formula for Wright type function [8, p. 24] $ze^{\mu,\delta}_{\alpha,\beta}(z)=e^{\mu-\alpha,\delta+\beta}_{\alpha,\beta}(z)-\frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)}$ we have

$$G_{\xi}(x,y,0,\eta)=\frac{(y-\eta)^{\delta-1}}{x}e^{0,\delta}_{1,\delta}\left(-\frac{|x|}{(y-\eta)^{\delta}}\right)=-\frac{1}{y-\eta}e^{1,0}_{1,\delta}\left(-\frac{x}{(y-\eta)^{\delta}}\right). \tag{6}$$

We calculate the inner integral in the first summand of equality (5)

$$\int\limits_0^l G_{\xi}(x,y,0,\eta)dx=-\int\limits_0^l\frac{1}{y-\eta}e^{1,0}_{1,\delta}\left(-\frac{x}{(y-\eta)^{\delta}}\right)dx=\left[\frac{x}{y-\eta}e^{2,0}_{1,\delta}\left(-\frac{x}{(y-\eta)^{\delta}}\right)\right]_{x=0}^{x=l}=$$

$$= \frac{l}{y-\eta} e_{1,\delta}^{2,0}\Big(-\frac{l}{(y-\eta)^{\delta}}\Big). \tag{7}$$

Applying the autotransformation formula for Wright type function one more time from relation (7) we finally obtain

$$\int\limits_0^l G_{\xi}(x,y,0,\eta)dx = \frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} - (y-\eta)^{\delta-1} e_{1,\delta}^{1,\delta}\Big(-\frac{l}{(y-\eta)^{\delta}}\Big). \tag{8}$$

Substituting relation (8) into formula (5), we have

$$\int\limits_0^y \varphi(\eta)\Big[\frac{(y-\eta)^{\delta-1}}{\Gamma(\delta)} - (y-\eta)^{\delta-1} e_{1,\delta}^{1,\delta}\Big(-\frac{l}{(y-\eta)^{\delta}}\Big)\Big] d\eta = F(y), \tag{9}$$

where

$$F(y) = \psi(y) - \int\limits_0^{\infty} \tau(\xi) \int\limits_0^l G(x,y,\xi,0)dxd\xi + \int\limits_0^y\int\limits_0^{\infty} f(\xi,\eta)\int\limits_0^l G(x,y,\xi,\eta)dxd\xi d\eta.$$

Owing to the definition of fractional integro-differentiation, we wright equation (9) in the form

$$D_{0y}^{-\delta}\varphi(y) - \int\limits_0^y \varphi(\eta)(y-\eta)^{\delta-1} e_{1,\delta}^{1,\delta}\Big(-\frac{l}{(y-\eta)^{\delta}}\Big)d\eta = F(y). \tag{10}$$

Taking into account the notations $\omega_{\mu}(y) = y^{\mu-1} e_{1,\delta}^{1,\mu}(-\frac{l}{y^{\delta}})$, $(f*h)(y) = \int\limits_0^y f(y-\eta)h(\eta)d\eta$ and

$$g(y) = D_{0y}^{-\delta}\varphi(y) \tag{11}$$

relation (10) can be rewritten as follows

$$g(y) - (\varphi * \omega_{\delta})(y) = F(y). \tag{12}$$

Applying formula $\omega_{\mu-\varepsilon}(y) = D_{0y}^{\varepsilon}\omega_{\mu}(y)$ from equality (12) we obtain

$$g(y) - \varphi * D_{0y}^{-\delta}\omega_0(y) = F(y). \tag{13}$$

Taking into consideration the Laplace convolution property $f * D_{0y}^{-\delta} g(y) = g * D_{0y}^{-\delta} f(y)$ from formula (13), we obtain

$$g(y) - (g * \omega_0)(y) = F(y). \tag{14}$$

Relation (14) is Volterra integral equation of the second type. Solution of equation (14) can be represented in the form

$$g(y) = F(y) + (F * R)(y), \tag{15}$$

where

$$R(y) = \sum_{n=0}^{\infty} y^{\delta(n+1)-1} e_{1,\delta}^{1,\delta(n+1)}\Big(-\frac{l(n+1)}{y^{\delta}}\Big).$$

From the notation (11) due to the equality (15) we have

$$\varphi(y) = D^{\delta}_{0y} g(y)$$

or

$$\varphi(y) = D^{\delta}_{0y}[F(y) + (F * R)(y)]. \tag{16}$$

Theorem. *We assume that* $\tau(x) \in C[0,\infty[, y^{1-\alpha}\psi(y) \in C[0,T], y^{1-\alpha} f(x,y) \in C(\overline{\Omega}), f(x,y)$ *satisfies Helder condition with respect to x. Then solution of problem (2), (3) for equation (1) can be represented in the form of (4), where* $\varphi(y)$ *is determined by relation (16).*

References

1. Nakhushev A.M.Drobnoe ischislenie i ego primenenie [Fractional calculus and its application]. Moscow, Fizmatlit, 2003. 272p.

2. Uchaikin V.V. Metod drobnykh proizvodnykh [Method of fractional derivatives]. Ul'yanovsk, Artishok, 2008. 512 p.

3. Losanova F.M. Nelokal'naya kraevaya zadacha s operatorom Kaputo [Nonlocal boundary value problem with Caputo operator]. Izvestiya VUZov Severo-Kavkazskii region - University news North Caucasian Region, 2010, № 5 (159), pp. 22-25.

4. Losanova F.M. Nelokal'naya kraevaya zadacha dlya nagruzhennogo uravneniya s operatorom Kaputo [Nonlocal boundary value problem for a loaded equation with Caputo operator]. Materialy Vserossiiskoi nauchnoi konferentsii molodykh uchenykh «Sovremennye voprosy matematicheskoi fiziki, matematicheskoi biologii i informatiki» [Proceedings of the Russian Scientific Conference of Young Researchers "Modern issues of Mathematical Physics, Mathematical Biology and Informatics"], 2014, pp. 80-81.

5. Gekkieva S.Kh. Kraevaya zadacha dlya obobshchennogo uravneniya perenosa s drobnoi proizvodnoi v polubeskonechnoi oblasti [A boundary value problem for a generalized transfer equation with fractional derivative in a semi-infinite domain]. "Pontryaginskie chteniya - XIII". Sbornik materialov - Proceedings of "Pontryaginskie chteniya - XIII Voronezh, VGU, 2002. p. 37.

6. Nachusheva Z.A. 1-ya i 2-ya kraevye zadachi v integral'noi postanovke dlya parabolicheskogo uravneniya vtorogo poryadka [1st and 2nd boundary value problems in integral setting for the second order parabolic equation] Differentsial'nye uravneniya - Differential Equations, 1990, V. 26, № 1, pp. 1982-1992.

7. Nachushev V.A. Differentsial'nye uravneniya matematicheskikh modelei nelokal'nykh protsessov [Differential equations of nonlocal process mathematical models]. Moscow, Nauka, 2006, 173 p.

8. Pschu A.V. Uravneniya v chastnykh proizvodnykh drobnogo poryadka [Partial differential equation of fractional order]. Mowcow, Nauka, 2005, 199 p.

For citation: Losanova F. M. Problem with Samarskii condition for fractional diffusion equation in the half-string. *Bulletin KRASEC. Physical and Mathematical Sciences* 2015, vol. **11**, issue **2**, 15-18. DOI: 10.18454/2313-0156-2015-11-2-15-18

Original article submitted: 19.09.2015