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PHYSICS

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PRESERVATION OF THE THIRD ADIABATIC INVARIANT OF MOTION IN THE EQUATORIAL PLANE OF MAGNETIC FIELD WITH WEAK AXIAL ASYMMETRY

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The paper considers the question of preservation of the third adiabatic invariant of motion of charged particles $v_{II} = 0$ (equatorial plane) in flow and in canonical forms in magnetic fields having weak asymmetry. Transition to the coordinate system rotating with drift angular velocity allows us to reduce the problem to the one already been solved, namely, to the problem of preservation of the third adiabatic invariant in an axially symmetric but time-variable magnetic field.

Key words: adiabatic invariant of motion, weak asymmetry of magnetic field, drift approximation

Introduction

The papers [1]-[3] consider the question on the preservation of the third adiabatic invariant of charged particle motion in the equatorial plane of axially symmetric magnetic field. It was shown that the equivalence of the canonical form $I = \frac{1}{2\pi}\oint p_\varphi dq \sim RU_\varphi$ and of the flow one $\Phi \sim RA_\varphi$ as invariants in drift approximation was provided by the constancy of the generalized momentum $P_\varphi = mRU_\varphi + (e/c)RA_\varphi$ with the accuracy of drift approximation constant. The following symbols were used in the relations: p_φ and q_φ is the generalized momentum and the corresponding generalized coordinate; U_φ and A_φ is drift velocity and vector potential with respect to a cyclic coordinate φ, R is the radius vector of a guiding center (cylindrical coordinate system). In its turn, the constancy of the third adiabatic invariant in canonical I and in flow Φ forms with the

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desired accuracy degree held when the standard condition of adiabatic invariance was fulfilled $\frac{T_{dr}}{H}|\frac{dH}{dt}| \ll 1$, where T_{dr} is the drift period with respect to a coordinate φ. It was shown in the papers [1]-[3], that in drift approximation, this allows us to neglect both the effect of a disturbance term $\ddot{R} = c\frac{d}{dt}\frac{E}{H}$ (radial acceleration) on particle motion and the cumulative effect of this disturbance in the magnetic field. Thus, preservation of the third invariant in the I and Φ forms was provided by the possibility to consider the electric drift velocity v_E as a constant in a variable field, i.e. $v_E = \text{const}$ (in the case of a steady field this constant is equal to zero). The paper considers in drift approximation the charged particle motion in the equatorial plane of an asymmetric magnetic field. We will suppose that the asymmetry is limited only by the condition of preservation of the third adiabatic invariant. Therefore, in such fields the first adiabatic invariant μ is known to be a constant $T_{dr} \gg T_l$), T_l is the period of Larmor gyration.

The system of drift equations has the following general form [4]-[6]

$$\begin{aligned} \vec{U}_{qp} = \frac{d\vec{R}}{dt} = v_{II}\frac{\vec{H}}{H} + \frac{c}{H^2}\left[\vec{E},\vec{H}\right] + \frac{mc\left(2v_{II}^2 + v_\perp^2\right)}{2eH^3}\left[\vec{H},\vec{\bigtriangledown}H\right], \\ \frac{d\tilde{\varepsilon}}{dt} = e\left(\vec{E}\frac{d\vec{R}}{dt}\right) + \mu\frac{\partial H}{\partial t}, rot\vec{H} = 0. \end{aligned} \qquad (1)$$

1) has the following notations: $\vec{U_{dr}}$ is the guiding center drift velocity, v_{II} and $v_\perp$ are the velocity parallel and perpendicular components of particle v, μ is the first adiabatic invariant, $\tilde{\varepsilon} = \frac{mv^2}{2} = \frac{m\left(v_{II}^2 + v_\perp^2\right)}{2}$ is the particle energy. The constancy of μ in system (1) holds with the accuracy of drift approximation, that is with the accuracy of $\varepsilon \ll 1$, where $\varepsilon \sim \rho_l/R$ is the infinitesimal order of drift approximation, ρ_l is the Larmor radius, R is the characteristic dimension of a magnetic system.

