MSC 32A07, 43A10, 32A07

Research Article

On new sharp theorems for multifunctional BMOA type spaces in bounded pseudoconvex domains

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We provide new equivalent expressions in the unit ball and pseudoconvex domains for multifunctional analytic BMOA type space. We extend in various directions a known theorem of atomic decomposition of BMOA type spaces in the unit ball.

Keywords: unit ball, analytic functions, analytic spaces, pseudoconvex domain, Hardy spaces, Bergman spaces, BMOA type spaces

DOI: 10.26117/2079-6641-2020-32-3-102-113

Original article submitted: 30.04.2020

Revision submitted: 08.08.2020

For citation. Shamoyan R. F., Tomashevskaya E. B. On new sharp theorems for multifunctional BMOA type spaces in bounded pseudoconvex domains. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2020, **32**: 3, 102-113. DOI: 10.26117/2079-6641-2020-32-3-102-113

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1. Introduction

The well-known theorem on atomic decomposition of BMOA type in the unit ball (see [17]) says if $f \in \text{BMOA}$ then f can be decomposed into BMOA type atoms (see [17]). Also various similar type theorems from [17] say that each analytic function from Bergman or Hardy or Bloch spaces can be also decomposed into elementary functions (atoms) and similar type atomic decomposition theorems to mentioned for BMOA type spaces are valid also for these analytic classes in the unit ball. Some similar type theorems are valid also in bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary (see, for example, [3], [5], [8] and various references there). Let B be unit ball and let $H(B)$ be the space of all analytic functions in B .

The problem we consider is well-known for functional spaces in $\mathbb{R}^n$ (the problem of equivalent norms) (see, for example [6], [13], [17]). Let $X, (X_j)$ be quasinormed analytic function space in a fixed product domain and (or) in $\mathbb{C}^n$ (normed or quasinormed) we wish to find equivalent expression for $\|f_1 \dots f_m\|_X; m \in \mathbb{N}$ (Note these are closely

Funding. The study was carried out at the expense of the authors.

connected with spaces on product domains since often if $f(z_1 \dots z_m) = \prod_{j=1}^m f_j(z_j)$, then $\|f\|_X = \prod_{j=1}^m \|f_j\|_{X_j}$. These results also as we'll see extend some well-known assertions on atomic decomposition of A_α^p type spaces.

To study such group of functions it is natural, for example, to ask about structure of each $\{f_j\}_{j=1}^m$ of this group.

We define multifunctional analytic BMOA type in the ball

$$\widetilde{BMOA}_{\alpha, \beta, \tau}^p = \left\{ (f_1, \dots, f_m), f_j \in H(B), j = 1, \dots, m, m \in \mathbb{N}, \right. \\ \left. \left(\sup_{w \in B} \left(\int_B \frac{|\tilde{f}(z)|^p (1 - |z|)^\alpha dV(z)}{|1 - \bar{w}z|^\beta} \right) \times (1 - |w|)^\tau < \infty \right) \right\},$$

where $0 < p < \infty, \alpha > -1, \beta \geq 0, \tau \geq 0$, and $\tilde{f} = \prod_{j=1}^m f_j$ and where dV is a normalized Lebesgue measures on B . We denote in this paper various positive constants by c, C_1, C_β .

And the question is can we decompose each $(f_j), j = 1, \dots, m$ from $\widetilde{BMOA}_{\alpha, \beta, \tau}^p$ into atoms. We will show that under certain integral condition (which vanishes for $m = 1$). We can decompose each (f_j) into various atoms "H^p atoms", "Bergman atoms" and also BMOA atoms. But one member from this group should be cut into BMOA atom.

This extend a known result on atomic decomposition of BMOA space in the unit ball in various directions (see [17]). Such type theorems have various applications. We refer to [7], [2], [4], [3], [6] for various such type theorems in various domains for various spaces of analytic functions.

Moreover we extend our result then even to bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$.

We note that both our theorems have parallel, very similar proofs, we prefer to concentrate on this ball case then add various remarks on how to change the proof to more general case of bounded pseudoconvex domains with smooth boundary in $\mathbb{C}^n$.

