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TEACHING MATERIALS

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SOLUTIONS OF THE PROBLEMS FOR MATHEMATICAL COMPETITION «VITUS BERING – 2016»

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The paper considers the solutions of the problems for the Mathematical Competition «Vitus Bering – 2016» for high school students. It was held at Kamchatka State University in April 2016.

Key words: mathematical competition for high school students

Introduction

The paper contains neither results of original research no a review of such results. It is devoted to the Mathematical Competition «Vitus Bering – 2016» for high school students which was held at Vitus Bering Kamchatka State University at the beginning of April 2016. The paper also presents the problems of the Competition and the solutions. It was the second Mathematical Competition held at the Physical-Mathematical Faculty in 2015/2016. The paper [1] describes the problems of the first Competition which took place in November 2015.

Subject Competitions are effective and time-tested means to attract talented students to the science and to organize the selection for higher educational establishments.

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Initiation of the Common State Examination to select students for higher educational institutions increased the interest of these establishments to organize their own Competitions. The prize-winners of Competitions get extra credits to their credits for the Common State Examination when entering a corresponding higher educational institution.

The Mathematical Competition «Vitus Bering – 2016» had only one round and included 6 problems of diverse complexity for high schools students of 9-11 grades. The participants of the Competition had 3 hours to complete the tasks. Some typical problems from the workbook [2] were chosen by the organizers when preparing the Competition tasks.

Further, we present the tasks of the Competitions and their solutions.

Competition tasks

1) (10 credits) There is a sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ... in which $a_{n+2} = a_{n+1} + a_n$ for any $n = 1, 2, \ldots$ Find the greatest common factor of the numbers a_{100} and a_{99}.

2) (10 credits) In an online store having less than 20 smartphone models, the number of six-inch smartphones is multiple of the number of five-inch smartphones the number of which, in its turn, is three times less than that of the four-inch smartphones. If the number of six-inch smartphones is doubled, their number will exceed the number of five-inch smartphones by 14. How many four-inch smartphones are there in the online store?

3) (10 credits) There is a polynomial $J(x) = x^2 + x + 2016$. The polynomial $H(x) = x^3 + ax^2 + bx + c$ has only three different real roots, and the polynomial $H(J(x))$ does not have any real roots. Prove that $H(2016) > 1/64$.

4) (10 credits) There is a function $f(x) = \dfrac{1}{\sqrt[3]{1-x^3}}$. Find $\underbrace{f(\ldots f(f(\sqrt[3]{2016})))}_{2016}$.

5) (10 credits) Find all prime numbers p, q such that $p + q = (p-q)^3$.

6) (10 credits) Prove that the product $(n+1)(n+2)..(2n-1)2n$ is divided by $n!$.

7) (10 credits) For which a, has the system

$$\begin{cases} y = 2 - \sqrt{a^2 - 9 + 6x - x^2} \\ -x^2 + 3x + yx + 3y + 18 = 0 \end{cases}$$

2 solutions?

Solutions

1) For $n = 98$ we obtain $a_{100} = a_{99} + a_{98}$. To find the greatest common factor (GCF), we apply the Euclidean algorithm to this expression. Then the remainder of the division is $r_1 = a_{100} : a_{99} = a_{98}$.

   For $n = 97$ we obtain $a_{99} = a_{98} + a_{97}$. We apply the Euclidean algorithm and the remainder of the division is $r_2 = a_{99} : a_{98} = a_{97}$.

   Continuing in a similar way, for $n = 1$ we obtain $a_3 = a_2 + a_1$ and the remainder is $r_{98} = a_3 : a_2 = a_1$. As long as $a_2 = 1$ and $a_1 = 1$, then $r_{99} = 0$ and the Euclidean algorithm is completed.

   Thus, GCF (a_1, a_2) = GCF(a_2, a_3) = ... = GCF(a_{100}, a_{99}) = 1.

   Answer. 1.