In the equatorial plane of a constant asymmetric field, the drift equation of system (1) ($v_{II} = 0$) in a curvilinear coordinate system X_1, X_2, X_3 has the following form:

$$\frac{d\vec{R}}{dt} = \frac{mcv_\perp^2}{2eH^3}\left[\vec{H},\vec{\bigtriangledown}H\right] = -\frac{v_\perp^2}{2\omega_l H}\frac{\partial H}{\partial X_3}\vec{e_2} + \frac{v_\perp^2}{2\omega_l H}\frac{\partial H}{\partial X_2}\vec{e_3}, \qquad (2)$$

where $\vec{e_1},\vec{e_2},\vec{e_3}$ are unit vectors aligned along the field $\vec{e_1} = \vec{H}/H$, the principal normal and binormal to a field line, and X_1, X_2, X_3 are the corresponding curvilinear coordinates, X_1 coordinate coincides with a field line.

Before we pass on to the calculation and analysis of the preservation of the third adiabatic invariant in the I and Φ forms, we consider the possibility of its existence in magnetic fields the asymmetry of which is limited only by the conditions of application of drift approximation. As long as the canonical form $I = \frac{1}{2\pi}\oint p_\varphi dq \sim RU_\varphi$ minimally assumes quasi-periodicity with respect to the generalized coordinate q_φ (X_3 in our case), and approximation does not postulate such a periodicity, it may turn out that the third invariant does not have any sense in an asymmetric field. As long as in a steady case the field asymmetry results in a drift with respect to the coordinate X_2 (an analogue of R in a cylindrical coordinate system) that is across the field lines, a particle may leave a trap before it makes a periodic movement with respect to the generalized coordinate q_i (X_3), for example, in the region of quasi-trapping of the Earth magnetosphere. In this case, the notion of a flow Φ does not have sense due to the open drift trajectory.

Moreover, field asymmetry may result in the distortion of the equatorial plane, and, as a consequence, in the occurence of a velocity component along the field line, and to the particle turnoff from the equatorial plane ($v_{II} \neq 0$). Consequently, limitation of the asymmetry by the condition of application of drift approximation is not enough not only for the analysis of the accuracy of adiabatic invariant preservation but for its existence in general.

In accordance with equation (2), the guiding center for particles with pitch angles $\alpha = \pi/2$ moves along the line of the constant H (drift velocity vector is perpendicular both to $\vec{H}$ and to $\vec{\bigtriangledown} H$).

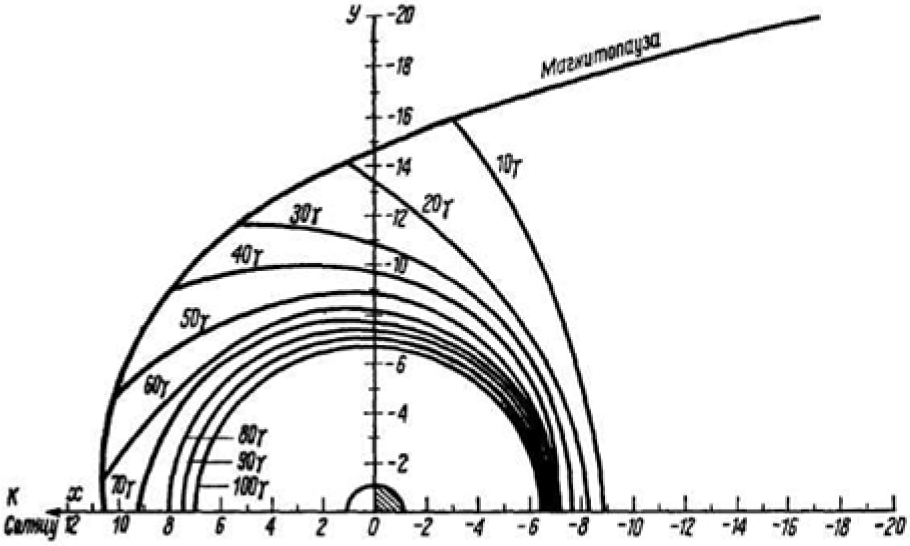

Figure. Drift trajectories of charged particles (pitch angle is $\alpha = \pi/2$), calculated according to experimental average isolines of the magnetic field in the Earth magnetosphere equatorial plane [7]. Numbers by the corresponding isolines show the magnetic field values.