2. Main results

In this section we formulate and prove our main results. We say integral condition (A) is valid if

$$f_1(w_1) \dots f_m(w_n) = c_\beta \int_B \frac{\prod_{j=1}^m f_j(z) (1 - |z|)^{\tilde{\beta}} dV(z)}{\prod_{j=1}^m (1 - w_j \bar{z})^{\frac{\tilde{\beta} + n + 1}{m}}}, \quad (A)$$

where c_β is a Bergman representation constant and where $w_j \in B$ and $\tilde{\beta} > \beta_0$, β_0 is large enough positive number, $f_j \in H(B), j = 1, \dots, m$.

Note it is well known that for $m = 1$ this integral condition vanishes (see [4]).

Theorem 1. Assume (A) is valid, $f_j \in H(B)$. Let $p \leq 1, \beta \geq 0, \tau \geq 0, \alpha > -1, j = 1, \dots, m, m \in \mathbb{N}$. There is $\alpha, \alpha = \alpha(p, \tilde{\alpha}, \tilde{\beta}, \tilde{\tau}, \tilde{\alpha}_j)$. So that each $f_j, f_j \in \widetilde{BMOA}_{\alpha, \beta, \tau}^p, j = 1, \dots, m$, can be decomposed into or " H^p atoms" or "Bergman $A_{\alpha_j}^p$ atoms" for some α_j or $BMOA_{\tilde{\alpha}, \tilde{\beta}, \tilde{\tau}}^p$ atoms (for some parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\tau}$) in the unit ball. And always one member of (f_j) group must be decomposed if (A) is valid into $BMOA_{\tilde{\alpha}, \tilde{\beta}, \tilde{\tau}}^p$ atoms in the unit ball in $\mathbb{C}^n$.

And moreover we have

$$\begin{aligned} \left(\sup_{z \in B} \right) \int_B \frac{\prod_{j=1}^m |f_j(w)|^p (1 - |w|)^\alpha dV(w)}{|1 - \bar{w}z|^\beta} \cdot (1 - |z|)^\tau \asymp \\ \asymp \prod_{i=1}^k \|f_i\|_{BMOA_{\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\tau}_i}^p}^p \cdot \prod_{i=k+1}^s \|f_i\|_{H^p}^p \cdot \prod_{i=s+1}^m \|f_i\|_{A_{\alpha_i}^p}^p. \end{aligned}$$

Remark 1.

We refer to [17] for atoms of various spaces in the unit ball.

Remark 2.

Note if $m = 1$ (A) vanishes by known results on integral representation theorem on we obtain a known result on atomic decomposition of BMOA type spaces (see [17]).

Remark 3.

Very similar results are valid for Bergman and Hardy spaces (see [13]-[16]).

Proof of theorem 1

Let us remark our theorem is a corollary of two new estimates. The first one is easy and follows from known uniform estimates (see [7], [2], [4], [3], [8], [6], [10], [1]) of Hardy, Bergman analytic function spaces.

For example we for $\tilde{\alpha} = \tilde{\alpha}(\vec{p}, \alpha_{k-1}, \dots, \alpha_{m-1}, \alpha, \beta, \tau, n)$ have obviously

$$\begin{aligned} \left(\sup_{w \in B} \right) \int_B \frac{\prod_{i=1}^m |f_i|^{p_i} (1 - |z|)^{\tilde{\alpha}} dV(z)}{|1 - \bar{w}z|^\beta} \times (1 - |w|)^\tau \leq \\ \leq c \left(\sup_{z \in B} \right) \left(\int_B \frac{|f_m(z)|^{p_m} (1 - |z|)^\alpha dV(z)}{|1 - \bar{w}z|^\beta} \times (1 - |w|)^\tau \right) \times \\ \times \left(\sup_z \right) |f_1(z)|^{p_1} (1 - |z|)^n \dots \left(\sup_z \right) |f_k(z)|^{p_k} (1 - |z|)^n \times \\ \times \left(\sup_z \right) |f_{k+1}(z)|^{p_{k+1}} (1 - |z|)^{(\alpha_{k+1} + n + 1)} \dots \left(\sup_z \right) |f_{m-1}(z)|^{p_{m-1}} (1 - |z|)^{(\alpha_{m-1} + n + 1)} \leq \\ \leq c_1 \|f_1\|_{H^{p_1}}^{p_1} \dots \|f_k\|_{H^{p_k}}^{p_k} \cdot \|f_{k+1}\|_{A_{\alpha_{k+1}}^{p_{k+1}}}^{p_{k+1}} \dots \|f_{m-1}\|_{A_{\alpha_{m-1}}^{p_{m-1}}}^{p_{m-1}} \cdot \|f_m\|_{BMOA_{\alpha, \beta, \tau}^{p_m}}, \end{aligned} \quad (B)$$

