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MATHEMATICS

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CHARACTERISTIC PROBLEMS FOR THE LOADED WAVE EQUATION WITH SPECIFIC SHIFT

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The paper investigates the characteristic problems for the loaded wave equation with a special shift. A theorem on the uniqueness of the solution of the Goursat problem was proved and the necessary conditions for its solvability were found.

Key words: Goursat problem, Goursat condition, wave equation, loaded equation, characteristics

Introduction

Nakhushev A.M. was the first one [2] who paid attention to the topicality of investigation of loaded differential equations with partial derivatives [1] when they contain a trace of the solution sought on the variety of the kind $(\xi, 0)$, where ξ is the characteristic coordinate. In the paper the author gave an example of a degenerate hyperbolic equation $u_{yy} - y^2 u_{xx} + u_x + \lambda u(\xi, 0) = 0$, for which the effect of characteristic inequality of Darboux second problem that present in this equation when $\lambda = 0$ is eliminated. The papers [3]-[7] are devoted to the investigation of boundary problems for linear loaded (in the sense mentioned above) strictly and weakly hyperbolic differential equations of the second order with two independent variables when the input media are the characteristics and the domain contains a parabolic degeneracy line interval.

Statement of the problem

We consider the equation

$$u_{yy} - u_{xx} + \lambda \operatorname{sign} y \cdot u(x - |y|, 0) = 0 \tag{1}$$

in the domain $\Omega = \{(x,y) : |y| < x < 1 - |y|\}$.

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We begin from Goursat problem for the equation (1). Goursat conditions are as follows:

$$u\left(\frac{x}{2}, \operatorname{sign} y \frac{x}{2}\right) = \varphi^{\pm}(x), \quad 0 \le x \le 1. \tag{2}$$

The equation (1) in the domain $\Omega^- = \Omega \cap (y < 0)$ is equivalent to the equation

$$u_{yy} - u_{xx} = \lambda u(x+y, 0), \tag{3}$$

and it is in characteristic coordinates

$$\xi = x + y, \eta = x - y \tag{4}$$

has the form

$$v_{\xi\eta} = -\frac{\lambda}{4} v(\xi, \xi), \tag{5}$$

where

$$v(\xi, \eta) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right). \tag{6}$$

Domain $\triangle = \{(\xi, \eta), 0 < \xi < \eta < 1\}$ is the image of Ω^- under the mapping (4). According to (6), the condition (2) turns into the condition

$$v(0, \eta) = \varphi^-(\eta), \quad 0 \le \eta \le 1. \tag{7}$$

It is easy to show by direct computation that the function

$$v(\xi, \eta) = \frac{\lambda}{4} \int_{\xi}^{\eta} v(t,t)(\eta - t) dt$$

is the solution of the equation (5), satisfying Cauchy homogeneous conditions

$$v(\xi, \xi) = 0, \quad (v_\xi - v_\eta)\big|_{\eta=\xi} = 0.$$

This fact enables us to introduce the following definition.

We call the function $u(x,y)$ a generalized solution of the equation (3) in the domain Ω^-, if it may be represented by the formula

$$u(x,y) = \frac{\tau(\xi) + \tau(\eta)}{2} - \frac{1}{2} \int_{\xi}^{\eta} \nu(t) dt + \frac{\lambda}{4} \int_{\xi}^{\eta} (\eta - t) \tau(t) dt, \tag{8}$$

where $\tau(x) \in C(\bar{J}) \cap C^1(J)$, $\nu(x)$ is a continuous J function integrated on a unit interval.

