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ON MODEL OF LOADED HYPERBOLIC-PARABOLIC PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER

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The paper investigates models of loaded mixed hyperbolic-parabolic equations with characteristic and noncharacteristic change of the type. For the proposed equation models, boundary value problems are considered, and explicit solutions are worked out.

Key words: equation model, loaded equation, hyperbolic-parabolic equation, boundary value problem

Introduction

The paper considers models of loaded equations of mixed hyperbolic-parabolic type both with characteristic and noncharacteristic change of the type. The first section investigates the model of a loaded hyperbolic-parabolic equation changing its type on a characteristic line. The second section studies a model of a loaded hyperbolic-parabolic type with noncharacteristic change of the type. The third section considers a mixed boundary value problem for an equation of a plane wave in the rectangular plane. For the proposed equation models of mixed types, boundary value problems are considered and explicit solutions are worked out.

In the paper [1], Nakhushev A.M. suggested a method of approximate solution of boundary value problems for differential equations. It is based on the reduction to loaded equations ([2]). Local and nonlocal boundary value problems for such a type of equations at $y > 0$ were considered in the papers [3] - [5].

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Model of a loaded hyperbolic-parabolic equation with characteristic change of the type

The following equation may serve as a model of loaded hyperbolic-parabolic partial derivative equation of second order with characteristic change of the type:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial}{\partial y} \int\limits_0^r u(x,y)dx, \ \ 0 < x < r, \ 0 \le y \le \beta_2, \\ \frac{\partial^2 u}{\partial x \partial y} = \beta \frac{\partial}{\partial y} \int\limits_0^r u(x,y)dx, \ \ 0 < x < r, \ \beta_1 \le y \le 0, \end{cases} \tag{1}$$

where $\alpha = const, r = const < \infty, \beta_2 = const > 0, \beta = const, \beta_1 = const < 0$, $u = u(x,y)$ is the unknown function.

We denote by $\delta(y) = \int\limits_0^r u(x,y)dx$; $\Omega = \{(x,y) : 0 < x < r, \beta_1 < y < \beta_2\}$, $\bar{\Omega}$ is the closure of the area Ω, $\Omega^+ = \Omega \cap \{y > 0\}$, $\Omega^- = \Omega \cap \{y < 0\}$.

Definition. *Regular solution of equation (1) is the function $u = u(x,y)$ from the class $C(\bar{\Omega}) \cap C^2(\Omega^-) \cap C^{2,1}_{x,y}(\Omega^+)$, $\delta(y) \in C[\beta_1, \beta_2] \cap C^1]\beta_1, \beta_2[$, satisfying equation (1) in $\Omega^+ \cup \Omega^-$.*

Equation (1) at $y > 0$ coincides with Targ equation [6, p. 75]. In [7], the authors worked out a solution of equation (1) in the class $C(\bar{\Omega}) \cap C^1(\Omega)$ with significant constraints on the solution at the area boundary.

When $\beta_1 < y < 0$, in has the following form:

$$\frac{\partial^2 u}{\partial x \partial y} = \beta \delta'(y), \tag{2}$$

where $\delta'(y) = \int\limits_0^r u_y(x,y)dx$.

It follows from (2) that

$$u(x,y) = u(x,0) + u(0,y) - u(0,0) + \beta x[\delta(y) - \delta(0)], \tag{3}$$

i.e. (3) is the solution of Goursat problem for wave equation with the right-hand side $\beta \delta'(y)$.

Assume that

$$u(0,y) = \varphi_0(y), \quad \beta_1 \le y \le \beta_2, \tag{4}$$

where $\varphi_0(y)$ is the specified function from the class $C[\beta_1, \beta_2] \cap C^1]\beta_1, \beta_2[$.

We denote by $\tau(x) = u(x,0)$. Then $\tau(0) = \varphi_0(0)$, and (3) will have the following form

$$u(x,y) = \tau(x) + \varphi_0(y) - \varphi_0(0) + \beta x[\delta(y) - \delta(0)], \ \ \beta_1 \le y \le 0. \tag{5}$$

From (1) at $y > 0$ we conclude:

$$u(x,y) = \frac{\alpha x^2}{2} \delta'(y) + B(y)x + C(y), \tag{6}$$

where $B(y), C(y)$ are arbitrary functions from the class $C[0, \beta_2]$.

