

Assembling classical and dynamic inequalities accumulated on calculus of time scales

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The aim of this paper is to present reconciliation of some classical and dynamic inequalities by using calculus of time scales in more general, unified and extended form. We investigate here harmonious extensions and generalizations of Hermite–Hadamard and Rogers–Holder’s type inequalities in hybrid versions. The calculus of time scales combines continuous, discrete and quantum analogues.

Keywords: time scales, Hermite–Hadamard type inequality, Rogers–Holder’s type inequality

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Introduction

Motivated from the recent developments of the theory and applications of time scales, we will prove some results on time scales. The calculus of time scales was introduced by Stefan Hilger [5]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. The theory of time scales calculus is applied to harmonize results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The three popular branches of time scales calculus are studied as delta calculus, nabla calculus and diamond- α calculus. Many dynamic inequalities (see [1, 3, 8, 9, 10]) have been investigated by using this hybrid theory. Basic work on dynamic inequalities is done by Agarwal, Anastassiou, Bohner, Peterson, O’Regan, Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and \mathbb{T} is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

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Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adopted from monographs [3, 4].

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \rightarrow \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \rightarrow \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If \mathbb{T} has a left-scattered maximum M , then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \rightarrow \mathbb{R}$, the delta derivative f^Δ is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\varepsilon > 0$, there is a neighborhood U of t , such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t , and $f^\Delta(t)$ is called the delta derivative of f at t .

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be right-dense continuous (rd-continuous), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [3, 4].

DEFINITION 1. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$, provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t) \Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [2, 3, 4].

If \mathbb{T} has a right-scattered minimum m , then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$ and $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$. A function $f : \mathbb{T}_k \rightarrow \mathbb{R}$ is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^\nabla(t)$, if there exists $f^\nabla(t) \in \mathbb{R}$ such that given any $\varepsilon > 0$, there is a neighborhood V of t , such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \varepsilon |\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be left-dense continuous (ld-continuous), provided it is continuous at all left-dense points in \mathbb{T} and its right-sided limits exist (finite) at all right-dense points in \mathbb{T} . The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [2, 3, 4].

DEFINITION 2. A function $G : \mathbb{T} \rightarrow \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \rightarrow \mathbb{R}$, provided that $G^\nabla(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t) \nabla t = G(b) - G(a).$$

Now we present short introduction of the diamond- α derivative as given in [1, 12].

DEFINITION 3. Let \mathbb{T} be a time scale and $f(t)$ be differentiable on \mathbb{T} in the Δ and ∇ senses. For $t \in \mathbb{T}$, the diamond- α dynamic derivative $f^{\diamond\alpha}(t)$ is defined by

$$f^{\diamond\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha) f^\nabla(t), \quad 0 \leq \alpha \leq 1.$$

Thus f is diamond- α differentiable if and only if f is Δ and ∇ differentiable.

The diamond- α derivative reduces to the standard Δ -derivative for $\alpha = 1$, or the standard ∇ -derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem See [12]. Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be diamond- α differentiable at $t \in \mathbb{T}$ and we write $f^\sigma(t) = f(\sigma(t))$, $g^\sigma(t) = g(\sigma(t))$, $f^\rho(t) = f(\rho(t))$ and $g^\rho(t) = g(\rho(t))$. Then

(i) $f \pm g : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$, with

$$(f \pm g)^{\diamond\alpha}(t) = f^{\diamond\alpha}(t) \pm g^{\diamond\alpha}(t).$$

(ii) $fg : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$, with

$$(fg)^{\diamond\alpha}(t) = f^{\diamond\alpha}(t)g(t) + \alpha f^\sigma(t)g^\Delta(t) + (1 - \alpha)f^\rho(t)g^\nabla(t).$$

(iii) For $g(t)g^\sigma(t)g^\rho(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$, with

$$\left(\frac{f}{g}\right)^{\diamond\alpha}(t) = \frac{f^{\diamond\alpha}(t)g^\sigma(t)g^\rho(t) - \alpha f^\sigma(t)g^\rho(t)g^\Delta(t) - (1 - \alpha)f^\rho(t)g^\sigma(t)g^\nabla(t)}{g(t)g^\sigma(t)g^\rho(t)}.$$

DEFINITION 4. [See [12]] Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \rightarrow \mathbb{R}$. Then the diamond- α integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s) \Delta s + (1 - \alpha) \int_a^t h(s) \nabla s, \quad 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on \mathbb{T} .

Theorem See [12]. Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are \diamond_α -integrable functions on $[a, b]_{\mathbb{T}}$. Then

- (i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;
- (ii) $\int_a^t cf(s) \diamond_\alpha s = c \int_a^t f(s) \diamond_\alpha s$;
- (iii) $\int_a^t f(s) \diamond_\alpha s = - \int_t^a f(s) \diamond_\alpha s$;
- (iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;
- (v) $\int_a^a f(s) \diamond_\alpha s = 0$.