Since $u_y = u_\xi - u_\eta$, then from (8) it is easy to see that

$$u(x, 0) = \tau(x), \quad \lim_{t \to 0^-} u_y = \nu(x). \tag{9}$$

We consider now the equation (1) in the domain $\Omega^+ = \Omega \backslash (y > 0)$. In this domain it has the following form

$$u_{yy} - u_{xx} + \lambda u(x - y, 0) = 0. \tag{10}$$

In (10) we proceed to coordinate characteristics

$$\xi = x - y, \quad \eta = x + y. \tag{11}$$

Than it has the following form

$$v_{\xi\eta} = \frac{\lambda}{4} v(\xi, \xi),$$

and the domain is mapped into $\triangle$. It is clear that this time

$$u_y = u_\eta - u_\xi, \quad v(\xi, \eta) = u\left(\frac{\xi + \eta}{2}, \frac{\eta - \xi}{2}\right).$$

According to the generalized solution of the equation (10) in the domain Ω^+, we call any function of the form

$$u(x,y) = \frac{\tau(\xi) + \tau(\eta)}{2} + \frac{1}{2}\int_\xi^\eta \nu(t)dt - \frac{\lambda}{4}\int_\xi^\eta (\eta - t)\tau(t)dt, \tag{12}$$

The equations (4) and (11) admit a unified notation

$$\xi = x - |y|, \quad \eta = x + |y|. \tag{13}$$

Taking this into consideration, we introduce the following definition.

Generalized solution of the equation (1) in the domain Ω is any function $u(x,y)$ presented in the form

$$u(x,y) = \frac{\tau(x - |y|) + \tau(x + |y|)}{2} + \frac{\operatorname{sign} y}{2}\int_{x-|y|}^{x+|y|} \nu(t)dt -$$

$$- \frac{\lambda \operatorname{sign} y}{4}\int_{x-|y|}^{x+|y|} (x + |y| - t)\tau(t)dt, \tag{14}$$

where $\tau(x) \in C^1(J) \cap C(\bar{J})$, and $\nu(x)$ is continuous and integrated into J.

Satisfying (14) the condition of (2), taking into account that $\tau(0) = \varphi^\pm(0)$, and supposing that $\varphi^\pm(x) \in C^1(\bar{J})$, we have

$$\tau(x) = \varphi^+(x) + \varphi^-(x) - \varphi^+(0),$$

$$\nu(x) = \frac{\lambda}{2}\int_0^x \tau(t)dt + [\varphi^+(x) - \varphi^-(x)]'.$$

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$\varphi^\pm(x)$.

Thus, we have proved the following theorem.

Theorem 1. *If $\varphi^\pm(x) \in C^1(\bar{J})$, the Goursat problem (2) for the equation (1) in the domain Ω has a single generalized solution $u(x,y)$ and it has the following property*

$$u_y(x,0) = \frac{\lambda}{2}\int_0^x \tau(t)dt + [\varphi^+(x) - \varphi^-(x)]',$$

$$u(x,0)=\varphi^{+}(x)+\varphi^{-}(x)-\varphi^{+}(0).$$

Now for the equation (1) we consider the problem with the data on mutually disjoint characteristics

$$u\left(\frac{x}{2},-\frac{x}{2}\right)=\varphi_0(x),\quad 0\le x\le 1, \tag{15}$$

$$u\left(\frac{1+x}{2},\frac{1-x}{2}\right)=\varphi_1(x),\quad 0\le x\le 1. \tag{16}$$

We satisfy (14) the conditions of (15), (16). Then we have

$$\varphi_0(x)=\frac{\tau(0)+\tau(x)}{2}-\frac{1}{2}\int\limits_0^x \nu(t)dt+\frac{\lambda}{4}\int\limits_0^x (x-t)\tau(t)dt, \tag{17}$$

$$\varphi_1(x)=\frac{\tau(1)+\tau(x)}{2}+\frac{1}{2}\int\limits_x^1 \nu(t)dt-\frac{\lambda}{4}\int\limits_x^1 (1-t)\tau(t)dt. \tag{18}$$

From (17), (18) it is clear that

$$2\varphi_0'(x)=\tau'(x)-\nu(x)+\frac{\lambda}{2}\int\limits_0^x \tau(t)dt, \tag{19}$$

$$2\varphi_1'(x)=\tau'(x)-\nu(x)+\frac{\lambda}{2}(1-x)\tau(x). \tag{20}$$

That means that

$$4[\varphi_1'(x)-\varphi_0'(x)]=\lambda[(1-x)\tau(x)-\int\limits_0^x \tau(t)dt].$$