From (6) and (4) we obtain directly that $C(y) = \varphi_0(y)$. Assume that

$$u(r,y) = \varphi_r(y), \ \ 0 \le y \le \beta_2, \tag{7}$$

where $\varphi_r(y)$ is the specified function from the class $C[0,\beta_2]\cap C^1[0,\beta_2[,\quad \tau(r)=\varphi_r(0)$.

Then from (6) and (7) we receive directly that

$$B(y)=\frac{\varphi_r(y)-\varphi_0(y)}{r}-\frac{\alpha r}{2}\delta'(y).$$

Then (6) has the following form:

$$u(x,y)=\frac{\alpha x(x-r)}{2}\delta'(y)+\frac{x}{r}\varphi_r(y)+\frac{r-x}{r}\varphi_0(y),\quad 0\le y\le\beta_2. \tag{8}$$

From (8) at $y\to+0$ we obtain that

$$\tau(x)=\frac{\alpha x(x-r)}{2}\delta'(0)+\frac{x}{r}\varphi_r(0)+\frac{r-x}{r}\varphi_0(0). \tag{9}$$

Taking into account that the solution is sought in the class $C(\bar{\Omega})$, and substituting $\tau(x)$ from (9) into (5), at $\beta_1\le y\le 0$ we obtain

$$u(x,y)=\frac{\alpha x(x-r)}{2}\delta'(0)+\frac{x}{r}\varphi_r(0)+\frac{r-x}{r}\varphi_0(0)+\varphi_0(y)-\varphi_0(0)+\beta x[\delta(y)-\delta(0)]. \tag{10}$$

It follows from (10) that

$$\delta(y)=\frac{-\alpha r^3}{12}\delta'(0)+\frac{r}{2}[\varphi_r(0)-\varphi_0(0)]+r\varphi_0(y)+\frac{\beta r^2}{2}[\delta(y)-\delta(0)],$$

from which at $2-\beta r^2\neq 0$ we obtain directly

$$\delta(y)=\frac{2r\varphi_0(y)}{2-\beta r^2}+\frac{r[\varphi_r(0)-\varphi_0(0)]}{2-\beta r^2}-\frac{\alpha r^3\delta'(0)}{6(2-\beta r^2)}-\frac{\beta r^2\delta(0)}{2-\beta r^2}. \tag{11}$$

From (11) at $y\to-0$ we have

$$\delta(0)=\frac{2r\varphi_0(0)}{2-\beta r^2}+\frac{r[\varphi_r(0)-\varphi_0(0)]}{2-\beta r^2}-\frac{\alpha r^3\delta'(0)}{6(2-\beta r^2)}-\frac{\beta r^2\delta(0)}{2-\beta r^2},$$

or

$$\delta(0)=-\frac{\alpha r^3}{12}\delta'(0)+\frac{r}{2}[\varphi_r(0)+\varphi_0(0)].$$

It follows from (11) that

$$\delta'(y)=\frac{2r}{2-\beta r^2}\varphi_0'(y),\ \beta_1\le y\le 0,$$

$$\delta'(0)=\frac{2r}{2-\beta r^2}\varphi_0'(0). \tag{12}$$

Taking into account (12) we obtain

$$\delta(0)=-\frac{\alpha r^4}{6(2-\beta r^2)}\varphi_0'(0)+\frac{r}{2}[\varphi_r(0)+\varphi_0(0)], \tag{13}$$

$$\delta(y)=\frac{2r\varphi_0(y)}{2-\beta r^2}+\frac{r}{2}\varphi_r(0)-\frac{r(2+\beta r^2)}{2(2-\beta r^2)}\varphi_0(0)-\frac{\alpha r^4}{6(2-\beta r^2)}\varphi_0'(0),\ \beta_1\le y\le 0. \tag{14}$$