where $0 < p_i < \infty, i = 1, \dots, m$, based on well-known uniform estimates for H^p, A_α^p spaces in the unit ball in $\mathbb{C}^n$. The uniform estimate for $BMOA_{\alpha, \beta, \tau}^p$ is also known so we can add to this group also

$$\prod_{j=1}^m \|f_j\|_{BMOA_{\alpha_j, \beta_j, \tau_j}^{p_j}}^{p_j}.$$

for some values of parameters $\alpha_j, \beta_i, \tau_j$.

Namely based on known estimate from below of Bergman kernel on Bergman ball (see [17]) we have the following uniform estimate for $BMOA_{\alpha, \beta, \tau}^p$ spaces

$$\begin{aligned} \left(\sup_{w \in B} \right) |f(w)|^p (1 - |w|)^s &\leq \\ &\leq \tilde{c} \left(\sup_{w \in B} \right) \int_B \frac{|f(v)|^p (1 - |v|)^\alpha dV(v)}{|1 - \bar{w}v|^\beta} \times (1 - |w|)^\tau = \\ &= \|f\|_{BMOA_{\alpha, \beta, \tau}^p}^p, \quad (C) \end{aligned}$$

where $s = \alpha - \beta + \tau + n + 1, \tau > 0, \beta > 0, \alpha > -1$. This result with same proof is valid also in pseudoconvex domains. For the case when $p_i = p, i = 1, \dots, m$, we under integral condition (A) wish to show the reverse to this (B).

Namely that the following estimate is true

$$\begin{aligned} \prod_{j=1}^m \|f_j\|_{H^p}^p \cdot \prod_{i=m+1}^{\tilde{m}} \|f_i\|_{A_{\alpha_i}^p} \cdots \prod_{i=\tilde{m}}^{\tilde{m}} \|f_i\|_{BMOA_{\alpha_i, \beta_i, \tau_i}^p} &\leq \\ &\leq \tilde{c} \left(\sup_{w \in B} \right) \int_B \frac{\prod_{i=1}^{\tilde{m}} |f_i|^p (1 - |w|)^{\tilde{\alpha}} dV(w)}{|1 - \bar{w}z|^\beta} \cdot (1 - |z|)^\tau; \quad (B1) \end{aligned}$$

for some values of parameters $(\alpha_1, \dots, \tilde{\alpha}_1, \dots, \tilde{\alpha}, \beta, \beta_1, \dots, \beta_m, \tau, \tau_1, \dots)$.

Moreover (B1) coincide with (B) for $p_i = p, i = 1, \dots, m$, under integral condition (A).

From (B1) we set immediately that the following is valid answer to our question. Each f_i from $\widehat{BMOA}_{\alpha, \beta, \tau}^p$ can be decomposed into "BMOA atoms" or " A_α^p atoms" or " H^p atoms" according to one functional known result in the unit ball in $\mathbb{C}^n$. Note also (A) vanishes for $m = 1$ and we get a well-known one functional result on atomic decomposition of BMOA spaces in the unit ball in $\mathbb{C}^n$.

Let use show now (B1) since (B) is clear. It is based on following simple Lemma (decomposition lemma (see [2], [3])).

Lemma. Let $s > -1, r, t \geq 0, (r + t - s) > n + 1$. Then we have that the following estimates are valid.

$$\int_B \frac{(1 - |w_1|)^s dV(w_1)}{|1 - \bar{w}w_1|^r |1 - w_1z|^t} \leq \begin{cases} \frac{1}{|1 - \bar{z}\bar{w}|^{r+t-s-n-1}}, & r - s, t - s < n + 1; \\ \frac{1}{(1 - |z|^2)^{r-s-n-1} |1 - \bar{z}w|^t}, & t - s < n + 1 < r - s; \\ \frac{1}{|1 - \bar{z}w|^t (1 - |z|)^{r-s-n-1}}, & t - s < n + 1 < r - s. \end{cases}$$

We now use this Lemma to show easily that (B1) is valid. We have easily the following chain of simple estimates in two unit ball in $\mathbb{C}^n$ based on (A).