The figure illustrates the calculated drift trajectories of charged particles moving in the magnetic equator plane [7]. Within $\sim 4R_3$ the trajectories are circular (dipole approximation of the magnetic field). The particle trajectories as they drift across the midnight meridian at distances of more than $7R_3$ do not close around the Earth and reach the magnetosphere side boundaries (magnetopause). Magnetic field asymmetry yields that drift trajectory along the line $= \mathrm{const}$ passes the Earth the most closely at magnetic midnight crossing the midday meridian at distances of $\sim 10R_3$. Therefore, if in some region of the equatorial plane of a stationary magnetic field, the isolines $H = \mathrm{const}$ are closed, then preservation of the third adiabatic invariant both in canonical and flow forms is trivial. However, it makes sense to examine the motion in the equatorial plane in detail to obtain some general statements and to pass on to a more complicated case of charged particle motion with $v_{II} \neq 0$ in a different from axially symmetric field $\left(\dfrac{\partial H}{\partial \varphi} \neq 0\right)$.

We should note that in the case of an axially symmetric magnetic field, the coordinate satisfying the periodic motion of a charged particle in the equatorial plane corresponds to the angle Φ of cylindrical coordinate system (R,φ). For such fields, the coordinate lines X_2 and X_3 coincide with the coordinate lines R and φ, respectively

Moving coordinate system K'

In the cylindrical coordinate system which will be hereafter called K system, drift equation (2) is decomposed as follows:

$$U_R = \frac{dR}{dt} = -\frac{v_\perp^2}{2\omega_{\text{л}} H}\frac{1}{R}\frac{\partial H}{\partial \varphi}, U_\varphi = R\frac{d\varphi}{dt} = \frac{v_\perp^2}{2\omega_{\text{л}} H}\frac{\partial H}{\partial R}. \tag{3}$$

Lagrangian of such a particle will have the following form:

$$L = \frac{mU_{dr}^2}{2} + \frac{e\left(\vec{U}_{dr}\vec{A}\right)}{c} = \frac{m}{2}\left(\vec{R} + \left[\vec{\varphi},\vec{R}\right]\right)^2 + \frac{e}{c}\left\{\left(\left[\vec{\varphi},\vec{R}\right]\vec{A}_\varphi\right) + \left(\vec{R}\vec{A}_R\right)\right\}, \tag{4}$$

where $\vec{A} = \vec{A}_R + \vec{A}_\varphi$ are the components of vector potential and drift velocities are determined by (3). From Lagrange equation for a generalized momentum $P_\varphi = mR^2\varphi + (e/c)RA_\varphi$ we obtain:

$$\frac{dP_\varphi}{dt} = \frac{e}{c}\frac{\partial\left(\vec{U}_{dr}\vec{A}\right)}{\partial\varphi} \neq 0.$$

and the generalized momentum P_φ in an asymmetric magnetic field is not the motion constant.

To analyze the values $R^2\varphi$ and RA_φ, included into the generalized momentum P_φ, we turn to a new coordinate system K' which rotates relatively the cylindrical system K with the angular velocity φ, related locally with the angular velocity of the guiding center [8]. The rotation center of K' coincides with the center of the cylindrical coordinate center K. For the coordinates of the moving system with unit vectors $\vec{e'_R}$ and $\vec{e'_\varphi}$ we may wright: $R' = R$, φ'=const. As long as in K' system the relative velocity of the guiding center $\vec{V}_{rel}$ is equal to $\vec{R}$, and the transport one is $\vec{V}_{tr} = \left[\vec{\varphi},\vec{R}\right]$, then

$$\vec{U}_{dr} = \vec{U}_{abc} = \vec{V}_{rel} + \vec{V}_{tr} = \vec{R} + \left[\vec{\varphi},\vec{R}\right].$$

From the physical point of view, the transition to the rotating coordinate system means the following. Firstly, due to $\vec{V}_{tr}$, the electric field $\vec{E}_1$ is induced in K' system which is determined as follows:

$$\vec{E}_1 = \frac{1}{c}\left[\left[\vec{\varphi},\vec{R}\right],\vec{H}\right] = \frac{1}{c}\left[\vec{U}_\varphi,\vec{H}\right]. \tag{5}$$