Substituting (12) - (14) into (10), in view that $\delta(y)-\delta(0)=\frac{2r}{2-\beta r^2}[\varphi_0(y)-\varphi_0(0)]$, we obtain that the solution of equation (1) at $\beta_1 \le y \le 0$, satisfying the condition (4) is defined by the formula

$$u(x,y)=\frac{2-\beta r^2+2\beta rx}{2-\beta r^2}\varphi_0(y)+\frac{x}{r}\varphi_r(0)-\frac{x(2+\beta r^2)}{r(2-\beta r^2)}\varphi_0(0)+\frac{\alpha rx(x-r)}{2-\beta r^2}\varphi_0'(0), \qquad (15)$$

and (9) has the form

$$\tau(x)=\frac{\alpha rx(x-r)}{2-\beta r^2}\varphi_0'(0)+\frac{x}{r}\varphi_r(0)+\frac{r-x}{r}\varphi_0(0). \qquad (16)$$

By a direct check we prove that (15) satisfies the equation (1) and the condition (4). Based on (8) we have

$$\delta(y)=-\frac{\alpha r^3}{12}\delta'(y)+\frac{r}{2}\varphi_r(y)+\frac{r}{2}\varphi_0(y), \quad 0\le y\le \beta_2,$$

or, at $\alpha \ne 0$,

$$\delta'(y)+\lambda\delta(y)=f(y), \qquad (17)$$

where $\lambda=\frac{12}{\alpha r^3}$, $\quad f(y)=\frac{\lambda r}{2}[\varphi_r(y)+\varphi_0(y)]=\frac{6}{\alpha r^2}[\varphi_r(y)+\varphi_0(y)]$.

Solving the Cauchy problem (13) for equation (17), we obtain

$$\delta(y)=e^{-\lambda y}\delta(0)+\int\limits_0^y e^{-\lambda(y-\eta)}f(\eta)d\eta, \quad 0\le y\le\beta_2,$$

from which it follows that

$$\delta'(y)=-\lambda e^{-\lambda y}\delta(0)+f(y)-\lambda\int\limits_0^y e^{-\lambda(y-\eta)}f(\eta)d\eta, \quad 0\le y\le\beta_2. \qquad (18)$$

Substituting (18) into (8), we obtain that the solution of equation (1) at $0\le y\le\beta_2$, satisfying the conditions (4), (7), is defined by the formula

$$u(x,y)=x(x-r)\left[\left(\frac{\alpha r\varphi_0'(0)}{2-\beta r^2}-\frac{3}{r^2}[\varphi_r(0)+\varphi_0(0)]\right)e^{-\frac{12y}{\alpha r^3}}+\frac{3}{r^2}[\varphi_r(y)+\varphi_0(y)]\right]-$$

$$-\frac{36x(x-r)}{\alpha r^5}\int\limits_0^y e^{-\frac{12(y-\eta)}{\alpha r^3}}[\varphi_r(\eta)+\varphi_0(\eta)]d\eta+\frac{x}{r}\varphi_r(y)+\frac{r-x}{r}\varphi_0(y), \quad 0\le y\le\beta_2. \qquad (19)$$

It is easy to check that (19) satisfies equation (1), and that from (19) at $y\to+0$ we obtain (16).

Thus, we have confirmed the following

Theorem 1. *If $\beta r^2\ne 2$, then a unique regular solution $u(x,y)$ of equation (1), satisfying the boundary conditions*

$$u(0,y)=\varphi_0(y), \quad \varphi_0(y)\in C[\beta_1,\beta_2]\cap C^1[0,\beta_2[\cap C^2]\beta_1,0], \quad \beta_1\le y\le\beta_2,$$

$$u(r,y)=\varphi_r(y),\quad \varphi_r(y)\in C[0,\beta_2]\cap C^1[0,\beta_2[,\quad 0\leq y\leq \beta_2,$$

where $\varphi_0(y)$, $\varphi_r(y)$ *are specified functions, is defined by the formulas*

$$u(x,y)=\frac{2-\beta r^2+2\beta rx}{2-\beta r^2}\varphi_0(y)+\frac{x}{r}\varphi_r(0)-\frac{x(2+\beta r^2)}{r(2-\beta r^2)}\varphi_0(0)+\frac{\alpha rx(x-r)}{2-\beta r^2}\varphi_0'(0),\ \beta_1\leq y\leq 0,$$