We show that under (A) integral condition for some $\tilde{\alpha}$

$$\left(\sup_{w \in B} \right) \int_B \frac{\prod_{j=1}^m |f_j(w)|^p (1 - |w|)^{\tilde{\alpha}} dV(w)}{|1 - \bar{w}z|^{\tilde{\beta}}} \cdot (1 - |z|)^{\tilde{\tau}} \asymp$$

$$\asymp \prod_{i=1}^k \|f_i\|_{BMOA_{\alpha_i, \beta_i, \tau_i}^p}^p \cdot \prod_{i=k+1}^s \|f_i\|_{H^p}^p \cdot \prod_{i=s+1}^m \|f_i\|_{A_{s_i}^p}^p.$$

As result each f_i can be decomposed into $H^p, A_{\alpha}^p, BMOA^p$ atoms if $\tilde{f} = (f_1, \dots, f_m) \in BMOA_{\tilde{\alpha}, \tilde{\beta}, \tilde{\tau}}^p$. For $m = 1$ (A) vanishes and we set a known result on decomposition of $BMOA$ type space into atoms (see [17]).

Using (see [17])

$$\left(\int_B \frac{|f(z)|(1 - |z|)^t dV(z)}{|1 - \bar{z}w|^{\alpha}} \right)^q \leq (q \leq 1) \leq$$

$$\leq c \int_B \frac{|f(z)|^q \times (1 - |z|)^{tq + (q-1)(n+1)}}{|1 - \bar{z}w|^{\alpha q}} dV(z).$$

and Lemma (part 2) twice we have that from (A). (we consider f_1, f_2, f_3, f_4 the general case of many functions is the same).

$$\left(\int_B |f_1(w)|^p \times (1 - |w|)^{\alpha_1} dV(w) \right) \cdot \|f_2\|_{H^p}^p \cdot \|f_4\|_{BMOA_{\alpha, \nu, \tau}^p}^p \cdots \|f_3\|_{BMOA_{x, y, z}^p}^p \leq$$

$$\leq (m = 4) \leq$$

$$\leq c \left(\sup_{\tilde{w}_4 \in B} \right) \int_B \frac{|f_1(z)|^p \cdot |f_2(z)|^p \cdot |f_3(z)|^p \cdot |f_4(z)|^p \times (1 - |z|)^{p\beta + (n+1)(p-1)}}{(1 - |z|)^{\frac{p(\beta+n+1)}{m} - n + \frac{p(\beta+n+1)}{m} - (\alpha_1 + n + 1)}} dV(z) \times$$

$$\times (1 - |z|)^{\lambda_1} \times$$

$$\times \int_B \frac{(1 - |w_4|)^{\alpha} dV(w_4) \times (1 - |\tilde{w}_4|)^{\tau}}{|1 - \bar{w}_4 z|^{\frac{p(\beta+n+1)}{m}} |1 - w_4 \tilde{w}_4|^{\nu}} dV(z) \leq \tilde{c} \left\| \prod_{j=1}^4 |f_j| \right\|_{BMOA_{a, b, c}^p}^p \quad (G);$$

where $b = \nu, \nu - \alpha \leq n + 1, \lambda_1 = x - y + z + n + 1 - p(\beta + n + 1)/m$, and $c = \tau, y \geq z$ and $a = \alpha_1 + \alpha + 2n + 1 + \lambda_1$, generally $a = \sum \alpha_j + \sum \tilde{\alpha}_j + (m - 2)n + 1$.

Let us note as we discussed already above now the reverse of (G) follows directly from (C) and well-known uniform estimates of Hardy and Bergman spaces. We leave this easy computation interested reader.

The proof of our theorem is now complete.

Remark 4. Let $\{a_k\}$, where $k \geq 0$ be a fixed r -lattice in the unit ball B (see, for example, [14]). We here must also note, uniform estimates are well-known also for various Herz type spaces $H_{\alpha}^{p, q}$ (see [15]), so in (B) a group with norms of Herz spaces $\tilde{H}_{\alpha}^{p, q}$ can easily also be included.

