DOI: 10.18454/2079-6641-2016-14-3-7-13

MATHEMATICS

MSC 34A34, 70H33, 70S10, 34A05

CONSERVATION LAWS AND SIMILARITY REDUCTION OF THE ZOOMERON EQUATION

S. Reza Hejazi, A. Naderifard, S. Rashidi

Department of Mathematical Sciences, Shahrood University of Technology, 3619995161, Shahrood, Semnan, Iran.

E-mail: ra.hejazi@gmail.com

In this study, we consider a 4−th order (1+1)-dimensional PDE called Zoomeron equation. Some conservation laws are derived based on direct method. We also derived some similarity solutions using the symmetries.

Key words: Zoomeron equation, Lie point symmetries, conservation laws, multiplier, similarity solution.

МАТЕМАТИКА

УДК 517.956

ЗАКОНЫ СОХРАНЕНИЯ И ПОДОБИЯ УРАВНЕНИЯ ЗУМЕРОНА

С. Р. Хейязи, А. Надерифард, С. Рашиди

Шахрудский технологический университет, математическое отделение, 3619995161, Шахруд, Семнан, Иран

E-mail: ra.hejazi@gmail.com

В данном исследовании мы рассматриваем уравнение 4-го порядка (1 + 1) -мерном ФДЭ называемое уравнением Зумерона. Некоторые законы сохранения выводятся на основе прямого метода. Мы также получены некоторые свойства подобия решений с использованием симметрий.

Ключевые слова: уравнение Зумерона, точечные симметрии Ли, законы сохранения, мультипликатор, подобие решения.

Introduction

The Zoomeron equation is a 4−th order non-linear single PDE in the form of:

$$\left(\frac{u_{xt}}{u}\right)_{tt}-\left(\frac{u_{xt}}{u}\right)_{xx}+2(u^2)_{xt}=0. \tag{1}$$

In the study of DEs, conservation laws play significant roles not only in obtaining in-depth understanding of physical properties of various systems, but also in constructing of their exact solutions. They described physical conserved quantities such as mass, energy, momentum and angular momentum, as well as charge and other constant of motion. They are important for investigating integrability and linearization mapping and for stablishing existence and uniqueness of solutions. They are also used in the analysis of stability and global behaviour of solutions. In addition they play an essential role in the developement of numerical methods and provide an essential starting point for find non-locally related systems and potential variables. Moreover, the structure of conservation laws is coordinate-free, as a point or contact transformation maps a conservation laws into a conservation laws. A systematic way of constructing the conservation laws of a sytem of DEs that admits a variational principle is via Noether's theorem. Its application allows physicists to gain powerful insights into any general theory in physics just by analyzing the various transformations that would make the form of the laws involved invariant. For instance, the invariant of physical systems with respect to spatial translation, rotation and time translation respectively give rise to the well known conservation laws of linear momentum, angular momentum and energy. Among the generalization of Noether's theorem an Ibragimov's theorem [9], based on the self-adjontness of DEs allows to find independent conservation laws for a system of PDEs. This method is very limited because of the self-adjointness. But the direct method has no any limitation and is applicable for any system of DEs. Thus, in this paper we use the second method for finding some conservation laws for the Eq. (1). The objective of this article is to look for conservation laws and exact solutions for solving the (1+ 1)-dimensional Zoomeron equation, where $u(x,t)$ is the amplitude of the relevant wave mode. This equation is one of incognito equation.

According to our recent search, there are a few article about this equation. We only know that this equation was introduced by Calogero and Degasperis [1, 4, 5].

The paper is organized in the following manner. In section 2 the conservation laws of the Zoomeron equation are obtained in direct method and were expressed in section 3 similarity reductions and explicit solutions.