We need the following results.

The generalized Hermite–Hadamard type inequality for a weight function [1] states:

Theorem. Let \mathbb{T} be a time scale and $a, b \in \mathbb{T}$. Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a continuous convex function and let $w : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a continuous function such that $w(x) \geq x$ for all $x \in \mathbb{T}$ and $\int_a^b w(x) \diamond_{\alpha} x > 0$. Then

$$f(y_{w,\alpha}) \leq \frac{\int_a^b w(x) f(x) \diamond_{\alpha} x}{\int_a^b w(x) \diamond_{\alpha} x} \leq \frac{b - y_{w,\alpha}}{b - a} f(a) + \frac{y_{w,\alpha} - a}{b - a} f(b), \quad (1)$$

$$\text{where } y_{w,\alpha} = \frac{\int_a^b w(x) x \diamond_{\alpha} x}{\int_a^b w(x) \diamond_{\alpha} x}.$$

The following inequality from (2) is called the weighted power mean inequality in literature.

Note that if $-\infty \leq r < s \leq \infty$, then

$$M_n^{[r]} \leq M_n^{[s]}, \quad (2)$$

where $M_n^{[r]} = \frac{\sum_{k=1}^n w_k x_k^r}{\sum_{k=1}^n w_k}$, (see, e.g., [7, page 15]).

The generalized Jensen's inequality [1] states:

Theorem. Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a, b]_{\mathbb{T}}, (c, d))$ and $h \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |h(s)| \diamond_{\alpha} s > 0$. If $\Phi \in C((c, d), \mathbb{R})$ is convex, then generalized Jensen's inequality is

$$\Phi \left(\frac{\int_a^b |h(s)| g(s) \diamond_{\alpha} s}{\int_a^b |h(s)| \diamond_{\alpha} s} \right) \leq \frac{\int_a^b |h(s)| \Phi(g(s)) \diamond_{\alpha} s}{\int_a^b |h(s)| \diamond_{\alpha} s}. \quad (3)$$

If Φ is strictly convex, then the inequality \leq can be replaced by $<$.

Main results

Our first result concerning extended Hermite–Hadamard's inequality on time scales starts in this section.

Theorem 1. Let $w, f, g \in C([a, b]_{\mathbb{T}}, [0, +\infty))$ be \diamond_{α} -integrable functions such that f^p and g^q are concave on $[a, b]_{\mathbb{T}}$ with $\int_a^b w(x) \diamond_{\alpha} x > 0$. If $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then

$$\begin{aligned} & \left[\frac{b - y_{w,\alpha}}{b - a} f(a) + \frac{y_{w,\alpha} - a}{b - a} f(b) \right]^p \left[\frac{b - y_{w,\alpha}}{b - a} g(a) + \frac{y_{w,\alpha} - a}{b - a} g(b) \right]^q \\ & \leq \frac{1}{\left(\int_a^b w(x) \diamond_{\alpha} x \right)^2} \left(\int_a^b w(x) f^p(x) \diamond_{\alpha} x \right) \left(\int_a^b w(x) g^q(x) \diamond_{\alpha} x \right), \quad (4) \end{aligned}$$

$$\text{where } y_{w,\alpha} = \frac{\int_a^b w(x) x \diamond_{\alpha} x}{\int_a^b w(x) \diamond_{\alpha} x}.$$

Proof. From reverse Hermite–Hadamard's inequality (1), we have

$$\frac{b - y_{w,\alpha}}{b - a} f^p(a) + \frac{y_{w,\alpha} - a}{b - a} f^p(b) \leq \frac{\int_a^b w(x) f^p(x) \diamond_{\alpha} x}{\int_a^b w(x) \diamond_{\alpha} x} \quad (5)$$

and

$$\frac{b-y_{w,\alpha}}{b-a}g^q(a) + \frac{y_{w,\alpha}-a}{b-a}g^q(b) \leq \frac{\int_a^b w(x)g^q(x) \diamond_\alpha x}{\int_a^b w(x) \diamond_\alpha x}. \quad (6)$$

From inequalities (5) and (6), we have

$$\begin{aligned} & \left[\frac{b-y_{w,\alpha}}{b-a}f^p(a) + \frac{y_{w,\alpha}-a}{b-a}f^p(b) \right] \left[\frac{b-y_{w,\alpha}}{b-a}g^q(a) + \frac{y_{w,\alpha}-a}{b-a}g^q(b) \right] \\ & \leq \frac{1}{\left(\int_a^b w(x) \diamond_\alpha x \right)^2} \left(\int_a^b w(x)f^p(x) \diamond_\alpha x \right) \left(\int_a^b w(x)g^q(x) \diamond_\alpha x \right). \end{aligned} \quad (7)$$