In the unit ball it can be easily shown that (the same in pseudoconvex domains) is valid (see [9], [17], [10], [1]).

$$(\sup_{z \in B}) |f(z)|(1-|z|)^\tau \leq \tilde{c} \int_B \left(\int_{B(z,r)} |f(\tilde{z})|^p dV_\alpha(\tilde{z}) \right)^{q/p} dV(z) = \|f\|_{H_\alpha^{p,q}}^q$$

and

$$(\sup_{z \in B}) |f(z)|(1-|z|)^{\tilde{\tau}} \leq \tilde{c}_1 \sum_{k \geq 0} \left(\int_{B(z,r)} |f(\tilde{z})|^p dV_\alpha(\tilde{z}) \right)^{q/p} dV(z) = \|f\|_{\tilde{H}_\alpha^{p,q}}^q$$

for $0 < p, q < \infty, \alpha > -1$, for some $\tau, \tilde{\tau}$ and $\tau = \tau(p, q, \alpha); \tilde{\tau} = \tilde{\tau}(p, q, \alpha)$ and in $(B), (B1)$ these group of spaces can also be easily added. The following estimates can be used for cases when some f_j in such spaces in proof of theorem if $p = q$ (all this can be valid in bounded pseudodomains) (See [17]).

$$1) \int_{B(z,r)} \frac{(1-|w|)^\alpha dV(w)}{|1-w\tilde{w}|^\beta} \leq c \frac{1}{|1-z\tilde{w}|^{\beta-n-1}} (1-|z|)^\alpha, z, w \in B, \beta > n+1, \alpha \geq 0.$$

$$2) \sum_{k \geq 0} \frac{(1-|a_k|)^\tau}{|1-a_k z|^\beta} \leq \frac{\tilde{c}_1}{(1-|z|)^{\beta-\tau}}, \tau > 1, \beta > \tau.$$

$$3) \int_{B(z,r)} \frac{(1-|z|)^\alpha dV(z)}{|1-\tilde{w}z|^\beta} \leq \frac{\tilde{c}}{|1-wa_k|^{\beta-\alpha-n-1}}, w, a_k \in B, k = 1, 2, \dots,$$

$$\beta > \alpha + n + 1, \alpha > -1.$$

We now also formulate the same result with very similar proof for pseudoconvex domains with smooth boundary in $\mathbb{C}^n$.

We need some well-known basic definitions in bounded strongly pseudoconvex domains and function spaces on them and some preliminary facts on function theory in such type domains.

Throughout this paper $H(\Omega)$ denotes the space of all holomorphic functions on an open set $\Omega \subset \mathbb{C}^n$.

We follow notation from [14]. Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, let $\delta(z) = \text{dist}(z, \partial\Omega)$.

Then there is a neighborhood U of $\bar{\Omega}$ and $\rho \in C^\infty(U)$ such that $\Omega = \{z \in U : \rho(z) > 0\}$, $|\nabla \rho(z)| \geq c > 0$ for $z \in \partial\Omega$, $0 < \rho(z) < 1$ for $z \in \Omega$ and $-\rho$ is strictly plurisubharmonic in a neighborhood U_0 of $\partial\Omega$. Note that $\delta(z) \asymp \rho(z), z \in \Omega$. Then there is an $r_0 > 0$ such that the domains $\Omega_r = \{z \in \Omega : \rho(z) > r\}$ are also smoothly bounded strictly pseudoconvex domains for all $0 \leq r \leq r_0$. Let $d\sigma_r$ be the normalized surface measure on $\partial\Omega_r$ and dv the Lebesgue measure on Ω . The following mixed norm spaces were investigated in [14]. For $0 < p < \infty, 0 < q \leq \infty, \delta > 0$ and $k = 0, 1, 2, \dots$ set

$$\|f\|_{p,q,\delta;k} = \left(\sum_{|\alpha| \leq k} \int_0^{r_0} \left(r^\delta \int_{\partial\Omega_r} |D^\alpha f|^p d\sigma_r \right)^{q/p} \frac{dr}{r} \right)^{1/q}, 0 < q < \infty$$

and weighted Hardy space ($A_0^{p,\infty} = H^p$)

$$\|f\|_{p,\infty,\delta;k} = \sup_{0 < r < r_0} \sum_{|\alpha| \leq k} \left(r^\delta \int_{\partial\Omega_r} |D^\alpha f|^p d\sigma_r \right)^{1/q}, 0 < q < \infty,$$

where D^α is a derivative of f (see [14]). The corresponding spaces $A_{\delta;k}^{p,q} = A_{\delta;k}^{p,q}(\Omega) = \{f \in H(\Omega) : \|f\|_{p,q,\delta;k} < \infty\}$ are complete quasi normed spaces, for $p, q \geq 1$ they are Banach spaces. We mostly deal with the case $k = 0$, when we write simply $A_\delta^{p,q}$ and $\|f\|_{p,q,\delta}$. We also consider this spaces for $p = \infty$ and $k = 0$, the corresponding space is denoted by $A_\delta^{\infty,p} = A_\delta^{\infty,p}(\Omega)$ and consists of all $f \in H(\Omega)$ such that

$$\|f\|_{\infty,p,\delta} = \left(\int_0^{r_0} \left(\sup_{\partial\Omega_r} |f| \right)^p r^{\delta p - 1} dr \right)^{1/p} < \infty.$$