$$u(x,y)=x(x-r)\left[\left(\frac{\alpha r\varphi_0'(0)}{2-\beta r^2}-\frac{3}{r^2}[\varphi_r(0)+\varphi_0(0)]\right)e^{-\frac{12y}{\alpha r^3}}+\frac{3}{r^2}[\varphi_r(y)+\varphi_0(y)]\right]-$$

$$-\frac{36x(x-r)}{\alpha r^5}\int\limits_0^y e^{-\frac{12(y-\eta)}{\alpha r^3}}[\varphi_r(\eta)+\varphi_0(\eta)]d\eta+\frac{x}{r}\varphi_r(y)+\frac{r-x}{r}\varphi_0(y),\quad 0\leq y\leq\beta_2.$$

Model of a loaded hyperbolic-parabolic equation with noncharacteristic change of the type

As a model of a loaded equation changing its type on a noncharacteristic line, we consider the following equation

$$\begin{cases} u_{xx}+a_1\frac{\partial}{\partial y}\int\limits_0^r u(x,y)dx=0,\ y>0,\\ u_{xx}-u_{yy}=0,\quad y<0\end{cases}\tag{20}$$

In the area Ω, limited by the segments AA_0, BB_0, A_0B_0 of the straight lines $x=0$, $x=r$, $y=\beta_2$, respectively, at $y>0$, and wave equation characteristics $AC: x+y=0$, $BC: x-y=r$ at $y<0$.

We denote the parabolic and hyperbolic parts of the area Ω by Ω^+, Ω^-, respectively, and the interval $0<x<r$ of the straight line $y=0$, by J, by $\delta(y)=\int\limits_0^r u(x,y)dx$, $0\leq y\leq\beta_2$.

Definition. *As a regular solution of equation (20) we understand the function* $u=u(x,y)$ *from the class* $C(\bar{\Omega})\cap C^2(\Omega^-)\cap C^{2,1}_{x,y}(\Omega^+)$, $\delta(y)\in C[0,\beta_2]\cap C^1[0,\beta_2[$, *satisfying equation (20) in* $\Omega^+\cup\Omega^-$.

The analogue of Tricomi problem for equation (20) in the area Ω is

Problem T. *Find in the area* Ω *a regular solution* $u(x,y)$ *of equation* (20), *satisfying the boundary conditions*:

$$u(0,y)=\varphi_0(y),\qquad u(r,y)=\varphi_r(y),\qquad 0\leq y\leq\beta_2,\tag{21}$$

$$u(x/2,-x/2)=\psi(x),\qquad 0\leq x\leq r,\tag{22}$$

where $\varphi_0(y),\varphi_r(y),\psi(x)$ *are the specified functions, and* $\varphi_0(0)=\psi(0)$.

Let there be a solution $u(x,y)$ of the problem T. We denote by

$$\tau(x)=u(x,0),\quad \nu(x)=u_y(x,0),\tag{23}$$

then

$$\tau(0)=\varphi_0(0),\qquad \tau(r)=\varphi_r(0).\tag{24}$$

Passing on to the limit in equation (20) at $y \to +0$, and taking into account that $u_y(x,0) \in L(J)$, we obtain functional accuracy taken from the parabolic side Ω^+ of a mixed area Ω

$$\tau''(x) + a_1 \int_0^r \nu(\xi)d\xi = 0. \tag{25}$$

Solution of the Cauchy problem (23) for equation (20) in Ω^- can be presented in the following form [8]:

$$u(x,y) = \frac{\tau(x+y)+\tau(x-y)}{2} - \frac{1}{2}\int_{x+y}^{x-y} \nu(\xi)d\xi. \tag{26}$$

If we write the condition (22) by formula (26), in Ω^- we obtain:

$$\tau(x) + \tau(0) - \int_0^x \nu(\xi)d\xi = 2\psi(x), \quad 0 \le x \le r. \tag{27}$$

Differentiating (27), taking into account that $\tau(0) = \varphi_0(0) = \psi(0)$, from Ω^- we obtain

$$\nu(x) = \tau'(x) - 2\psi'(x). \tag{28}$$

From (25) and (28) we obtain the equation

$$\tau''(x) + a_1 \int_0^r \tau'(\xi)d\xi = 2a_1 \int_0^r \psi'(\xi)d\xi,$$

or taking into account (24)

$$\tau''(x) = g, \tag{29}$$

where $g = 2a_1[\psi(r) - \psi(0)] - a_1[\varphi_r(0) - \varphi_0(0)] = 2a_1\psi(r) - a_1[\varphi_r(0) + \varphi_0(0)]$.