From (2), we note that

$$\left[\frac{b-y_{w,\alpha}}{b-a}f^p(a) + \frac{y_{w,\alpha}-a}{b-a}f^p(b) \right]^{\frac{1}{p}} \geq \frac{b-y_{w,\alpha}}{b-a}f(a) + \frac{y_{w,\alpha}-a}{b-a}f(b) \quad (8)$$

and

$$\left[\frac{b-y_{w,\alpha}}{b-a}g^q(a) + \frac{y_{w,\alpha}-a}{b-a}g^q(b) \right]^{\frac{1}{q}} \geq \frac{b-y_{w,\alpha}}{b-a}g(a) + \frac{y_{w,\alpha}-a}{b-a}g(b). \quad (9)$$

Therefore

$$\begin{aligned} & \left[\frac{b-y_{w,\alpha}}{b-a}f^p(a) + \frac{y_{w,\alpha}-a}{b-a}f^p(b) \right] \left[\frac{b-y_{w,\alpha}}{b-a}g^q(a) + \frac{y_{w,\alpha}-a}{b-a}g^q(b) \right] \\ & \geq \left[\frac{b-y_{w,\alpha}}{b-a}f(a) + \frac{y_{w,\alpha}-a}{b-a}f(b) \right]^p \left[\frac{b-y_{w,\alpha}}{b-a}g(a) + \frac{y_{w,\alpha}-a}{b-a}g(b) \right]^q. \end{aligned} \quad (10)$$

From inequalities (7) and (10), we obtain the desired result. \square

Remark 1. Let $\mathbb{T} = \mathbb{R}$ and $w \equiv 1$. Then (4) reduces to

$$\frac{(f(a) + f(b))^p (g(a) + g(b))^q}{2^{(p+q)}} \leq \frac{1}{(b-a)^2} \left(\int_a^b f^p(x) dx \right) \left(\int_a^b g^q(x) dx \right). \quad (11)$$

The inequality (11) may be found in [11].

Our second result concerning generalized Rogers–Holder’s inequality on time scales is given.

Theorem 2. Let $\Phi : I_{\mathbb{T}} \rightarrow \mathbb{R}$ be a convex function for $I_{\mathbb{T}} = I \cap \mathbb{T}$, where I is an interval and \mathbb{T} is a time scale. Let $h_i, w, g \in C([a, b]_{\mathbb{T}}, (0, +\infty))$ ($i = 1, \dots, p$) be \diamond_α -integrable functions with $\sum_{i=1}^p h_i(x) = 1$ and $W = \int_a^b w(x) \diamond_\alpha x > 0$. Let L_1 and L_2 be two nonempty disjoint subsets such that $L_1 \cup L_2 = \{1, \dots, p\}$. Then

$$\begin{aligned} & \frac{\int_a^b w(x)\Phi(g(x)) \diamond_\alpha x}{W} \geq \frac{\int_a^b \sum_{i \in L_1} h_i(x)w(x) \diamond_\alpha x}{W} \Phi \left(\frac{\int_a^b \sum_{i \in L_1} h_i(x)w(x)g(x) \diamond_\alpha x}{\int_a^b \sum_{i \in L_1} h_i(x)w(x) \diamond_\alpha x} \right) \\ & + \frac{\int_a^b \sum_{i \in L_2} h_i(x)w(x) \diamond_\alpha x}{W} \Phi \left(\frac{\int_a^b \sum_{i \in L_2} h_i(x)w(x)g(x) \diamond_\alpha x}{\int_a^b \sum_{i \in L_2} h_i(x)w(x) \diamond_\alpha x} \right) \geq \Phi \left(\frac{\int_a^b w(x)g(x) \diamond_\alpha x}{W} \right). \end{aligned} \quad (12)$$

If the function Φ is concave, then the reverse inequalities hold in (12).

Proof. Since $\sum_{i=1}^n h_i(x) = 1$, we may write

$$\frac{\int_a^b w(x)\Phi(g(x)) \diamond_{\alpha} x}{W} = \frac{\int_a^b \sum_{i \in L_1} h_i(x)w(x)\Phi(g(x)) \diamond_{\alpha} x}{W} + \frac{\int_a^b \sum_{i \in L_2} h_i(x)w(x)\Phi(g(x)) \diamond_{\alpha} x}{W}. \quad (13)$$