This field “compensates” gradient drift velocity occurring due to $(\partial H/\partial R)$. Multiplying vectorially (5) by $\vec{H}$, we obtain the velocity in a moving system:

$$\vec{V}_{E_1} = c\frac{\left[\vec{E}_1,\vec{H}\right]}{H^2} = \frac{\left[\left[\vec{U}_\varphi,\vec{H}\right],\vec{H}\right]}{H^2} = -\vec{U}_\varphi.$$

Secondly, since there is no field dependence on φ' in such a system, that is $(\partial H/\partial\varphi') = 0$, and the magnetic field H is asymmetric, then a partial derivative $\partial H'/\partial t$ not equal to zero ($H' = H$) arises. That results in the generation of one more electric field $\vec{E}_2$ orthogonal to $\vec{E}_1$. Since for the given $\vec{R}'$ the field in a point changes due to the rotation of the coordinate system K' in an asymmetric field H, then in the given point of a moving coordinate system $\partial H'/\partial t \sim rot\vec{E}_2 \neq 0$. Therefore, the guiding center motion in system K' corresponds to the motion in crossed electric and magnetic fields and is defined by the following equation:

$$\frac{d\vec{R}'}{dt} = \frac{c}{H^2}\left[\vec{E}_1,\vec{H}\right] + \frac{c\mu}{eH'}\frac{\partial H'}{\partial R'}\vec{e}_{\varphi'}. \tag{6}$$

Equation (6) corresponds to the equation describing the guiding center motion in a variable axially symmetric field (no dependence on φ'), plus the electric field induced due to the transport velocity. Consequently, relatively the kinematic relations, the transition to the description of charged particle motion in a moving coordinate system is equivalent to the transition to an axial variable field. If we take into account the values of drift velocities (3), the electric field $\vec{E}_1$ (5) and the equality $H' = H$, then we may write for the change of a radius-vector R' in system K':

$$\frac{d\vec{R}'}{dt} = \frac{c}{H^2}\left[\vec{E}_2,\vec{H}\right]. \tag{7}$$

Multiplying (7) by $\vec{H}$, we obtain the electric field $\vec{E}_2$:

$$\vec{E}_2 = -\frac{1}{c}\left[\vec{R},\vec{H}\right] = -\frac{v_\perp^2}{2\omega_л HR}\frac{\partial H}{\partial\varphi}\vec{e}_{\varphi'} \tag{8}$$

R value from (2) was substituted into the latest expression. Thus, expression (8) shows that generation of the electric field in system K' is determined by axial asymmetry.

Third adiabatic invariant in canonical form

We consider the Lagrangian L' in a moving coordinate system [9]. According to Lorentz transforms for the 4-vector of field potential $\vec{A}^i = \left(\psi,\vec{A}\right)$, where ψ is the scalar potential and $i = (0,1,2,3)$, we obtain in system K' for a nonrelativistic case

$$\psi' = -\frac{U_\varphi A'_\varphi}{c}, A'_\varphi = A_\varphi, A'_R = A_R.$$

The prime on the 4-vector components refers to system K', velocity U_φ is along A_φ. It is clear from 4-vector transformations that the components of vector potential $\vec{A}$ in a moving coordinate system remain but the electric field potential ψ is induced, the occurrence of which is associated with $\vec{E}_1$. The vertex electric field $\vec{E}_2$ does not have a potential and is defined by time partial derivative from the magnetic field. Taking that into account, the Lagrangian (4) in system K' has the form:

$$L' = \frac{m\vec{V}_{rel}^2}{2} + m\left(\vec{V}_{rel}\left[\dot{\vec{\varphi}},\vec{R}\right]\right) + \frac{m\left[\dot{\vec{\varphi}},\vec{R}\right]^2}{2} + \frac{e\left(\vec{V}_{rel}\vec{A}\right)}{c} - e\psi. \tag{9}$$