The Dirichlet problem (24) for the equation (29) has a unique solution

$$\tau(x) = gx^2/2 + q_1 x + q_2, \tag{30}$$

where

$$q_1 = \frac{\varphi_r(0) - \varphi_0(0)}{r} - \frac{gr}{2} = \frac{2+a_1r^2}{2r}[\varphi_r(0) - \varphi_0(0)] - a_1 r[\psi(r) - \psi(0)] =$$

$$= \frac{2+a_1r^2}{2r}\varphi_r(0) - \frac{2-a_1r^2}{2r}\varphi_0(0) - a_1 r\psi(r), \qquad q_2 = \varphi_0(0).$$

That is

$$\tau(x) = a_1 x(x-r)\left(\psi(r) - \psi(0) - \frac{\varphi_r(0) - \varphi_0(0)}{2}\right) + \frac{x}{r}\varphi_r(0) - \frac{x-r}{r}\varphi_0(0) =$$

$$= a_1 x(x-r)\left(\psi(r) - \frac{\varphi_r(0)}{2} - \frac{\varphi_0(0)}{2}\right) + \frac{x}{r}\varphi_r(0) - \frac{x-r}{r}\varphi_0(0). \tag{31}$$

From (28) we obtain that

$$\nu(x) = gx - 2\psi'(x) + q_1 =$$

$$= \big(2a_1\psi(r) - a_1[\varphi_r(0) + \varphi_0(0)]\big)x - 2\psi'(x) + \frac{2+a_1r^2}{2r}\varphi_r(0) - \frac{2-a_1r^2}{2r}\varphi_0(0) - a_1r\psi(r). \quad (32)$$

It is clear from (30) and (32) that the homogeneous problem T will have in Ω^- only a trivial solution $u(x,y) \equiv 0$ as Cauchy problem solution $\tau(x) \equiv 0$, $\nu(x) \equiv 0$ for the equation (20).

When we have found $\tau(x)$ and $\nu(x)$, solution of the problem T in Ω^- is defined by formula (26).

Consider the solution of the problem T in Ω^+. It is necessary to find the solution of the equation

$$u_{xx} = -a_1\delta'(y), \quad (33)$$

satisfying the boundary conditions (21) and the initial condition

$$u(x,0) = \tau(x), \qquad 0 \le x \le r, \quad (34)$$

where $\varphi_0(y), \varphi_r(y), \tau(x)$ are the known functions, and $\varphi_0(0) = \tau(0)$, $\varphi_r(0) = \tau(r)$.

General solution of equation (33) as solution of a standard differential equation with parameter y has the form

$$u(x,y) = -a_1\delta'(y)\frac{x^2}{2} + q_1(y)x + q_2(y), \quad (35)$$

where $q_1(y), q_2(y)$ are function derivatives from the class $C[0,h]$.

Satisfying (35) the conditions (21), we obtain

$$q_1(y) = \frac{\varphi_r(y) - \varphi_0(y)}{r} + \frac{a_1\delta'(y)r}{2}, \quad q_2(y) = \varphi_0(y),$$

and formula (35) has the form:

$$u(x,y) = \frac{a_1x(r-x)}{2}\delta'(y) + \frac{x}{r}\varphi_r(y) - \frac{x-r}{r}\varphi_0(y). \quad (36)$$