Applying integral Jensen's inequality on both terms on the right-hand side of inequality (13), we obtain

$$\begin{aligned} \frac{\int_a^b w(x)\Phi(g(x)) \diamond_{\alpha} x}{W} &\geq \frac{\int_a^b \sum_{i \in L_1} h_i(x)w(x) \diamond_{\alpha} x \Phi\left(\frac{\int_a^b \sum_{i \in L_1} h_i(x)w(x)g(x) \diamond_{\alpha} x}{\int_a^b \sum_{i \in L_1} h_i(x)w(x) \diamond_{\alpha} x}\right)}{W} \\ &\quad + \frac{\int_a^b \sum_{i \in L_2} h_i(x)w(x) \diamond_{\alpha} x \Phi\left(\frac{\int_a^b \sum_{i \in L_2} h_i(x)w(x)g(x) \diamond_{\alpha} x}{\int_a^b \sum_{i \in L_2} h_i(x)w(x) \diamond_{\alpha} x}\right)}{W}. \end{aligned} \quad (14)$$

By using the convexity of Φ , we have

$$\begin{aligned} \frac{\int_a^b w(x)\Phi(g(x)) \diamond_{\alpha} x}{W} &\geq \Phi\left(\frac{\int_a^b \sum_{i \in L_1} h_i(x)w(x)g(x) \diamond_{\alpha} x}{W} + \frac{\int_a^b \sum_{i \in L_2} h_i(x)w(x)g(x) \diamond_{\alpha} x}{W}\right) \\ &= \Phi\left(\frac{\int_a^b w(x)g(x) \diamond_{\alpha} x}{W}\right). \end{aligned} \quad (15)$$

This completes the proof of Theorem 2. \square

Remark 2. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $h_1(k), h_2(k), w(k) \in (0, +\infty)$ and $g(k) \in [0, +\infty)$ for $k \in \{1, 2, \dots, n\}$. Then (12) reduces to

$$\begin{aligned} \frac{\sum_{k=1}^n w(k)\Phi(g(k))}{W} &\geq \frac{\sum_{k=1}^n \sum_{i \in L_1} h_i(k)w(k)}{W} \Phi\left(\frac{\sum_{k=1}^n \sum_{i \in L_1} h_i(k)w(k)g(k)}{\sum_{k=1}^n \sum_{i \in L_1} h_i(k)w(k)}\right) \\ &\quad + \frac{\sum_{k=1}^n \sum_{i \in L_2} h_i(k)w(k)}{W} \Phi\left(\frac{\sum_{k=1}^n \sum_{i \in L_2} h_i(k)w(k)g(k)}{\sum_{k=1}^n \sum_{i \in L_2} h_i(k)w(k)}\right) \geq \Phi\left(\frac{\sum_{k=1}^n w(k)g(k)}{W}\right). \end{aligned} \quad (16)$$

The continuous version of (16) may be found in [6].

Our third result concerning another generalized Rogers–Holder's inequality on time scales is investigated.

Theorem 3. Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. If $w, h_1, h_2, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ are such that $|w||f|^p$, $|w||g|^q$, $|wh_1||g|^q$, $|wh_2||g|^q$, $|wh_1fg|$, $|wh_2fg|$ and $|wfg|$ are \diamond_{α} -integrable

functions and $|h_1| + |h_2| = 1$, then

$$\begin{aligned} \left(\int_a^b |w(x)| |f(x)|^p \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x \right)^{\frac{1}{q}} &\geq \left(\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x \right)^{\frac{1}{q}} \\ &\quad \left[\left(\int_a^b |w(x)h_1(x)| |g(x)|^q \diamond_{\alpha} x \right)^{1-p} \left(\int_a^b |w(x)h_1(x)f(x)g(x)| \diamond_{\alpha} x \right)^p \right. \\ &\quad \left. + \left(\int_a^b |w(x)h_2(x)| |g(x)|^q \diamond_{\alpha} x \right)^{1-p} \left(\int_a^b |w(x)h_2(x)f(x)g(x)| \diamond_{\alpha} x \right)^p \right]^{\frac{1}{p}} \\ &\geq \int_a^b |w(x)| |f(x)g(x)| \diamond_{\alpha} x. \quad (17) \end{aligned}$$

In the case when $p \in (0, 1)$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x > 0$, we have

$$\begin{aligned} \int_a^b |w(x)| |f(x)g(x)| \diamond_{\alpha} x &\geq \left(\int_a^b |w(x)h_1(x)| |g(x)|^q \diamond_{\alpha} x \right)^{\frac{1}{q}} \left(\int_a^b |w(x)h_1(x)| |f(x)|^p \diamond_{\alpha} x \right)^{\frac{1}{p}} \\ &\quad + \left(\int_a^b |w(x)h_2(x)| |g(x)|^q \diamond_{\alpha} x \right)^{\frac{1}{q}} \left(\int_a^b |w(x)h_2(x)| |f(x)|^p \diamond_{\alpha} x \right)^{\frac{1}{p}} \\ &\geq \left(\int_a^b |w(x)| |f(x)|^p \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x \right)^{\frac{1}{q}}. \quad (18) \end{aligned}$$