The direct method for construction of coservation laws

In general, non-trivial local conservation laws arise obtain from linear combinations of the equations of the PDEs system with multipliers that yield non-trivial divergence expressions. In asking such expressions, the dependent variables and each of their derivatives that arise in PDEs system, or appear in the multipliers, are replaced by arbitrary functions [2, 8]. By their construction, such divergence expressions vanish on all solutions of the PDEs system. In particular, a set of multipliers $\{\xi_\sigma[U]\}_{\sigma=1}^N=\{\xi(x,U,\partial U,\cdots,\partial^l U)\}_{\sigma=1}^N$ yields a divergence expressions for PDEs system $R\{x;u\}$ if the identity

$$\xi_\sigma[U]R^\sigma[U]\equiv D_i\Phi^i[U], \tag{2}$$

holds for arbitrary functions $U(x)$.

A set of non-singular local multipliers $\{\xi_\sigma(x,U,\partial U,\cdots,\partial^l U)\}_{\sigma=1}^N$ yields a local conservation law for the PDEs system $R\{x,u\}$ if and only if the set of identities,

$$E_{U^j}(\xi(x,U,\partial U,\cdots,\partial^l U)R^\sigma(x,U,\partial U,\cdots,\partial^k U))\equiv 0 \tag{3}$$

holds for arbitrary functions $U(x)$.
We apply this method to obtain the local conservation laws of Eq. (1). Let

$$\begin{aligned} R[u] = \ & u^2u_{xttt} - uu_{tt}u_{xt} - 2uu_tu_{xtt} + 2u_t^2u_{xt} - u^2u_{xxxt} + uu_{xx}u_{xt} + 2uu_xu_{xxt} \\ & -2u_x^2u_{xt} + 4u_xu_tu^3 + 4u^4u_{xt}. \end{aligned} \tag{4}$$

In the following manner we explain the calculations to find the multpiliers and local conservation laws. First we search all local conservation law multipliers of the form zero order

$$\xi = \xi(x,t,U), \tag{5}$$

for the Eq. (4). Using the Euler operators

$$\begin{aligned} E_U = & \frac{\partial}{\partial U} - D_t\frac{\partial}{\partial U_t} - D_x\frac{\partial}{\partial U_x} + D_x^2\frac{\partial}{\partial U_{xx}} + D_t^2\frac{\partial}{\partial U_{tt}} + D_{xt}\frac{\partial}{\partial U_{xt}} \\ & -D_{xtt}\frac{\partial}{\partial U_{xtt}} - D_{xxt}\frac{\partial}{\partial U_{xxt}} + D_{xttt}\frac{\partial}{\partial U_{xttt}} + D_{xxxt}\frac{\partial}{\partial U_{xxxt}}, \end{aligned} \tag{6}$$

the determining equations (3) for the multipliers (4) becomes:

$$\begin{aligned} E_U\big[\xi(x,t,U)(U^2U_{xttt} - UU_{tt}U_{xt} - 2UU_tU_{xtt} + 2U_t^2U_{xt} - U^2U_{xxxt} + UU_{xx}U_{xt} + 2UU_xU_{xxt} \\ - 2U_x^2U_{xt} + 4U_xU_tU^3 + 4U^4U_{xt})\big] \equiv 0, \end{aligned} \tag{7}$$

where $U(x,t)$ are arbitrary functions. Equations (7) split with respect to each of dependent variables derivatives that arise in PDEs system (except dependent variables) such as U_t, U_x, U_{xx}, U_{xt}, $U_{xxt},\cdots$ to yield the over-determined linear PDEs system given by (8).

$$\text{zeroth characteristic} = \begin{cases} U_t & 12\xi_xU^3 + 4\xi_{xU}U^4 + 8\xi_{xtt}U + 3\xi_{Uxtt}U^2 - 2\xi_{xxx}U = 0, \\ U_{xt} & 24\xi U^3 + 8\xi_UU^4 - 7\xi_{xx}U + 7\xi_{tt}U + 3\xi_{UUU}U^2 - 3\xi_{xxU}U^2 = 0, \\ \vdots & \vdots \\ U_{xttt} & 2\xi_UU^2 + 6\xi U = 0. \end{cases} \tag{8}$$