Also, for $\delta > -1$, the space $A_\delta^\infty = A_\delta^\infty(\Omega)$ consist of all $f \in H(\Omega)$ such that

$$\|f\|_{A_\delta^\infty} = \sup_{z \in \Omega} |f(z)| \rho(z)^\delta < \infty,$$

and the weighted Bergman space $A_\delta^p = A_\delta^p(\Omega) = A_{\delta+1}^{p,p}(\Omega)$ consists of all $f \in H(\Omega)$ such that

$$\|f\|_{A_\delta^p} = \left(\int_\Omega |f(z)|^p \rho^\delta(z) dv(z) \right)^{1/p} < \infty.$$

We denote by K_β the weighted Bergman kernel on Ω (see [7], [13]).

Since $|f(z)|^p$ is subharmonic (even plurisubharmonic) for a holomorphic f , we have $A_s^p(\Omega) \subset A_t^\infty(\Omega)$ for $0 < p < \infty, sp > n$ and $t = s$. Also, $A_s^p(\Omega) \subset A_s^1(\Omega)$ for $0 < p \leq 1$ and $A_s^p(\Omega) \subset A_t^1(\Omega)$ for $p > 1$ and t sufficiently large. Therefore we have an integral representation

$$f(z) = C_\beta \int_\Omega f(\xi) K(z, \xi) \rho^t(\xi) dv(\xi), \quad f \in A_t^1(\Omega), z \in \Omega, \quad (*)$$

where $K(z, \xi)$ is a kernel of type t , that is a smooth function on $\Omega \times \Omega$ such that $|K(z, \xi)| \leq C |\tilde{\Phi}(z, \xi)|^{-(n+1+t)}$, where $\tilde{\Phi}(z, \xi)$ is so called Henkin-Ramirez function for Ω . Note that (*) holds for functions in any space X that embeds into A_t^1 . We review some facts on $\tilde{\Phi}$ and refer reader to [15] for details. This function is C^∞ in $U \times U$, where U is a neighborhood of $\bar{\Omega}$, it is holomorphic in z , and $\tilde{\Phi}(\zeta, \zeta) = \rho(\zeta)$ for $\zeta \in U$. Moreover, on $\bar{\Omega} \times \bar{\Omega}$ it vanishes only on the diagonal $(\zeta, \zeta), \zeta \in \partial\Omega$. Locally, it is up to a non vanishing smooth multiplicative factor equal to the Levi polynomial of ρ . From now on the work with a fixed Henkin-Ramirez function $\tilde{\Phi}$.

The following lemmas from [3], [16] will be used by us during the proof partially.

Lemma 1.

Let $\tilde{\alpha}, \beta > -1, s > 0, y \in \Omega, 0 < t < t_0 = t_0(\lambda, r)$ then

$$\int_{\{x:r(x)=t\}} \left| K_{\alpha}(x,y) \right|^s d\sigma(x) \asymp [r(y)+t]^{n-q}; n < q;$$

and

$$\sup_{w \in \Omega} \left| K_{\alpha}(z,w) \right| \delta^v(z) \leq C \delta^{-\tilde{\alpha}+v}(z); \quad (S)$$

$v \geq 0, v - \tilde{\alpha} < 0$ and

$$\int_{\Omega} \left| K_{\alpha}(x,y) \right|^s \left(r(x) \right)^{\beta} dv(x) \asymp \left(r(y) \right)^{n-q+\beta+1}; n-q+\beta+1 < 0;$$

and $r(y) \asymp \delta(y), y \in \Omega; q = \alpha s$.

Lemma B.