2) Assume that m_4, m_5, m_6 is the number of four-, five- and six-inch telephones. From the statement of the problem, we have the following limitations:

$$\begin{cases} m_4+m_5+m_6<20, \\ m_6=k\cdot m_5, \\ m_4=3m_5, \\ 2m_6=m_5+14. \end{cases}$$

We apply the second and the third equations to transform the first and the fourth expressions

$$\begin{cases} 3m_5+m_5+k\cdot m_5<20, \\ 2k\cdot m_5=m_5+14, \end{cases} \Leftrightarrow \begin{cases} (4+k)m_5<20, \\ (2k-1)m_5=14=7\cdot 2. \end{cases}$$

Proceeding from the problem situation, k and m_5 should be integral numbers. Then from the latest equation, we obtain that $2k-1=7$ and $m_5=2$, whence $k=4$ and $m_5=2$.

These values satisfy the inequality $(4+k)m_5<20$.

Thus, the number of four-inch smartphones is $m_4=3\cdot 2=6$.

Answer. 6.

3) As long as the polynomial $H(x)$ has three real roots, it can be represented in the form

$$H(x)=(x-x_1)(x-x_2)(x-x_3),$$

where x_i are the real numbers.

We consider a polynomial

$$H(J(x))=(J(x)-x_1)(J(x)-x_2)(J(x)-x_3),$$

which does not have real roots according to the problem situation.

Consequently, discriminant of each factor of the form

$$J(x)-x_i=x^2+x+2016-x_i,$$

should be not less than null.

We calculate the discriminant

$$D=1-4(2016-x_i)<0,$$
$$(2016-x_i)>\frac{1}{4},$$

then the polynomial value is $H(2016)=(2016-x_1)(2016-x_2)(2016-x_3)>\dfrac{1}{64}$.

Which was to be proved.

4) Based on the problem situation, $f(x)=\dfrac{1}{\sqrt[3]{1-x^3}}$.

We consider $f(f(x))=\dfrac{1}{\sqrt[3]{1-\left(\frac{1}{\sqrt[3]{1-x^3}}\right)^3}}=\dfrac{1}{\sqrt[3]{1-\frac{1}{1-x^3}}}=\dfrac{1}{\sqrt[3]{\frac{-x^3}{1-x^3}}}=-\dfrac{\sqrt[3]{1-x^3}}{x}$

$$f(f(f(x)))=\frac{1}{\sqrt[3]{1-\left(-\frac{\sqrt[3]{1-x^3}}{x}\right)^3}}=\frac{1}{\sqrt[3]{1+\frac{1-x^3}{x^3}}}=\frac{1}{\sqrt[3]{\frac{1}{x^3}}}=x,$$

$\underbrace{f(f(f(f(x))))}_{4}=\dfrac{1}{\sqrt[3]{1-x^3}}$ и т. д.

Thus, superposition of the functions $f(x)$ possesses only three values

$$\underbrace{f(\ldots(f(f(x))))}_{n}=\begin{cases}\dfrac{1}{\sqrt[3]{1-x^3}}, & n=1,\ 4,\ 7,\ \ldots\\ -\frac{\sqrt[3]{1-x^3}}{x}, & n=2,\ 5,\ 8,\ \ldots\\ x, & n=3,\ 6,\ 9,\ \ldots\end{cases}$$

As long as 2016 is multiple of 3, we obtain that $\underbrace{f(...f(f(x))))}_{2016}=x$.

Whence $\underbrace{f(...f(f(\sqrt[3]{2016}))))}_{2016}=\sqrt[3]{2016}$

Answer. $\sqrt[3]{2016}$.

5) From the problem situation p and q are prime numbers, consequently, they are integral numbers. Then $p+q$ and $p-q$ are also integral numbers. We change $p-q=m$, where m is an integer, then $p=m+q$ and the basic relation $p+q=(p-q)^3$ is written in the form

$$2q+m=m^3.$$

We divide the left and the right parts by m:

$$\frac{2q}{m}+1=m^2,$$
$$\frac{2q}{m}=m^2-1. \tag{1}$$

As long as m is an integral number, then m^2-1 are integral numbers and, consequently, $\dfrac{2q}{m}$ is an integer. As long as q is a prime number, then $m=1$ or $m=2$, or $m=q$. We substitute the value $m=1$ into expression (1). Then $2\cdot q=0$, whence $q=0$. This value does not fit, since it is not an integral number. Substituting the value $m=2$ into expression (1), we obtain an integral number $q=3$ and, estimating the value $p=m+q=5$, we also obtain an integer. For $m=q$ the value $q^2=3$ and q is not an integer.