Proof. Taking $n = 2$ and putting $\Phi(x) = x^p$, $x > 0$ and letting $|w(x)|$ and $|g(x)|$ be replaced by $|w(x)| |g(x)|^q$ and $|f(x)| |g(x)|^{-\frac{q}{p}}$, respectively, in the (3), we obtain

$$\begin{aligned} &\frac{\int_a^b |w(x)| |g(x)|^q \left(|f(x)| |g(x)|^{-\frac{q}{p}} \right)^p \diamond_{\alpha} x}{\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x} \\ &\geq \frac{\int_a^b |w(x)h_1(x)| |g(x)|^q \diamond_{\alpha} x}{\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x} \left(\frac{\int_a^b |w(x)h_1(x)| |g(x)|^q |f(x)| |g(x)|^{-\frac{q}{p}} \diamond_{\alpha} x}{\int_a^b |w(x)h_1(x)| |g(x)|^q \diamond_{\alpha} x} \right)^p \\ &\quad + \frac{\int_a^b |w(x)h_2(x)| |g(x)|^q \diamond_{\alpha} x}{\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x} \left(\frac{\int_a^b |w(x)h_2(x)| |g(x)|^q |f(x)| |g(x)|^{-\frac{q}{p}} \diamond_{\alpha} x}{\int_a^b |w(x)h_2(x)| |g(x)|^q \diamond_{\alpha} x} \right)^p \\ &\geq \left(\frac{\int_a^b |w(x)g(x)|^q |f(x)| |g(x)|^{-\frac{q}{p}} \diamond_{\alpha} x}{\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x} \right)^p. \quad (19) \end{aligned}$$

Therefore (17) follows from (19).

For $p \in (0, 1)$, take $P = \frac{1}{p} > 1$ and $Q = \frac{1}{1-p} > 1$ and letting $|f(x)|$ and $|g(x)|$ be replaced by $|f(x)g(x)|^p$ and $|g(x)|^{-p}$, respectively, in the inequality (17), we obtain

$$\begin{aligned} & \left(\int_a^b |w(x)| |f(x)g(x)|^{pP} \diamond_{\alpha} x \right)^{\frac{1}{P}} \left(\int_a^b |w(x)| |g(x)|^{-pQ} \diamond_{\alpha} x \right)^{\frac{1}{Q}} \geq \left(\int_a^b |w(x)| |g(x)|^{-pQ} \diamond_{\alpha} x \right)^{\frac{1}{Q}} \\ & \left[\left(\int_a^b |w(x)h_1(x)| |g(x)|^{-pQ} \diamond_{\alpha} x \right)^{1-P} \left(\int_a^b |w(x)h_1(x)| |f(x)g(x)|^p |g(x)|^{-p} \diamond_{\alpha} x \right)^P \right. \\ & \left. + \left(\int_a^b |w(x)h_2(x)| |g(x)|^{-pQ} \diamond_{\alpha} x \right)^{1-P} \left(\int_a^b |w(x)h_2(x)| |f(x)g(x)|^p |g(x)|^{-p} \diamond_{\alpha} x \right)^P \right]^{\frac{1}{P}} \\ & \geq \int_a^b |w(x)| |f(x)|^p |g(x)|^p |g(x)|^{-p} \diamond_{\alpha} x. \quad (20) \end{aligned}$$

Hence, we get the inequality (18). This completes the proof of Theorem 3. \square

Remark 3. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$ and $h_1(k), h_2(k), f(k), g(k) \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$. Then (17) reduces to

$$\begin{aligned} & \left(\sum_{k=1}^n f^p(k) \right)^{\frac{1}{p}} \left(\sum_{k=1}^n g^q(k) \right)^{\frac{1}{q}} \geq \left(\sum_{k=1}^n g^q(k) \right)^{\frac{1}{q}} \left[\left(\sum_{k=1}^n h_1(k) g^q(k) \right)^{1-p} \left(\sum_{k=1}^n h_1(k) f(k) g(k) \right)^p \right. \\ & \left. + \left(\sum_{k=1}^n h_2(k) g^q(k) \right)^{1-p} \left(\sum_{k=1}^n h_2(k) f(k) g(k) \right)^p \right]^{\frac{1}{p}} \geq \sum_{k=1}^n f(k) g(k) \quad (21) \end{aligned}$$

and (18) reduces to

$$\begin{aligned} \sum_{k=1}^n f(k) g(k) & \geq \left(\sum_{k=1}^n h_1(k) g^q(k) \right)^{\frac{1}{q}} \left(\sum_{k=1}^n h_1(k) f^p(k) \right)^{\frac{1}{p}} \\ & + \left(\sum_{k=1}^n h_2(k) g^q(k) \right)^{\frac{1}{q}} \left(\sum_{k=1}^n h_2(k) f^p(k) \right)^{\frac{1}{p}} \geq \left(\sum_{k=1}^n f^p(k) \right)^{\frac{1}{p}} \left(\sum_{k=1}^n g^q(k) \right)^{\frac{1}{q}}. \quad (22) \end{aligned}$$

The continuous versions of inequalities (21) and (22) may be found in [6].

