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On controllability and the pursuit problem for linear discrete systems

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In this paper, it is considered linear discrete control and pursuit game problems. Control vectors are subjected to total constraints those are a discrete analogue of the integral constraints. Necessary and sufficient conditions of solvability of the 0-controllability problem are obtained. The connection between 0-controllability and solvability of the pursuit problem is studied.

Keywords: pursuit, evader, control, 0-controllability, game, strategy, integral constraint, discrete constraint

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Introduction

In the paper controllability and pursuit problems for linear systems with discrete time are studied. The item for the case of dynamic systems with continuous time was considered in [1]-[7]. Pursuit-evasion games with admissible controls of the players satisfying various constraints were also studied in many papers. For example, in [10] sufficient conditions of the pursuit termination in linear differential and discrete games with integral constraints on controls for both players are obtained. In [9] linear discrete games with many pursuers are studied in the case when the controls of the pursuer should satisfy the integral constraint. In [9] the following two cases are considered for admissible controls of the pursuers: (a) controls satisfy the integral constraint, and (b) controls satisfy a geometrical constraint. (In both cases, the origin must be an interior point of the control region of the pursuer.) The basis result of [9] is sufficient conditions of termination of pursuit for arbitrary initial point. In the case of one pursuer, these conditions turn out to be necessary ones. In [8] structures of the reachability and

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controllability regions for linear discrete controlled systems with two types of control constraints are revealed in the case when the control space is one-dimensional and satisfies the Kalman controllability condition [4].

The aim of this paper is to identify the connection between the controllability of linear discrete systems and solvability the pursuit problem in linear discrete game, when control vectors of the players are subjected to sum constraints.

Statement of the Problem

Here we consider the control system described by the equation

$$z(t+1) = Az(t) + Bu(t) \tag{1}$$

and the discrete pursuit game described by the equation

$$z(t+1) = Az(t) + Bu(t) + Cv(t) \tag{2}$$

where $z \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^l$, $t \in \mathbb{N}$, $\mathbb{N}$ is a set of nonnegative integers, A, B, C are constant matrices of corresponding dimensions, u is the control parameter of the pursuer, v is that of the evader.

Definition 1. The sequences $u(\cdot) : \mathbb{N} \to \mathbb{R}^m$ and $v(\cdot) : \mathbb{N} \to \mathbb{R}^l$ subjected to the following total constraints

$$\|u(\cdot)\|_{l_p} = \left(\sum_{t=0}^{\infty} |u(t)|^p \right)^{1/p} \leq \rho,\ \rho > 0, \tag{3}$$

$$\|v(\cdot)\|_{l_p} = \left(\sum_{t=0}^{\infty} |v(t)|^p \right)^{1/p} \leq \sigma,\ \sigma \geq 0, \tag{4}$$

are referred to as admissible controls of the pursuer and the evader, respectively ($p > 1$).

The conditions (3) and (4) serve as discrete analogues of integral constraints, and will be called total constraints. If $\sigma = 0$, then clearly, the evader is absent, and (2) turns into (1). In this case the dynamical game becomes simply a control system with total constraints on control.

The sequence $u(t) = U(z(t),\ v(t))$, $t \in \mathbb{N}$, is referred to as the realization of the strategy U according to z_0 and $v(\cdot)$. The strategy U is called an admissible strategy of the pursuer if all its realizations with fixed z_0 satisfy the condition (3).

The strategy $V(t, z)$ of the evader, which satisfies the condition (4), can be defined analogously.

Definition 2. The function $U : \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m$ is called a strategy of the pursuer is for every initial point z_0, and control of the evader $v(\cdot)$ it is generate a unique trajectory $z(\cdot)$, such that

$$z(t+1) = Az(t) + BU\,(t,\ z(t), v(t)) + Cv(t),\ \ z(0) = z_0.$$

Definition 3. Pursuit is said to be completed in the game (2)-(4) from the initial position z_0 if there exists a strategy of the pursuer U such that for any control of the evader $v(\cdot)$ the trajectory $z(\cdot)$ generated by $(z_0, U, v(\cdot))$, satisfies the condition $z(t) = 0$ at some t, $t \in \mathbb{N}$.