Let $t-s < n+1 < r-s, s > -1, r, t \geq 0, r+t-s > n+1$

$$\int_{\Omega} \left(\delta^s(z) |K_r(z,w)| \cdot |K_t(z,w_1)| \right) dv(z) \leq \frac{C}{(\delta(w))^{r-s-n-1}} |K_t(w,w_1)|;$$

where $w, w_1 \in \Omega$.

The following substitutions of estimates must be made in the proof of pseudoconvex domains case and the rest is the repetition of unit ball proof (see for these estimates [1], [2], [3], [8], [9], [17]).

$$\begin{aligned} 1) & \left(\int_{\Omega} |f(w)| \cdot |K_{\tau}(z,w)| \delta^v(w) dV(w) \right)^p \leq \\ & \leq \tilde{c} \int_{\Omega} |f(w)|^p \cdot |K_{\tau}(z,w)|^p \delta^{vp+(n+1)p-(n+1)} dV(w), p \leq 1, v > -1, \tau > 0; \end{aligned}$$

$$2) \int_{\Omega} |K_{\tau}(z,w)| \cdot |K_{\tau_1}(\tilde{z}, \tilde{w})| \delta^r(\tilde{w}) dV(\tilde{w}) \leq \tilde{c} \delta^{\tau_1-r-(n+1)}(z) |K_{\tau}(z, \tilde{z})|,$$

$$r > -1, \tau, \tau_1 > 0, \tau - r < n+1, \tau_1 - r > n+1, \tau + \tau_1 - r - (n+1) > 0.$$

Also uniform estimates are valid for our spaces H^p, A_{α}^p , BMOA and Herz type analytic spaces. Namely the following estimates are valid for bounded pseudoconvex domains.

Namely (see [1], [2], [3], [9], [17])

$$|f(z)| \delta(z)^{\frac{n}{p}} \leq c \|f\|_{H^p}^p$$

$$|f(z)| \delta(z)^{\frac{n+\alpha+1}{p}} \leq \tilde{c} \|f\|_{A_{\alpha}^p(\Omega)}^p;$$

$$|f(z)|^p \delta(z)^{\alpha-\beta+\tau+n+1} \leq \tilde{\tilde{c}} \|f\|_{BMOA_{\alpha,\beta,\tau}^p}^p; z \in \Omega.$$

The last estimate (see [16]) is based on estimate from below of Bergman kernel. Similar uniform estimates are valid for Herz spaces in bounded pseudoconvex domains (see Remark 4). The general form of our theorem in bounded pseudoconvex domains has the following form (complete analogues are valid in tube domains).

Let $\widetilde{BMOA}_{\tilde{\alpha}, \tilde{\beta}, \tau}^p = \{(f_1, \dots, f_m); f_j \in H(\Omega), j = 1, \dots, m\}$,

$$\left(\sup_{z \in \Omega} \left(\int_{\Omega} |f_1(\tilde{w}) \dots f_m(\tilde{w})|^p \cdot |K_{\tau}(z, \tilde{w})| \delta^{\tilde{\alpha}}(\tilde{w}) dV(\tilde{w}) \right) \right) \delta^{\tilde{\beta}}(z) < \infty$$

for $0 < p \leq 1, \tau > 0, \tilde{\alpha} > -1, \tilde{\beta} \geq 0$.

Theorem 2. Let $p \leq 1, \tau > 0, \tilde{\alpha} > -1, \tilde{\beta} \geq 0$. Let $l \in \mathbb{N}$. Then for $v_k - u_k \leq n + 1, k = 1, \dots, l$

$$\|f_1 \dots f_l\|_{\widetilde{BMOA}_{\tilde{\alpha}, \tilde{\beta}, \tau}^p} \asymp \prod_{k=1}^m \|f_k\|_{A_{\alpha_k}^p} \cdot \prod_{k=m+1}^n \|f_k\|_{H^p} \cdot \prod_{k=n}^l \|f_k\|_{BMOA_{u_k, v_k, s}^p}$$

for some values $v_k, u_k, s_k; \alpha_j; j = 1, \dots, m, k = n, \dots, l$ of parameters, if for large enough β

$$f_1(w_1) \dots f_l(w_l) = \int_{\Omega} f_1(z) \dots f_l(z) \prod_{j=1}^l \left[K_{\frac{\beta+n+1}{m}}(z, w_j) \right] \delta^{\beta}(z) dV(z) \quad (R)$$

for $w_j \in \Omega, j = 1, \dots, l, \beta > \beta_0; \beta_0$ is large enough, $\alpha_k > -1, k = 1, \dots, m; s_k > 0; v_k \geq 0; u_k > -1, k = n, \dots, l$.