Hence, when $\lambda\neq 0$

$$(1-x)\tau(x)-\int\limits_0^x \tau(t)dt=\frac{4}{\lambda}[\varphi_1(x)-\varphi_0(x)]'. \tag{21}$$

If $\varphi_0(x)$ and $\varphi_1(x)\in C^2(J)$ then

$$(1-x)\tau'(x)-2\tau(x)=\frac{4}{\lambda}[\varphi_1(x)-\varphi_0(x)]'',$$

or

$$[(1-x)^2\tau(x)]'=\frac{4}{\lambda}(1-x)[\varphi_1(x)-\varphi_0(x)]''.$$

Thus,

$$\int\limits_1^x [(1-t)^2\tau(t)]'dt=\frac{4}{\lambda}\int\limits_1^x (1-t)[\varphi_1(t)-\varphi_0(t)]''dt.$$

Therefore,

$$\tau(x)=\frac{-4/\lambda}{(1-x)^2}\int\limits_x^1 (1-t)d[\varphi_1'(t)-\varphi_0'(t)]=$$

$$= \frac{4/\lambda}{(1-x)^2}(1-x)[\varphi_1(x)-\varphi_0(x)]' - \frac{4/\lambda}{(1-x)^2}\int_x^1 d[\varphi_1(t)-\varphi_0(t)] =$$

$$= \frac{4[\varphi_1'(x)-\varphi_0'(x)]}{\lambda(1-x)} + \frac{4[\varphi_1(x)-\varphi_0(x)]}{\lambda(1-x)^2} - \frac{4[\varphi_1(1)-\varphi_0(1)]}{\lambda(1-x)^2}.$$

Thus,

$$\tau(x) = \frac{(1-x)[\varphi_1'(x)-\varphi_0'(x)]+\varphi_1(x)-\varphi_1(1)+\varphi_0(1)-\varphi_0(x)}{\lambda(1-x)^2/4}. \qquad (22)$$

According to the condition (see (16)) $\tau(1)=\varphi_1(1)$. Taking this and (22) into consideration, we draw the conclusion, that the condition

$$\frac{\lambda}{4}\varphi_1(1) = \lim_{x\to 1}\frac{(1-x)[\varphi_1'(x)-\varphi_0'(x)]+\varphi_1(x)-\varphi_1(1)+\varphi_0(1)-\varphi_0(x)}{(1-x)^2}$$

is the necessary condition for problem solvability.

Applying the L'Hospital rule, the latest condition may be written in the following form

$$\lambda\varphi_1(1) = 2[\varphi_0''(1)-\varphi_1''(1)]. \qquad (23)$$

The condition (23) is the necessary condition for continuity $\tau(x)$ at the point $x=1$.

In accordance with (15) $\tau(0)=\varphi_0(0)$. Thus, from (22) when $x=0$ we have

$$\varphi_0(x) = \frac{4}{\lambda}[\varphi_1'(0)-\varphi_0'(0)+\varphi_1(0)-\varphi_1(1)+\varphi_0(1)-\varphi_0(0)]. \qquad (24)$$

From (21) when $x\to 0$ we find

$$\varphi_0(0) = \frac{4}{\lambda}[\varphi_1'(0)-\varphi_0'(0)]. \qquad (25)$$

The equations (24) и (25) allow us to write

$$\varphi_1(0)-\varphi_1(1) = \varphi_0(0)-\varphi_0(1). \qquad (26)$$

The conditions (25) and (26) are the necessary conditions for the solvability of the problem (15), (16) for the equation (1).

Conclusions

Thus, we may consider the following theorem to be proved

Theorem 2. *If $\varphi_0(x)$ and $\varphi_1(x)$ belong to the class $C^1(\bar{J})\cap C^3]0,1]$ and*

$$\lambda\varphi_1(1) = 2[\varphi_0''(1)-\varphi_1''(1)],\ \lambda\varphi_0(0) = 4[\varphi_1'(0)-\varphi_0'(0)],$$

$$\varphi_1(0)-\varphi_1(1) = \varphi_0(0)-\varphi_0(1),$$

the problem (15), (16) for the equation (1) in the domain Ω has a unique generalized solution.

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