Theorem 4. Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. If $w, h_1, h_2, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ are such that $|w||f|^p$, $|w||g|^q$, $|wh_1||g|^q$, $|wh_2||g|^q$ and $|wfg|$ are \diamond_{α} -integrable functions with $\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x > 0$ and $|h_1| + |h_2| = 1$, then

$$\begin{aligned} \int_a^b |w(x)| |f(x)g(x)| \diamond_{\alpha} x & \leq \left(\int_a^b |w(x)h_1(x)| |f(x)|^p \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b |w(x)h_1(x)| |g(x)|^q \diamond_{\alpha} x \right)^{\frac{1}{q}} \\ & + \left(\int_a^b |w(x)h_2(x)| |f(x)|^p \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b |w(x)h_2(x)| |g(x)|^q \diamond_{\alpha} x \right)^{\frac{1}{q}} \\ & \leq \left(\int_a^b |w(x)| |f(x)|^p \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b |w(x)| |g(x)|^q \diamond_{\alpha} x \right)^{\frac{1}{q}}. \quad (23) \end{aligned}$$

In the case when $p \in (0, 1)$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $\int_a^b |w(x)| |g(x)|^q \diamond_\alpha x > 0$, we have

$$\begin{aligned} & \left(\int_a^b |w(x)| |f(x)|^p \diamond_\alpha x \right)^{\frac{1}{p}} \left(\int_a^b |w(x)| |g(x)|^q \diamond_\alpha x \right)^{\frac{1}{q}} \leq \left(\int_a^b |w(x)| |g(x)|^q \diamond_\alpha x \right)^{\frac{1}{q}} \\ & \quad \left[\left(\int_a^b |w(x) h_1(x)| |g(x)|^q \diamond_\alpha x \right)^{1-p} \left(\int_a^b |w(x) h_1(x) f(x) g(x)| \diamond_\alpha x \right)^p \right. \\ & \quad \left. + \left(\int_a^b |w(x) h_2(x)| |g(x)|^q \diamond_\alpha x \right)^{1-p} \left(\int_a^b |w(x) h_2(x) f(x) g(x)| \diamond_\alpha x \right)^p \right]^{\frac{1}{p}} \\ & \leq \int_a^b |w(x)| |f(x) g(x)| \diamond_\alpha x. \quad (24) \end{aligned}$$

Proof. Putting $\Phi(x) = \frac{1}{x^p}$, $x > 0$. Then clearly the function Φ is concave and letting $|w(x)|$ and $|g(x)|$ be replaced by $|w(x)| |g(x)|^q$ and $|f(x)|^p |g(x)|^{-q}$, respectively, in (3) for $n = 2$, we obtain

$$\begin{aligned} & \frac{\int_a^b |w(x)| |g(x)|^q (|f(x)|^p |g(x)|^{-q})^{\frac{1}{p}} \diamond_\alpha x}{\int_a^b |w(x)| |g(x)|^q \diamond_\alpha x} \\ & \leq \frac{\int_a^b |w(x) h_1(x)| |g(x)|^q \diamond_\alpha x}{\int_a^b |w(x)| |g(x)|^q \diamond_\alpha x} \left(\frac{\int_a^b |w(x) h_1(x)| |g(x)|^q |f(x)|^p |g(x)|^{-q} \diamond_\alpha x}{\int_a^b |w(x) h_1(x)| |g(x)|^q \diamond_\alpha x} \right)^{\frac{1}{p}} \\ & \quad + \frac{\int_a^b |w(x) h_2(x)| |g(x)|^q \diamond_\alpha x}{\int_a^b |w(x)| |g(x)|^q \diamond_\alpha x} \left(\frac{\int_a^b |w(x) h_2(x)| |g(x)|^q |f(x)|^p |g(x)|^{-q} \diamond_\alpha x}{\int_a^b |w(x) h_2(x)| |g(x)|^q \diamond_\alpha x} \right)^{\frac{1}{p}} \\ & \leq \left(\frac{\int_a^b |w(x) g(x)|^q |f(x)|^p |g(x)|^{-q} \diamond_\alpha x}{\int_a^b |w(x)| |g(x)|^q \diamond_\alpha x} \right)^{\frac{1}{p}}. \quad (25) \end{aligned}$$

Therefore (23) follows from (25).