As a corollary of this theorem easily we have that each (f_j) function from $\widetilde{BMOA}_{\tilde{\alpha}, \tilde{\beta}, \tau}^p$ can be decomposed into H^p and $A_{\alpha_k}^p$ atoms as far as integral (R) condition is valid for large enough β .

Remark 5. A group from analytic Herz spaces (see discussion in the unit ball above) can also be easily added in case of bounded pseudoconvex domains in $\mathbb{C}^n$.

We provide basic definitions of function theory in tubular domains (see [5]).

Let $T_{\Omega} = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n -dimensional Euclidean space V . $H(T_{\Omega})$ denotes the space of all holomorphic functions on T_{Ω} . Let

$$B_V(z, \omega) = C_V \Delta^{-(v + \frac{n}{r})}((z - \bar{\omega})/i)$$

is the Bergman reproducing kernel for $A_V^2(T_{\Omega})$. Where Δ^{α} is determinant function of our tubular domain (see[5]).

We define new analytic multifunctional $BMOA \in T_{\Omega}$ as follows

$$\widetilde{BMOA}_{\alpha, \beta, \tau}^p = \left\{ (f_1, \dots, f_m), f_j \in H(T_{\Omega}), j = 1, \dots, m, \right. \\ \left. \sup_{w \in T_{\Omega}} \int_{T_{\Omega}} |f_1(z) \dots f_m(z)|^p \Delta^{\alpha}(Imz) |B_{\tau}(z, w)|^{\beta} dV(z) < \infty, \right\},$$

where $p \in (0, \infty)$, $\alpha, \beta \geq 0$, $\tau \geq 0$. Note that obviously, if $\beta = 0, m = 1$ then we have classical Bergman space in tubular domain and the atomic decomposition of this space is well-known. By simple substitution of estimates of our proof in pseudoconvex domains we can obtain similar results in tubular domains under some additional condition on Bergman kernel $B_\tau(z, w)$.

Remark 6. Results of this note can be valid also for $p \geq 1$ case, we note obviously one part of these theorems based only on uniform estimates is valid for all positive values of p . As for the rest the complete analogues of our estimates which we used for $p \leq 1$ case exist also for $p \geq 1$ (see, for example, [3], [13], [14], [17] and references there).

Similar type sharp results are valid with similar proofs for multifunctional BMOA type spaces defined similarly, but in products of tubular and pseudoconvex domains.

Similar results are valid if you add weighted Hardy or Bloch type spaces into the group of analytic function spaces in pseudoconvex domains in our main theorem. (see for these classes [10,16,3,1]).

Competing interests. The authors declare that there are no conflicts of interest regarding authorship and publication.

Contribution and Responsibility. All authors contributed to this article. Authors are solely responsible for providing the final version of the article in print. The final version of the manuscript was approved by all authors.

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УДК 517.55+517.33

Научная статья

**О новых точных теоремах разложения для
многофункциональных пространств типа ВМОА в ограниченных
псевдовыпуклых областях**

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Мы приводим новые эквивалентные выражения в единичных шаровых и псевдовыпуклых областях для многофункционального аналитического пространства типа ВМОА. Мы расширяем в различных направлениях известную теорему атомарного разложения пространств типа ВМОА в единичном шаре.

Ключевые слова: единичный шар, аналитические функции, аналитические пространства, псевдовыпуклая область, пространства Харди, пространства Бергмана, пространства типа ВМОА.

DOI: 10.26117/2079-6641-2020-32-3-102-113

Поступила в редакцию: 30.04.2020

В окончательном варианте: 08.08.2020

Для цитирования. Shamoyan R. F., Tomashevskaya E. B. On new sharp theorems for multifunctional ВМОА type spaces in bounded pseudoconvex domains // *Вестник КРАУНЦ. Физ.-мат. науки.* 2020. Т. 32. № 3. С. 102-113. DOI: 10.26117/2079-6641-2020-32-3-102-113

Конкурирующие интересы. Авторы заявляют, что конфликтов интересов в отношении авторства и публикации нет.

Авторский вклад и ответственность. Все авторы участвовали в написании статьи и полностью несут ответственность за предоставление окончательной версии статьи в печать. Окончательная версия рукописи была одобрена всеми авторами.

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