The solution of (8) are the four sets of local multipliers given by (9),

$$\xi_1(x,t,u) = \frac{1}{u^3}, \quad \xi_2(x,t,u) = \frac{t}{u^3}, \quad \xi_3(x,t,u) = \frac{x}{u^3}, \quad \xi_4(x,t,u) = \frac{t^2+x^2}{2u^3}. \tag{9}$$

Similarly each ξ determines a non-trivial zeroth order local conservation law $D_t\Psi(x,t,U) + D_x\Phi(x,t,U) = 0$, with the characteristic from

$$\xi(x,t,U)R[U] = D_t\Psi(x,t,U) + D_x\Phi(x,t,U). \tag{10}$$

Table 1. Zeroth Order Local Conservation Laws.

Fluxes	Density	Zeroth Order Conservation Laws
0	1	$D_t 1 + D_x 0 = 0$
0	x	$D_t x + D_x 0 = 0$
1	0	$D_t 0 + D_x 1 = 0$
x	t	$D_t t + D_x x = 0$
0	t	$D_t 0 + D_x t = 0$

Table 2. First Order Local Conservation Laws

Fluxes	Density	First Order Conservation Laws
$xu_t + tu_t + u$	$-xu_x - u - tu_x$	$D_t(-xu_x - u - tu_x) + D_x(xu_t + tu_t + u) = 0$
x^2	$-2xt + x + x^2$	$D_t(-2xt + x + x^2) + D_x(x^2) = 0$
$-2xt + u_t + x$	$t^2 - u_x - t$	$D_t(t^2 - u_x - t) + D_x(-2xt + u_t + x) = 0$
$-2t + uu_t$	$-uu_x$	$D_t(-uu_x) + D_x(-2t + uu_t) = 0$

Using in (10) the expression (5) for ξ and doing a similar calculations table (1) is obtained. Now we search all local conservation law multipliers of the form first order,

$$\xi = \xi(x,t,U,U_x,U_t), \tag{11}$$

for the Eq. (4). Using the corresponding Euler operators the determining equations (3) for the multipliers (4) become:

$$E_U\left[\xi(x,t,U,U_x,U_t)(U^2U_{xttt} - UU_{tt}U_{xt} - 2UU_tU_{xtt} + 2U_t^2U_{xt} - U^2U_{xxxt} + UU_{xx}U_{xt} + 2UU_xU_{xxt} \right.$$
$$\left. - 2U_x^2U_{xt} + 4U_xU_tU^3 + 4U^4U_{xt})\right] \equiv 0. \tag{12}$$

Equations (12) split with respect to each of dependent variables derivatives that arise in PDEs system (except dependent variables and first order derivatives of them) such as U_{xx}, U_{tt}, U_{xt}, U_{xxt},... to yield the over-determined linear PDEs system given by (13).

$$\text{first characteristic} = \begin{cases} U_{xt}U_{ttt} & -\xi_{U_t}U = 0, \\ U_{ttt} & -\xi U_x - U\xi_x - UU_x\xi_U = 0, \\ \vdots\ \vdots & \\ U_{xxtt} & 2\xi_{U_x}UU_t - 3U^2U_x\xi_{UU_t} - 3U^2\xi_{xU_t} = 0. \end{cases} \tag{13}$$

The solution of (13) ($\xi(x,t,U,U_x,U_t)$) are the same as given by, (9). Each $\xi(x,t,U,U_x,U_t)$ determines a non-trivial first order local conservation law $D_t\Psi(x,t,U,U_x,U_t)$ $+D_x\Phi(x,t,U,U_x,U_t) = 0$, with the characteristic from

$$\xi(x,t,U,U_x,U_t)R[U] = D_t\Psi(x,t,U,U_x,U_t) + D_x\Phi(x,t,U,U_x,U_t). \tag{14}$$

After placement (11) in (14) and doing some tedious calculations table (2) is obtained. To find the second order multipliers we start by the multiplier of the form,

$$\xi = \xi(x,t,U,U_x,U_t,U_{xx},U_{tt},U_{xt}).$$

We can find ξ with the same expression such as (8), and (13). After tedious calculation we get second order conservation laws. The results are comming in table (3).