The aim of the pursuer is to realize the equality $z(t) = 0$ as earlier as possible. Thus we deal with pursuit problem. The pursuer uses a strategy, and the evader uses any control $v(\cdot)$. By Definition 2 at any time t the value of the strategy of the pursuer is constructed based on the state $z(t)$ and value of the control parameter of the evader $v(t)$.

Definition 4. The system (1) is called 0-controllable if for any point z_0 there exists a control $u(\cdot) : \mathbb{N} \to \mathbb{R}^m$ such that the trajectory $z(\cdot)$ of the system (2) with the initial position $z(0) = z_0$ satisfies the equality $z(t) = 0$ at some t, $t \in \mathbb{N}$.

Definition 5. The evasion is possible in the game (2)-(4) from all initial positions z_0 if there exists a strategy of the evader V such that for any control of the pursuer u the trajectory $z(\cdot)$ generated by (z_0, u, V), satisfies the condition $z(t) \neq 0$ for all t, $t \in \mathbb{N}$.

Main Results

Let $S_h^d = \{x \in \mathbb{R}^d \mid |x| \leq h\}$ denotes the ball with radius h in the space $\mathbb{R}^d$.

Theorem 1. *If there is a number $\mu > 1$ such that*

$$\mu \sigma C S_l^1 \subset \rho int B S_1^m \tag{5}$$

then the following statements are equivalent:

(a) the system (1) is 0-controllable,

(b) pursuit can be completed in the game (2) from any initial point of the space $\mathbb{R}^n$.

Proof. Let pursuit can be completed in the game (2) from any initial point of the space $\mathbb{R}^n$. Then there exists a strategy of the pursuer such that for any strategy of the evader, in particular, if $v(k) = 0$ for all $k \geq 0$, pursuit can be completed in the game (2) from any initial point of the space $\mathbb{R}^n$. This means the system (1) is 0-controllable. Hence, (b) implies (a).

We now show that the statement (a) implies (b). Let the hypothesis of the theorem hold and the system (1) be 0-controllable. We shall have established the theorem if we prove that there exists an admissible strategy $U = U(t, z, v)$ of the pursuer such that $z(t) = 0$ at some $t \geq 0$ for the trajectory $z(\cdot)$ generated by the strategy $U = U(t, z, v)$, the control $v(\cdot)$, and initial position z_0.

We construct the strategy of the pursuer as follows. For each vector $v \in \mathbb{R}^l$, we define the vector $U_1(v) \in \mathbb{R}^m$ by the equation $BU_1 = -Cv$ requiring that

$$|U_1(v)| \leq \frac{\rho}{\mu \sigma} |v|. \tag{6}$$

According to the inclusion (5), such choice of $U_1(v)$ is possible. Indeed, from (5) we obtain that $C\mathbb{R}^l \subset B\mathbb{R}^m$. Again from (5) we obtain that for any $v \in \mathbb{R}^l$ there exists $u_v \in S_1^m$ such that

$$\mu \sigma C \frac{v}{|v|} = \rho B u_v.$$

This implies that

$$B \frac{\rho |v|}{\mu \sigma} u_v = Cv.$$

Then

$$U_1(v) = \frac{\rho|v|}{\mu\sigma}u_v$$

is the desired vector.