From (36) taking into account the notation $\delta(y) = \int\limits_0^r u(x,y)dx$ and after simple transformations we obtain

$$\delta(y) = \frac{a_1r^3}{12}\delta'(y) + \frac{r}{2}\varphi_r(y) + \frac{r}{2}\varphi_0(y),$$

or at $a_1 \neq 0$

$$\delta'(y) - \lambda\delta(y) = f(y), \quad (37)$$

where $\lambda = \frac{12}{a_1r^3}$, $f(y) = -\frac{\lambda r}{2}[\varphi_r(y) + \varphi_0(y)]$, and taking into account (12) we have

$$\delta(0) = \int\limits_0^r u(\xi,0)d\xi = \int\limits_0^r \tau(\xi)d\xi = \int\limits_0^r \Big(g\xi^2/2 + q_1\xi + q_2\Big)d\xi = \frac{r^3}{6}g + \frac{r^2}{2}q_1 + rq_2 =$$

$$= -\frac{a_1 r^3}{6}[\psi(r)-\psi(0)] + \frac{a_1 r^3}{12}[\varphi_r(0)-\varphi_0(0)] + \frac{r}{2}[\varphi_r(0)+\varphi_0(0)].$$

Solving the Cauchy problem for equation (37), we obtain

$$\delta(y) = e^{\lambda y}\delta(0) + \int_0^y e^{\lambda(y-\eta)} f(\eta)d\eta,$$

from which it follows that

$$\delta'(y) = \lambda e^{\lambda y}\delta(0) + f(y) + \int_0^y \lambda e^{\lambda(y-\eta)} f(\eta)d\eta.$$

It is clear that for the homogeneous problem T $\delta'(y)=0$.

From (36) at $y\to 0$ we have

$$\tau(x) = u(x,0) = -a_1 x(x-r)\frac{\delta'(0)}{2} + \frac{x}{r}\varphi_r(0) - \frac{x-r}{r}\varphi_0(0), \tag{38}$$

where

$$\delta'(0) = \lambda\delta(0) + f(0) = -2[\psi(r)-\psi(0)] + [\varphi_r(0)-\varphi_0(0)].$$

It is clear from (32) and (38) that $\lim\limits_{y\to+0} u(x,y) = \lim\limits_{y\to-0} u(x,y) = \tau(x)$.

Next, solution of the problem T in Ω^+ is defined by formula (36). It is obvious, that a homogeneous problem, corresponding to the problem T, will have only a trivial solution $u(x,y)\equiv 0$ in Ω^+.

Thus, we have proved

Theorem 2. *If $\varphi_0(y), \varphi_r(y) \in C[0,\beta_2]\cap C^1]0,\beta_2[$, $\psi(x)\in C^1[0,r]\cap C^2]0,r[$, the problem T has a unique solution which in Ω^- is defined by the formula*

$$u(x,y) = \psi(x-y) - \psi(x+y) + a_1(x+y)(x+y-r)\psi(r) -$$

$$-\frac{x+y}{2r}\left(a_1 r(x+y) - a_1 r^2 - 2\right)\varphi_r(0) + \left[1 - \frac{x+y}{2r}\left(a_1 r(x+y) - a_1 r^2 + 2\right)\right]\varphi_0(0),$$

and in Ω^+ it is defined by the formula

$$u(x,y) = \frac{x(r-x)}{2r^2} e^{\frac{12y}{a_1 r^3}}\left[(a_1 r^2+6)[\varphi_r(0)+\varphi_0(0)] - 2a_1 r^2\psi(r)\right] +$$

$$+\frac{x}{r}\left[1 - \frac{3(r-x)}{r}\right]\varphi_r(y) + \frac{r-x}{r}\left[1-\frac{3x}{r}\right]\varphi_0(y) - \frac{36x(r-x)}{a_1 r^5}\int_0^y e^{\frac{12(y-\eta)}{a_1 r^3}}[\varphi_r(\eta)-\varphi_0(\eta)]d\eta.$$

Mixed boundary value problem for an equation of a plane wave in a rectangular area

In the area $D=\Omega^+\cup J$ we consider a loaded wave equation

$$\frac{\partial^2 u}{\partial x^2} = \alpha\frac{\partial^2}{\partial y^2}\int_0^r u(x,y)dx, \tag{39}$$

where $\alpha = const > 0$.

Definition. *As a regular solution of equation (39) we understand a function $u = u(x,y)$ from the class $C(\bar{D}) \cap C^2(D)$ satisfying equation (39) in D.*

Problem S. *Find a regular solution $u(x,y)$ of equation (39) satisfying the conditions (21) and the conditions*

$$\int_0^r u(x,0)dx = \bar{\tau}, \quad \int_0^r u_y(x,0)dx = \bar{\nu}, \tag{40}$$

where $\varphi_0(y), \varphi_r(y)$ are the specified continuous functions, $\bar{\tau}, \bar{\nu}$ are the specified constants.