Table 3. Second Order Local Conservation Laws

Fluxes	Density	Second Order Conservation Laws
$xu_{xt}-u_t+tu_{xt}+u_x$	$-xu_{xx}-tu_{xx}$	$D_t(-xu_{xx}-tu_{xx})+D_x(xu_{xt}-u_t+tu_{xt}+u_x)=0$
$u_xu_{tt}+u_tu_{xt}$	$-u_xu_{xt}-u_tu_{xx}$	$D_t(-u_xu_{xt}-u_tu_{xx})+D_x(u_xu_{tt}+u_tu_{xt})=0$
$xu_{tt}+tu_{tt}+u_t$	$-xu_{xt}-u_t-tu_{xt}$	$D_t(-xu_{xt}-u_t-tu_{xt})+D_x(xu_{tt}+tu_{tt}+u_t)=0$
$uu_{xt}+u_xu_t$	$-uu_{xx}-u_x^2$	$D_t(uu_{xt}+u_xu_t)+D_x(-uu_{xx}-u_x^2)=0$
$uu_{tt}+u_t^2-u_{xt}$	$-uu_{xt}-u_tu_x+u_{xx}$	$D_t(-uu_{xt}-u_tu_x+u_{xx})+D_x(uu_{tt}+u_t^2-u_{xt})=0$
xu_t+tu_t+u	$-xu_x-u-tu_x$	$D_t(-xu_x-u-tu_x)+D_x(xu_t+tu_t+u)=0$
x^2+xt+t^2	$-2xt-\frac{1}{2}t^2$	$D_t(-2xt-\frac{1}{2}t^2)+D_x(x^2+xt+t^2)=0$
$-u_{tt}-u_t-uu_t$	$u_{tx}+u_x+uu_x$	$D_t(u_{tx}+u_x+uu_x)+D_x(-u_{tt}-u_t-uu_t)=0$
$-x-u_xu_{tx}-u_{tt}u_t$	$t+u_xu_{xx}+u_tu_{xt}$	$D_t(t+u_xu_{xx}+u_tu_{xt})+D_x(-x-u_xu_{tx}-u_{tt}u_t)=0$

Similarity reduduction and exact solution

In this section, we obtain similarity solution of the Zoomeron equation using Lie symmetries [3, 6, 7, 8].

Classical similarity solutions

First of all, let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned} x &\to x+\varepsilon\xi(x,t,u),\\ t &\to t+\varepsilon\tau(x,t,u),\\ u &\to u+\varepsilon\phi(x,t,u), \end{aligned}$$

with a small parameter $\varepsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$X=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\phi(x,t,u)\frac{\partial}{\partial u}. \tag{15}$$

The symmetry group of Eq. (1) will be generated by the vector field of the form (15). Thus, this equation admits X as a symmetry operator if the condition

$$X^{(4)}(1)\Big|_{(1)}=0,$$

is satisfied on solutions of Eq (1). Applying the fourth prolongation and solving the determininig equation one can demonstrate the equation (1) admits the following Lie algebra:

$$X_1=\frac{\partial}{\partial t}+\frac{\partial}{\partial x},\qquad X_2=\frac{\partial}{\partial x},\qquad X_3=x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}. \tag{16}$$

We make some discussion on the Zoomeron equation based on the vector fields (1).

Similarity solution of X_1

For the generator X_1, we have

$$u=v(r,q), \tag{17}$$

where $q=t$, $r=t-x$ are the group-invariants. Substituting (17) into (1), one can get

$$-4v(r)^3\left(\left(\frac{d}{dr}v(r)\right)^2+\left(\frac{d^2}{dr^2}v(r)\right)v(r)\right)=0. \tag{18}$$

Consequently, the exact solution of (1) can be written as follows

$$u(x,t)=\pm\sqrt{2a(t-x)+2b}, \tag{19}$$

where a,b are arbitrary constants.