For $p \in (0, 1)$, take $P = \frac{1}{p} > 1$ and $Q = \frac{1}{1-p} > 1$ and letting $|f(x)|$ and $|g(x)|$ be replaced by $|f(x) g(x)|^p$ and $|g(x)|^{-p}$, respectively, in the inequality (23), we obtain

$$\begin{aligned} & \int_a^b |w(x)| |f(x)|^p |g(x)|^p |g(x)|^{-p} \diamond_\alpha x \\ & \leq \left(\int_a^b |w(x) h_1(x)| |f(x) g(x)|^{pP} \diamond_\alpha x \right)^{\frac{1}{P}} \left(\int_a^b |w(x) h_1(x)| |g(x)|^{-pQ} \diamond_\alpha x \right)^{\frac{1}{Q}} \\ & \quad + \left(\int_a^b |w(x) h_2(x)| |f(x) g(x)|^{pP} \diamond_\alpha x \right)^{\frac{1}{P}} \left(\int_a^b |w(x) h_2(x)| |g(x)|^{-pQ} \diamond_\alpha x \right)^{\frac{1}{Q}} \\ & \leq \left(\int_a^b |w(x)| |f(x) g(x)|^{pP} \diamond_\alpha x \right)^{\frac{1}{P}} \left(\int_a^b |w(x)| |g(x)|^{-pQ} \diamond_\alpha x \right)^{\frac{1}{Q}}. \quad (26) \end{aligned}$$

Hence, we get the inequality (24). This completes the proof of Theorem 4. \square

Remark 4. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$ and $h_1(k), h_2(k), f(k), g(k) \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$. Then (23) reduces to

$$\begin{aligned} \sum_{k=1}^n f(k)g(k) &\leq \left(\sum_{k=1}^n h_1(k)f^p(k) \right)^{\frac{1}{p}} \left(\sum_{k=1}^n h_1(k)g^q(k) \right)^{\frac{1}{q}} \\ &+ \left(\sum_{k=1}^n h_2(k)f^p(k) \right)^{\frac{1}{p}} \left(\sum_{k=1}^n h_2(k)g^q(k) \right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^n f^p(k) \right)^{\frac{1}{p}} \left(\sum_{k=1}^n g^q(k) \right)^{\frac{1}{q}} \end{aligned} \quad (27)$$

and (24) reduces to

$$\begin{aligned} \left(\sum_{k=1}^n f^p(k) \right)^{\frac{1}{p}} \left(\sum_{k=1}^n g^q(k) \right)^{\frac{1}{q}} &\leq \left(\sum_{k=1}^n g^q(k) \right)^{\frac{1}{q}} \left[\left(\sum_{k=1}^n h_1(k)f(k)g(k) \right)^p \left(\sum_{k=1}^n h_1(k)g^q(k) \right)^{1-p} \right. \\ &\quad \left. + \left(\sum_{k=1}^n h_2(k)f(k)g(k) \right)^p \left(\sum_{k=1}^n h_2(k)g^q(k) \right)^{1-p} \right]^{\frac{1}{p}} \leq \sum_{k=1}^n f(k)g(k). \end{aligned} \quad (28)$$

The continuous versions of inequalities (27) and (28) may be found in [6].

Theorem 5. Let $w, h_1, h_2, g \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be \diamond_{α} -integrable functions with $|h_1| + |h_2| = 1$. Let $p, q \in \mathbb{R} - \{0\}$ such that $p \leq q$ and $M_r(u, v) = \left(\frac{\int_a^b |u(x)||v(x)|^r \diamond_{\alpha} x}{\int_a^b |u(x)| \diamond_{\alpha} x} \right)^{\frac{1}{r}}$. Then

$$(1) \quad M_q(|w|, |g|) \geq [M_1(|h_1|, |w|)M_p^q(|h_1| \cdot |w|, |g|) + M_1(|h_2|, |w|)M_p^q(|h_2| \cdot |w|, |g|)]^{\frac{1}{q}} \geq M_p(|w|, |g|). \quad (29)$$

$$(2) \quad M_p(|w|, |g|) \leq [M_1(|h_1|, |w|)M_q^p(|h_1| \cdot |w|, |g|) + M_1(|h_2|, |w|)M_q^p(|h_2| \cdot |w|, |g|)]^{\frac{1}{p}} \leq M_q(|w|, |g|). \quad (30)$$