We obtain the equations of guiding center motion in system K'. To do that, we represent the total differential of the Lagrangian (9) in the following form introducing the notation for potential energy $W_n = W_n\left(\vec{R},\vec{V}_{rel}\right)$,

$$dL' = m\left(\vec{V}_{rel}\cdot d\vec{V}_{rel}\right) + m\left(d\vec{V}_{rel}\left[\dot{\vec{\varphi}},\vec{R}\right]\right) + m\left(dR\left[\vec{V}_{rel}\dot{\vec{\varphi}}\right]\right) +$$

$$+m\left[\left[\dot{\vec{\varphi}},\vec{R}\right]\dot{\vec{\varphi}}\right]d\vec{R} - \frac{\partial W_n}{\partial \vec{R}}d\vec{R} - \frac{\partial W_n}{\partial \vec{V}_{rel}}d\vec{V}_{rel}.$$

Differentiating separately the latest expression with respect to $\vec{V}_{rel}$, and then to $\vec{R}$, from Lagrange equation $\frac{d}{dt}\frac{\partial L'}{\partial V_{rel}} = \frac{\partial L'}{\partial R}$, we obtain the desired equation relatively the charged particle velocity in a moving coordinate system

$$m\frac{d\vec{V}_{rel}}{dt} = \frac{d}{dt}\frac{\partial W_n}{\partial \vec{V}_{rel}} - \frac{\partial W_n}{\partial \vec{R}} + m\left[\vec{R},\ddot{\vec{\varphi}}\right] + 2m\left[\vec{V}_{rel},\dot{\vec{\varphi}}\right] + m\left[\left[\dot{\vec{\varphi}},\vec{R}\right],\dot{\vec{\varphi}}\right]. \qquad (10)$$

In (10) the last three terms came about from the transition to a noninertial coordinate system K' and are, naturally, equal to the force generated due to the irregularity in speed of rotation ($\ddot{\varphi} \neq 0$), to Coriolis force and to the centrifugal force. Developing the vector products and multiplying (10) scalarwise by unit vectors $\vec{e}_{R'}$ and $\vec{e}_{\varphi'}$ of system K', we obtain:

$$m\dot{\vec{V}}_{rel} = -\frac{\partial W_n}{\partial R} + \frac{d}{dt}\frac{\partial W_n}{\partial V_{rel}} + mR\dot{\varphi}^2, 2m\dot{R}\dot{\varphi} + mR\ddot{\varphi} = 0. \qquad (11)$$

It follows from the second equation of (11) by integration that

$$R^2\dot{\varphi} = const. \qquad (12)$$

Therefore, the coordinate system K' rotates in such a way that the square radius-vector of a particle in this system ($R' = R$) multiplied by its rotation angular velocity is a constant. However, the angular velocity K' is locally associated with the angular velocity of particle rotation in system K, so for a particle in cylindrical coordinates, taking into account the equality $U_\varphi = R\dot{\varphi}$, we may write:

$$RU_\varphi = const. \qquad (13)$$

Based on the fact that the velocity U_φ is defined by the relation (3) with the accuracy of drift approximation, the constancy (13) is realized with the same accuracy degree.

The papers [1],[2] showed that in an axially symmetric variable magnetic field the cumulative effects, associated with radial acceleration, causing the breakdown of the third adiabatic invariant, may be neglected under the condition $U_R/U_\varphi \ll 1$, that corresponds to the requirement of radial drift velocity smallness in comparison to azimuthal velocity. As long as in a moving coordinate system K' the radial velocity U_R is defined by relation (7), the condition of smallness in relation to the transport velocity $V_{tr} = U_\varphi$ in that system may be represented as:

$$6\pi\frac{U_R}{U_\varphi} = 6\pi c\frac{E_2}{HU_\varphi} \ll 1$$

Substituting the value E_2 from (8) and from the second equation of(3) into the latest expression and omitting the minus, we obtain

$$\frac{1}{R}\frac{\partial H}{\partial \varphi}/\frac{\partial H}{\partial R} \ll 1$$

or

$$\frac{1}{R}\frac{\partial H}{\partial \varphi} \ll \frac{1}{6\pi}\frac{\partial H}{\partial R}, \tag{14}$$

that corresponds to the requirement of weak nonaxiality of the magnetic field, the radial gradient is much larger than the azimuthal one. We determine the type of the constant in expressions 12), (13). For asymmetric magnetic fields which satisfy drift approximation, the following is true:

$$\frac{\partial H}{\partial X_2} \cong \frac{\partial H}{\partial R} = \frac{H}{R_{cr}},$$

where R_{cr} is the radius of curvature of a field line [1],[2],[10]. We introduce the notation $k = R/R_{cr}$. However, in contrast to a symmetric field, in our case k is a weak function of angle φ. According to what has been said, we rewrite the latest expression:

$$\frac{\partial H}{\partial R} = \frac{H}{R}k(\varphi). \tag{15}$$

Substituting (15) into the second equation of (3), we express RU_φ

$$RU_\varphi = \frac{v_\perp^2}{2\omega_л}k(\varphi) = \frac{c}{e}\mu k(\varphi), \tag{16}$$

where $\mu = mv_\perp^2/2H$ is the first adiabatic invariant. Comparing (12) and (16) we obtain that $k(\varphi) = k \cong const$ with the accuracy $\mu \cong const$. Thus, knowing the explicit form of the const in (12) and (16), we can finally write:

$$RU_\varphi = \frac{c}{e}\mu k = const. \tag{17}$$

Therefore, the third adiabatic invariant in the canonical form remains in the fields with weak asymmetry according to (17). So,

$$I = \frac{m}{2\pi}\oint RU_\varphi d\varphi = mRU_\varphi = \frac{c}{e}\mu k = const. \tag{18}$$

Third adiabatic invariant in flow form

We consider the preservation of the third invariant in the flow form and the equivalence of the change of I by Φ. In the case of a symmetric field this change was provided by the constancy of the generalized momentum P_φ on the cyclic variable φ. However, if the cyclicity φ for an axial field is the basis of symmetry, then for drift equations the independence on φ is approximate with the accuracy degree with which a drift is separated from Larmor gyration. For this very reason the Lagrangian did not depend on ϕ and the generalized momentum P_φ was the constant with the same accuracy in a cylindrical coordinate system. In the case of an asymmetric field, $P_\varphi = mRU_\varphi + (e/c)RA_\varphi$

is not the constant of motion and $\dot{P}_\varphi \neq 0$. Based on the constancy of RU_φ, we represent the Lagrange equation P_φ in the following form:

$$\frac{d}{dt}P_\varphi = \frac{e}{c}\frac{d}{dt}RA_\varphi = \frac{e}{c}\frac{\partial}{\partial\varphi}\left(U_R A_R + U_\varphi A_\varphi\right) = \frac{e}{c}\left(U_R\frac{\partial A_R}{\partial\varphi} + U_\varphi\frac{\partial A_\varphi}{\partial\varphi}\right) \neq 0. \tag{19}$$

It was taken into account in (19) that drift velocities do not depend on φ with the accuracy $\mu = const$. Therefore, RA_φ does not remain when drifting in an asymmetric field. According to the definition of a magnetic flow we have:

$$\Phi = \iint\limits_s \vec{H}d\vec{S} = \oint \vec{A}d\vec{l} = \oint RA_\varphi d\varphi + \oint A_R dR.$$

According to the mean-value theorem, we transform the latest expression:

$$\Phi = 2\pi\langle RA_\varphi\rangle + \langle A_R\rangle \bigtriangleup R, \tag{20}$$

where $\bigtriangleup R = R_{\text{к}} - R_H$, R_H nitial is the initial value of radius R ($\varphi_{\text{н}} = 0$), $R_{\text{к}}$ is the final value of R ($\varphi_{\text{к}} = 2\pi$). That is why, for a flow in (20) to be an averaged value, quasi-periodicity of guiding center motion along the guiding center trajectory is needed at the least. At the same time, drift motion does not postulate the presence of such motion. We define $\bigtriangleup R$ in (20), dividing in (3) the first equation by the second one using the relation (16) ($k = const$)

$$\frac{1}{R}\frac{\partial H}{\partial\varphi} = -\frac{1}{kH}\frac{\partial H}{\partial\varphi}.$$

By integration we obtain:

$$\bigtriangleup R = R_{\text{н}} - R_{\text{к}} = R_{\text{н}}\left[\exp\left(-\frac{1}{k}\oint\frac{\partial H}{H\partial\varphi}d\varphi\right) - 1\right] \tag{21}$$