Just like in the previous cases, we assume that $\delta(y) = \int_0^r u(x,y)dx$. Then equation (39) can be rewritten in the form

$$u_{xx} = \alpha \delta''(y),$$

from which it follows that

$$u(x,y) = \alpha \delta''(y)\frac{x^2}{2} + A_1(y)x + B_1(y), \tag{41}$$

where $A_1(y)$ and $B_1(y)$ are the arbitrary functions of the argument y.

From (41) we obtain

$$\delta(y) = \int_0^r \alpha \delta''(y)\frac{x^2}{2} + A_1(y)x + B_1(y)dx = \alpha \delta''(y)\frac{r^3}{6} + A_1(y)\frac{r^2}{2} + B_1(y)r. \tag{42}$$

We rewrite (42) in the following form:

$$\delta''(y) - \frac{6}{\alpha r^3}\delta(y) = -\frac{6}{\alpha r^3}\left[A_1(y)\frac{r^2}{2} + B_1(y)r\right] = A_{12}(y),$$

or

$$\delta''(y) - c^2\delta(y) = A_{12}(y), \tag{43}$$

where $c^2 = \frac{6}{\alpha r^3}$, $\quad A_{12}(y) = -\frac{6}{\alpha r^3}\left[A_1(y)\frac{r^2}{2} + B_1(y)r\right]$.

Solution of equation (43) taking into account the conditions (40) can be presented in the form [9, p. 99]

$$\delta(y) = \sum_{i=1}^{m} \alpha_i U_i(y;c^2) + \int_0^y A_{12}(t)U_1(y-t;c^2)dt, \tag{44}$$

where $m = 2$, $\alpha_1 = \delta'(0) = \bar{\nu}$, $\alpha_2 = \delta(0) = \bar{\tau}$, U_i coincides with Barrett function:

$$U_1(y,c^2) = \sum_{k=1}^{\infty} \frac{c^{2(k-1)}y^{2k-1}}{\Gamma(2k)} = \frac{1}{c}\sum_{k=1}^{\infty} \frac{c^{2k-1}y^{2k-1}}{(2k-1)!} = \frac{1}{c}\sum_{k=0}^{\infty} \frac{c^{2k+1}y^{2k+1}}{(2k+1)!} = \frac{1}{c}\,\mathrm{sh}(cy),$$

$$U_2(y,c^2) = \sum_{k=1}^{\infty} \frac{c^{2k-2}y^{2k-2}}{\Gamma(2k-1)} = \sum_{k=1}^{\infty} \frac{c^{2k-2}y^{2k-2}}{(2k-2)!} = \sum_{k=0}^{\infty} \frac{c^{2k}y^{2k}}{(2k)!} = \mathrm{ch}(cy),$$

i.e. (44) has the form

$$\delta(y) = \bar{\tau}\,\mathrm{ch}(cy) + \frac{1}{c}\bar{\nu}\,\mathrm{sh}(cy) + \frac{1}{c}\int\limits_0^y \mathrm{sh}[c(y-t)]A_{12}(t)dt. \tag{45}$$

From (45) we obtain

$$\delta'(y) = \bar{\tau}c\,\mathrm{sh}(cy) + \bar{\nu}\,\mathrm{ch}(cy) + \int\limits_0^y \mathrm{ch}[c(y-t)]A_{12}(t)dt,$$

$$\delta''(y) = \bar{\tau}c^2\,\mathrm{ch}(cy) + \bar{\nu}c\,\mathrm{sh}(cy) + A_{12}(y) + c\int\limits_0^y \mathrm{sh}[c(y-t)]A_{12}(t)dt. \tag{46}$$

Substituting (46) into (41), we obtain

$$u(x,y) = \alpha\left[\bar{\tau}c^2\,\mathrm{ch}(cy) + \bar{\nu}c\,\mathrm{sh}(cy) + A_{12}(y) + c\int\limits_0^y \mathrm{sh}[c(y-t)]A_{12}(t)dt\right]\frac{x^2}{2} + A_1(y)x + B_1(y). \tag{47}$$