Since $\mu > 1$, by Theorem 1 [10], for any $z_0 \in \mathbb{R}^n$ there exists a control $\overline{u}(z_0, \cdot)$ such that

$$\sum_{t=0}^{\infty} |\overline{u}(z_0,t)|^p \leq \rho^p \left(1-\tfrac{1}{\mu}\right)^p \tag{7}$$

and the solution of the initial value problem

$$z(t+1) = Az(t) + B\overline{u}(z_0,t), \quad z(0) = z_0,$$

turns to zero at some step $t = t_*$. We now define for the initial position z_0, the strategy U as follows

$$U(t,z,v) = \begin{cases} \overline{u}(z_0,t) + U_1(v), & z \neq 0, \\ 0, & z = 0. \end{cases} \tag{8}$$

Then combining (4), (6) and (7), we have

$$\|U(t,z,v)\|_{l_p} \leq \|\overline{u}(z_0,t)\|_{l_p} + \|U_1(v(t))\|_{l_p} \leq$$

$$\leq \rho\left(1-\frac{1}{\mu}\right) + \frac{\rho}{\mu\sigma}\|v(\cdot)\|_{l_p} \leq \rho.$$

Hence, the strategy is admissible. Moreover, for the trajectory $z(\cdot)$ of the system (2) generated by the triple $(z_0, U, v(\cdot))$, we have

$$z(t+1) = Az(t) + BU(t,v(t)) + Cv(t) =$$

$$= Az(t) + B(\overline{u}(z_0,t) + U_1) + Cv(t) =$$

$$= Az(t) + B\overline{u}(z_0,t) + (BU_1 + Cv(t)).$$

Then by the choice of U_1 we obtain that

$$z(t+1) = Az(t) + B\overline{u}(z_0,t), \qquad z(0) = z_0. \tag{9}$$

As mentioned above that for the solution $z(\cdot)$ of the system (9), $z(t_*) = 0$ at some step t_*. Thus, pursuit can be completed in the game (2) from any initial position. This completes the proof of the theorem. □

Next theorem shows that for solvability of pursuit problem it is necessary the inclusion $C\mathbb{R}^l \subset B\mathbb{R}^m$ (compare with (5)).

Theorem 2. *If $C\mathbb{R}^l \not\subset B\mathbb{R}^m$, then in game (2) from all initial positions z_0, $z_0 \neq 0$ the evasion is possible.*

Proof. Let the pursuit apply an arbitrary control $u(t)$. From a condition $C\mathbb{R}^l \subset B\mathbb{R}^m$ there is a unit vector v_0, $v_0 \in \mathbb{R}^l$, such that $Cv_0 \notin B\mathbb{R}^m$. We'll write evader applying the following prestrategy:

$$v(t,z) = 0 \quad \textbf{if} \quad Az \notin B\mathbb{R}^m$$

and
$$v(t,z)=\frac{\sigma v_0}{2^{t+1}} \quad \textbf{if} \quad Az\in B\mathbb{R}^m.$$

Then
$$Bu(t)\in B\mathbb{R}^m,$$
$$Az(t)+Cv(t,z(t))\notin B\mathbb{R}^m,$$

thus
$$z(t+1)=Az(t)+Bu(t)+Cv(t,z(t))\notin B\mathbb{R}^m,$$

i.e., $z(t+1)\neq 0$ for all $t=0,1,2,...$, if only $z(0)\neq 0$.

Now we will show that all realization of prestrategy V (at fixed z_0) holds a condition (4), i.e. V is strategy evader with construction
$$\|v(\cdot)\|_{l_p}=\left(\sum_{t=0}^{\infty}|v(t)|^p\right)^{1/p}\leq\left(\sum_{t=0}^{\infty}\left|\frac{\sigma v_0}{2^{t+1}}\right|^p\right)^{1/p}\leq\sigma\left(\sum_{t=1}^{\infty}2^{-t}\right)^{1/p}=\sigma,$$

this completes the proof. □

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УДК 517.977.5+517.978.4 Научная статья

Об одной задаче управляемости и преследования в линейных дискретных системах

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В статье рассмотрена линейные дискретные игровые задачи управления и преследования. На векторы управления накладываются полные ограничения, которые являются дискретным аналогом интегральных ограничений. Получены необходимые и достаточные условия разрешимости проблемы 0-управляемости. Изучается связь между 0-управляемостью и разрешимостью задачи преследования.

Ключевые слова: преследования, убегания, управления, 0-управляемость, игры, стратегия, интегральное ограничение, дискретное ограничение.

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