Integration of the exponent power is performed from $\varphi = 0\,(R = R_{\text{н}})$ to $\varphi = 2\pi$ ($R = R_{\text{к}}$). However, according to (17), the guiding center periodically moves with the accuracy of drift approximation and returns to the initial point. That is why $R_{\text{н}} = R_{\text{к}}$ and $\langle\dfrac{1}{H}\dfrac{\partial H}{\partial\varphi}\rangle = 0$ the trajectory is closed and the motion is degenerate (symbol $\langle\rangle$ denotes cyclic angle φ averaging. Hence, $\bigtriangleup R = 0$ and $\langle A_R\rangle = 0$. That is why the invariant in the flow form Φ is constant with the same accuracy degree than that in the canonical form I in (18):

$$\Phi = 2\pi\langle RA_\varphi\rangle = const. \tag{22}$$

Therefore, as it follows from (22), drift period average value of the generalized momentum P_φ is the averaged value $P_\varphi = m\langle RU_\varphi\rangle + (e/c)\,\langle RA_\varphi\rangle = const$, where $\langle RU_\varphi\rangle = RU_\varphi$ according to (16).

It follows from the abovesaid that preservation of $R^2\dot{\varphi} = RU_\varphi$ in the magnetic field with weak asymmetry follows from the consideration of a moving coordinate system K', where the motion (from the point of view of kinematic relations) corresponded to axially symmetric time-variable field. However, preservation of the adiabatic invariant in the form RU_φ in these fields corresponded to the accuracy of drift approximation for short time intervals since the cumulative effects associated with R nonaveraging acceleration resulted in invariant breakdowns. So, having differentiated the first equation (3) on time for the constant field and neglecting second derivatives with respect to coordinates, we obtain:

$$\ddot{R} \approx \frac{v_\perp^2}{2\omega_{\text{л}} H}\frac{\dot{R}}{R}\frac{\partial H}{\partial\varphi} = -\frac{\dot{R}^2}{R} = -\frac{1}{R}\left(\frac{v_\perp^2}{2\omega_{\text{л}} HR}\right)^2\left(\frac{\partial H}{\partial\varphi}\right)^2.$$

Hence, in general case $\langle R\rangle \sim (\partial H/\partial\varphi)^2$, cumulative effects associated with nonaveraging of radial acceleration should result in the breakdown of the preservation of RU_φ. However, realization of the condition (14) allows us to neglect the acceleration of invariant envelope.

Time-variable source field

If we now consider a time-variable source magnetic field, that will not result in something principally new. According to Maxwell equation $rot\vec{E} \sim -\partial\vec{H}/\partial t$, vortex electric field $\vec{E}_3$ is induced in a variable field of a fixed coordinate system. It is summed up with electric fields induced in a rotating coordinate system due to the transport velocity $\vec{E}_1$ (5) and nonaxiality $\vec{E}_2$ (6). In this case, to preserve the third adiabatic invariant in canonical and flow forms, it is enough to fulfill the standard adiabaticity condition

$$\left|\frac{\partial\vec{H}}{\partial t}\right| \ll \frac{H}{T_{dr}}, \tag{23}$$

where T_{dr} is the drift period with respect to the coordinate φ.

Conclusions

On the example of charged particle motion in equatorial plane ($v_{II}=0$), we considered the question of preservation of the third adiabatic invariant of motion in flow and canonical forms in asymmetric magnetic fields. It was shown that - in the case when $\rho_{\text{л}} \ll R$ (drift approximation), in asymmetric fields, the transition to a rotation coordinate system with drift particle angular velocity $\dot{\varphi}_{dr}$ allows us to reduce the problem to the case of an axially symmetric but a time-variable magnetic field. The cumulative effects of radial acceleration may be neglected when the condition $U_R/U_\varphi \ll 1$ is fulfilled. That results in the requirement of presence of field weak axial asymmetry (5) and, as a consequence, in adiabatic invariant equivalence in J and Φ forms and in the possibility to consider them to be the constants; - in the case of a variable source magnetic field, the condition for preservation of the invariants in J and Φ forms and their constancy in the fields with weak asymmetry is the fulfillment of a standard adiabacity condition (23) together with (14).

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