Answer. $q=3,\ p=5$.

6) The product $(n+1)(n+2)..(2n-1)2n$ can be represented in the following form: $(n+1)(n+2)..(2n-1)2n=\frac{(2n)!}{n!}$. Consequently, we should say that the fraction $\frac{(2n)!}{n!}$ is divided evenly by $n!$, i. e. $\frac{(2n)!}{n!n!}$ is an integral number.

From the other hand, the number of combinations is $C_{2n}^{n}=\frac{(2n)!}{n!n!}$, and it is an integral number.

Which was to be proved.

7) We consider the first equation of the system $y=2-\sqrt{a^2-9+6x-x^2}$ and transform it $\sqrt{a^2-9+6x-x^2}=2-y$, $\sqrt{a^2-(3-x)^2}=2-y$, whence $\begin{cases}a^2-(3-x)^2=(2-y)^2,\\ 2-y\geq 0.\end{cases}$

Thus, we obtain an equation of the line $\begin{cases}(x-3)^2+(y-2)^2=a^2,\\ y\leq 2\end{cases}$ the graph of which is the lower semicircle with the center at $(3;2)$ and radius $r=|a|$.

We consider the second equation $-x^2+3x+yx+3y+18=0$ and transform it $(x+3)(-x+y+6)=0$. Thus, we obtain two straight lines with the equations $x=-3$, $y=x-6$. We plot these "variable-value"planes (Figure).

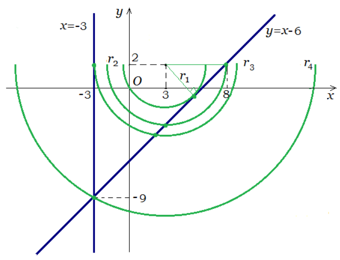

Figure. Plot for task 7.

The system has two solutions for $r \in (r_1; r_2] \cup [r_3; r_4) \cup (r_4; +\infty)$.

We find the values r_1, r_2, r_3, r_4: the value of r_1 as a leg of right isosceles triangle with a hypotenuse of 5 is $r_1 = 5 : \sqrt{2} = 2{,}5\sqrt{2}$, $r_2 = 5$, $r_3 = 6$. From a right triangle with the legs of 6 and 11, we find the hypotenuse $r_4 = \sqrt{6^2 + 11^2} = \sqrt{157}$.

Whence, $|a| \in (2{,}5\sqrt{2};\ 5] \cup [6; \sqrt{157}) \cup (\sqrt{157}; +\infty)$.

Thus, we obtain $a \in (-\infty; -\sqrt{157}) \cup (-\sqrt{157}; -6] \cup [-5; -2{,}5\sqrt{2}) \cup (2{,}5\sqrt{2};\ 5] \cup [6; \sqrt{157}) \cup (\sqrt{157}; +\infty)$.

Answer. $a \in (-\infty; -\sqrt{157}) \cup (-\sqrt{157}; -6] \cup [-5; -2{,}5\sqrt{2}) \cup (2{,}5\sqrt{2};\ 5] \cup [6; \sqrt{157}) \cup (\sqrt{157}; +\infty)$.

Conclusions

The Mathematical Competition took place in April 2016 at the Faculty of Physics and Mathematics of Vitus Bering Kamchatka State University for high school students. It was the second Competition within the academic year of 2015/2016. It was organized in order to attract the potential students to the physical and mathematical special subjects at higher educational institutions and to provide additional training before the Common State Examination and the State Final Examination. When participating in the entrance competitions to the Vitus Bering Kamchatka State University, the prize-winners of the Competition «Vitus Bering» get extra credits to their credits for the Unified State Exam.

The authors hope that the illustrated problems and their solutions will help the high school students to prepare for exams and to enter the chosen higher educational establishments.

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