Satisfying (47) the condition (21), we obtain directly

$$B_1(y) = \varphi_0(y).$$

With respect to $A_1(y)$ we obtain

$$\frac{\alpha r^2}{2}\left[\bar{\tau}c^2\,\mathrm{ch}(cy) + \bar{\nu}c\,\mathrm{sh}(cy) + A_{12}(y) + c\int\limits_0^y \mathrm{sh}[c(y-t)]A_{12}(t)dt\right] + A_1(y)r + B_1(y) = \varphi_r(y),$$

$$\frac{\alpha r^2}{2}A_{12}(y) + \frac{\alpha r^2}{2}c\int\limits_0^y \mathrm{sh}[c(y-t)]A_{12}(t)dt + A_1(y)r = \varphi_r(y) - \varphi_0(y) - \frac{\alpha r^2}{2}\left[\bar{\tau}c^2\,\mathrm{ch}(cy) + \bar{\nu}c\,\mathrm{sh}(cy)\right].$$

Taking into account that $A_{12}(y) = -\frac{3}{\alpha r}A_1(y) - \frac{6}{\alpha r^2}B_1(y)$, we obtain:

$$-\frac{3r}{2}A_1(y) + rA_1(y) - \frac{3cr}{2}\int\limits_0^y \mathrm{sh}[c(y-t)]A_1(t)dt =$$

$$= \varphi_r(y) + 2\varphi_0(y) + 3c\int\limits_0^y \mathrm{sh}[c(y-t)]\varphi_0(t)dt - \frac{\alpha r^2}{2}\left[\bar{\tau}c^2\,\mathrm{ch}(cy) + \bar{\nu}c\,\mathrm{sh}(cy)\right].$$

$$A_1(y) + 3c\int\limits_0^y \mathrm{sh}[c(y-t)]A_1(t)dt = f(y), \tag{48}$$

where

$$f(y) = \alpha r\left[\bar{\tau}c^2\,\mathrm{ch}(cy) + \bar{\nu}c\,\mathrm{sh}(cy)\right] - \frac{2}{r}[\varphi_r(y) + 2\varphi_0(y)] - \frac{6c}{r}\int\limits_0^y \mathrm{sh}[c(y-t)]\varphi_0(t)dt.$$

Equation (48) is the Volterra integral equation of second kind with respect to the unknown function $A_1(y)$. It is easy to check that a unique solution of equation (48) is defined by the formula

$$A_1(y) = f(y) - \frac{3c}{\sqrt{2}} \int_0^y \sin\left[\sqrt{2}c(y-t)\right] f(t)dt.$$

Thus, we have proved

Theorem 3. *Assume that* $\varphi_0(y), \varphi_r(y) \in C[0,\beta_2] \cap C^2]0,\beta_2[$. *Then the problem S has a unique solution defined by the formula*

$$u(x,y) = \alpha\left[\bar{\tau}c^2 \operatorname{ch}(cy) + \bar{\nu}c \operatorname{sh}(cy) + A_{12}(y) + c\int_0^y \operatorname{sh}[c(y-t)]A_{12}(t)dt\right]\frac{x^2}{2} + A_1(y)x + B_1(y),$$

where $A_{12}(y) = -\frac{6}{\alpha r^3}\left[A_1(y)\frac{r^2}{2} + B_1(y)r\right]$,

$$A_1(y) = f(y) - \frac{3c}{\sqrt{2}} \int_0^y \sin\left[\sqrt{2}c(y-t)\right] f(t)dt, \quad B_1(y) = \varphi_0(y),$$

$$f(y) = \alpha r\left[\bar{\tau}c^2 \operatorname{ch}(cy) + \bar{\nu}c \operatorname{sh}(cy)\right] - \frac{2}{r}[\varphi_r(y) + 2\varphi_0(y)] - \frac{6c}{r}\int_0^y \operatorname{sh}[c(y-t)]\varphi_0(t)dt.$$

Note. The solution $u(x,y)$ of the problem S by the variable y will belong to the same class than that of the function $\varphi_0(y), \varphi_r(y)$.

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