Similarity solution of X_2

For the generator X_2, we have

$$\exp(\varepsilon X_2)(x,t,u)=(x+\varepsilon,t,u), \tag{20}$$

with substituting $\bar{x}=x+\varepsilon$ and using (19), another exact solution of (1) can be written as follows

$$u(x,t)=\pm\sqrt{2a(t-x-\varepsilon)+2b}. \tag{21}$$

Similarity solution of X_3

For the generator X_3, we have

$$\exp(\varepsilon X_3)(x,t,u)=(e^{\varepsilon}x,e^{\varepsilon}t,e^{-\varepsilon}u), \tag{22}$$

with substituting $e^{\varepsilon}x=\bar{x},e^{\varepsilon}t=\bar{t},e^{-\varepsilon}u=\bar{u}$ and using (19), another exact solution of (1) can be written as follows

$$u(x,t)=\pm e^{-\varepsilon}\sqrt{2a(e^{-\varepsilon}t-e^{-\varepsilon}x)+2b}. \tag{23}$$

Traveling wave solutions

The most useful solution is the traveling wave solution associated with the space and time translation symmetries. Using the transformation

$$u(x,t)=f(\xi),\qquad \xi=x-ct \tag{24}$$

and substituting the expression () into (1) yields,

$$-c^3\left(\frac{-f''}{f}\right)''+c\left(\frac{f''}{f}\right)''-2c\left(f^2\right)''=0. \tag{25}$$

With integrating twice with respect to ξ, by setting the second integration constant equal to zero, we obtain the following non-linear ordinary differential equation

$$-c^3f''+cf''-2cf^3-Rf=0, \tag{26}$$

where R is integration constant.
So the solutions of Zoomeron equation can be obtained by (27),

$$u(x,t)=c_2\sqrt{\frac{-R}{c_2^2c-c-R}}\mathrm{JacobiSN}\Bigg(\left(\frac{\sqrt{c(c^2-1)(c+R)(x-ct)}}{c(c^2-1)}+c_1\right)\sqrt{\frac{-R}{c_2^2c-c-R}},\ \frac{c_2\sqrt{-(c+R)c}}{c+R}\Bigg), \tag{27}$$

where c_2 , c_1 are arbitrary constants and JacobiSN is an elliptic function.

Conclusion

In this paper we introduced a Lie group analysis for an important PDEs called Zoomeron equation. The Lie algebra of symmetries was found by a useful algorithm. We used the direct method to obtain fluxes and densities of conservation laws for the equation. Finally we used the symmetries to find the reduction forms of the Zoomeron equation.

References

[1] Calogero F., Degasperis A., "Non-linear evolution equations solvable by the inverse spectral transform I", *Nuovo Cimento B 32*, 1976, 201-242.

[2] Bluman G., Cheviakov A., Anco C., "Applications of Symmetry Methods to partial Differential Equations", *Appl. Math. Sci*, **168** (2010), 43-70.

[3] Craddock M., Lennox K., "Lie group symmetries as integral transforms of fundamental solutions", *J. Differential Equations*, **232(2)** (2007), 652-674.

[4] Abazari R., "The solitary wave solutions of Zoomeron equation", *Appl. Math. Sci.*, **5** (2011), 2943-2949.

[5] Alquran M., Al-Khaled K., "Mathematical methods for a reliable treatment of the (2+1)-dimensional Zoomeron equation", *Math. Sci.*, **6** (2012), 11-15.

[6] Craddock M., Platen E., "Symmety group methods for fundamental solutions", *J. Differential Equations*, **207(2)** (2007), 285-302.

[7] Liu H. Z. , Li J. B., "Symmetry reductions, dynamical behavior and exact explicit solutions to the Gordon types of equations", *J. Comput. Appl. Math.*, **257** (2014), 144-156.

[8] Olver P., *Applications of Lie Groups to Differential Equations*, Grad. Texts in Math.. V. 107, Springer, New York, 1993.

[9] Ibragimov N. H., "Nonlinear self-adjointness in constructing conservation laws", *Archives of ALGA*, **7-8** (2010-2011), 1-99.

Поступила в редакцию / Original article submitted: 30.05.2016