Proof. Putting $\Phi(x) = x^{\frac{q}{p}}$, $x > 0$. Then clearly the function Φ is convex and letting $|g(x)|$ be replaced by $|g(x)|^p$ and taking power $\frac{1}{q}$, the inequality (12) becomes

$$\begin{aligned} \frac{\int_a^b |w(x)| (|g(x)|^p)^{\frac{q}{p}} \diamond_{\alpha} x}{\int_a^b |w(x)| \diamond_{\alpha} x} &\geq \frac{\int_a^b |w(x)h_1(x)| \diamond_{\alpha} x}{\int_a^b |w(x)| \diamond_{\alpha} x} \left(\frac{\int_a^b |w(x)h_1(x)||g(x)|^p \diamond_{\alpha} x}{\int_a^b |w(x)h_1(x)| \diamond_{\alpha} x} \right)^{\frac{q}{p}} \\ &+ \frac{\int_a^b |w(x)h_2(x)| \diamond_{\alpha} x}{\int_a^b |w(x)| \diamond_{\alpha} x} \left(\frac{\int_a^b |w(x)h_2(x)||g(x)|^p \diamond_{\alpha} x}{\int_a^b |w(x)h_2(x)| \diamond_{\alpha} x} \right)^{\frac{q}{p}} \geq \left(\frac{\int_a^b |w(x)g(x)|^p \diamond_{\alpha} x}{\int_a^b |w(x)| \diamond_{\alpha} x} \right)^{\frac{q}{p}}. \end{aligned} \quad (31)$$

Therefore (29) follows from (31).

Putting $\Phi(x) = x^{\frac{p}{q}}$, $x > 0$. Then clearly the function Φ is concave and letting $|g(x)|$ be replaced by $|g(x)|^q$ and taking power $\frac{1}{p}$, the inequality (12) becomes

$$\begin{aligned} \frac{\int_a^b |w(x)| (|g(x)|^q)^{\frac{p}{q}} \diamond_{\alpha} x}{\int_a^b |w(x)| \diamond_{\alpha} x} &\leq \frac{\int_a^b |w(x)h_1(x)| \diamond_{\alpha} x}{\int_a^b |w(x)| \diamond_{\alpha} x} \left(\frac{\int_a^b |w(x)h_1(x)||g(x)|^q \diamond_{\alpha} x}{\int_a^b |w(x)h_1(x)| \diamond_{\alpha} x} \right)^{\frac{p}{q}} \\ &+ \frac{\int_a^b |w(x)h_2(x)| \diamond_{\alpha} x}{\int_a^b |w(x)| \diamond_{\alpha} x} \left(\frac{\int_a^b |w(x)h_2(x)||g(x)|^q \diamond_{\alpha} x}{\int_a^b |w(x)h_2(x)| \diamond_{\alpha} x} \right)^{\frac{p}{q}} \leq \left(\frac{\int_a^b |w(x)g(x)|^q \diamond_{\alpha} x}{\int_a^b |w(x)| \diamond_{\alpha} x} \right)^{\frac{p}{q}}. \end{aligned} \quad (32)$$

Therefore (30) follows from (32). This completes the proof of Theorem 5. \square

Remark 5. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$ and $h_1(k), h_2(k), g(k) \in (0, +\infty)$ for $k \in \{1, 2, \dots, n\}$ with $M_r(u_k, v_k) = \left(\frac{\sum_{k=1}^n w(k)g^r(k)}{\sum_{k=1}^n w(k)} \right)^{\frac{1}{r}}$. Then inequalities (29) and (30), respectively, reduce to

$$(1) \quad M_q(|w|, |g|) \geq [M_1(|h_1|, |w|)M_p^q(|h_1| \cdot |w|, |g|) + M_1(|h_2|, |w|)M_p^q(|h_2| \cdot |w|, |g|)]^{\frac{1}{q}} \geq M_p(|w|, |g|). \quad (33)$$

$$(2) \quad M_p(|w|, |g|) \leq [M_1(|h_1|, |w|)M_q^p(|h_1| \cdot |w|, |g|) + M_1(|h_2|, |w|)M_q^p(|h_2| \cdot |w|, |g|)]^{\frac{1}{p}} \leq M_q(|w|, |g|). \quad (34)$$

The continuous versions of inequalities (33) and (34) may be found in [6].

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Научная статья

Сборка классических и динамических неравенств, накопленных при исчислении временных масштабов

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Цель данной статьи — представить согласование некоторых классических и динамических неравенств с использованием исчисления шкал времени в более общей, унифицированной и расширенной форме. Здесь мы исследуем гармоничные расширения и обобщения неравенств типа Эрмита–Адамара и Роджерса–Гёльдера в гибридных версиях. Исчисление шкал времени сочетает в себе непрерывные, дискретные и квантовые аналоги.

Ключевые слова: шкалы времени, неравенство типа Эрмита–Адамара, неравенство типа Роджерса–Холдера.

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